Demonstratio Mathematica
Vol. 23 No. 1-4 1990

G.K.V. LIB., Hardwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

130274

Book Verification-2

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ISSN 1213

DEMONSTRATIO MATHEMATICA

VOL. XXIII
№ 1
1990

130274

I-M

130274

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

TECHNICAL UNIVERSITY OF WARSAW

POLITECHNIKA WARSZAWSKA

WARSZAWA

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EDITORIAL BOARD

Henryk Adamczyk, Maciej Mączyński *editor*,
Agnieszka Plucińska, Danuta Przeworska-Rolewicz,
Tadeusz Traczyk, Magdalena Tryjarska,
Zbigniew Żekanowski

SCIENTIFIC EDITORS

Elżbieta Ferenstein, Lech Polkowski
Magdalena Tryjarska

TECHNICAL EDITOR

Maria Mączyńska

Warsaw Technical University
Institute of Mathematics
DEMONSTRATIO MATHEMATICA
Pl. Politechniki 1
00-661 Warszawa, POLAND

Nakł. 475 + 30. Ark. wyd. 12,0. Ark. druk. 17. Papier offset. kl. III 70 g. Oddano do druku 1990.01.23. Druk ukończono w listopadzie 1990 r. Zamówienie 33/90. Cena zł. 12.000,—

Zakład Graficzny Politechniki Warszawskiej, ul. Kopińska 12/16, 02-321 Warszawa.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Contents

Jacek SKWIERCZYŃSKI: Dual variational formulation for a hyperbolic problem . . . . . 1—13

VU VAN HUNG: The stability of the difference schema approximating Cauchy's problem for the second order parabolic equation with variable coefficients in L_2 . . . . . 15—23

Józef SIDOROWICZ: Оценка переходной плотноси диффузионного процесса в тонкой трубке на гладком римановом многообразии 25—32

Ali A.A. DAW: Retracts of homogeneous-universal Post algebras and their generalizations . . . . . 33—38

Marian HOTLOŚ: Some theorems on doubly warped products . . . 39—58

Mieczysław KRÓL: Some properties of a quasi-diffusion vector-valued process . . . . . 59—67

Stanisław MIDURA: Sur les solutions des équations fonctionnelles qui déterminent certains sous-demi-groupes à deux paramètres du groupe L^1_4 . . . . . 69—81

Grażyna KOZŁOWSKA: Rearrangements of functions in some function spaces . . . . . 83—86

Jerzy MOTYL: Existence and uniqueness of solution of neutral functional-differential inclusions via dissipativity . . . . . 87—100

Andrzej RYCHLEWICZ: On the algebraic embeddability in the class of functions possessing connected graphs . . . . . 101—111

Dan Tudor VUZA: Uniform continuity and uniform convergence . . 113—121

Joanna SZYMAŃSKA-ZAGRODNY: Some sufficient conditions in constrained nonsmooth optimization . . . . . 123—132

Rüdiger GÜNTTNER: On the uniform approximation of continuous functions by the Euler means of Fourier series . . . . . 133—138

J. BARANOWICZ, S. WALCZAK: Necessary conditions for the existence of a solution of some extremal problems in the families of holomorphic functions . . . . . 139—154

Zygfryd KOMINEK: On approximately subadditive functions . . . 155—160

Jan KUREK: On the natural operators of vertical prolongation type on 1 — connection bundle . . . . . 161—173

Andrzej KASPERSKI: The modular spaces of multifunctions, II . . . 175—183

Jan STRABURZYŃSKI: The multi-valued Goursat problem for functional-differential equations of hyperbolic type . . . . . 185—190

NGUYEN VAN MAU: Interpolation problems induced by right and left invertible operators and its applications to singular integral equations . . . . . 191—212

Valeriu POPA: Theorems of unique fixed point for expansion mappings . . . . . 213—218

Jacek DĄBROWSKI: Asymptotic normality of a certain estimator in partially observed linear systems . . . . . 219—228

Tadeusz JAGODZIŃSKI: On Galerkin approximations to pseudo-parabolic problem . . . . . 229—239

Janina KOTUS: The oscillating trajectories and saddles at infinity of vector fields on open surfaces . . . . . 241—251

Manfred GOEBEL, Marian MAJCHROWSKI: Distributed and boundary control of a nonlinear elliptic mixed boundary value problem . . 253—266

Bogusław KACZMARSKI: On the category and Borel type of the set of points of one-sided non-differentiability . . . . . 267—270

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Cena zł 12.000,–

CC-0. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Contents

Marek ROCHOWSKI: A scheme of generating canonical frames in normal bundles of immersed manifolds in Euclidean space 271-300

T.D. NARANG: Metric projections in linear metric spaces 301-307

M. KIRANE, A. YOUKANA: A reaction-diffusion system modelling the post irridiation oxydation of an isotactic polypropylene 309-321

Shyam L. SINGH, Umesh C. GAIROLA: A coincidence theorem for three systems of transformations 323-327

Heinrich BEGEHR, CHEN XIU-GIU: Moment problem in $\mathbf{R}^n$ 329-333

Cornelia-Livia BEJAN: CR-submanifolds of hyperbolical almost Hermitian manifolds 335-343

Wiesława CHROMIK: Externally compatible odentities of algebras 345-355

B.G. PACHPATTE: On abstract second order differential equations 357-366

Hanna MATUSZCZYK: Vector fields and differential forms on premanifolds . . . 367-374

Francisco GÓMEZ: The number of generators of the algebra of Kähler differentials 375-383

Elżbieta ŁUCZAK-KUMOREK: The existence theorem for neutral functional-differential inclusions 385-393

Jan AMBROSIEWICZ: Powers of sets in linear groups 395-403

Piotr MULTARZYŃSKI, Wiesław SASIN: On the dimension of differential spaces 405-415

Czesław DYJAK: BV-solutions of nonlinear functional equations of higher order . 417-423

Michał KISIELEWICZ, Jerzy MOTYL: Controllability of the neutral functional-differential inclusion with M-dissipative right hand side 425-433

Andrzej GĘBAROWSKI: On a class of homogeneous conformally recurrent manifolds 435-445

James G. RAFTERY, Teo STURM: Completions of uniform universal algebras and strongly uniform BCK-algebras 447-467

H.W. PU, H.H. PU: Selective derivatives for monotone functions 469-475

Maciej SABLIK: Some remarks on Cauchy equation on a curve 477-490

Antoni SADOWSKI: A Goursat problem for Mangeron polyvibrating equation in $\mathbf{R}^n$, $n \geqslant 3$ 491-506

Ireneusz NABIAŁEK: Vector spaces of regular functional configurations 507-514

Anatolij DVUREČENSKIJ: Frame functions and completeness 515-519

Joanna GER: A closed epigraph theorem 521-528

Danuta PRZEWORSKA-ROLEWICZ: Remarks on multipliers for right inverses in D-algebras 529-539

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

cena zł 11,00 -

Digitized by Arya Samaj Foundation Chennai and eGangotri

Contents

130274

Józef SIDOROWICZ: О условном процессе с дифракцией на гладкой области в R^2 541-547

Andrzej WALENDZIAK: The Kuroš-Ore property, replaceable irredundant decompositions 549-556

Marek ROCHOWSKI: Parallel cross sections and canonical frames in normal bundles of immersed manifolds in Euclidean space 557-565

Jerzy PEŁCZEWSKI: Application of Legendre polynomials method for some optimal control problems 567-576

V. GUPTA, P.N. AGRAWAL, A. SAHAI, T.A.K. SINHA: L_p - approximation by combination of modified Szasz-Mirakyan operators 577-591

B.G. PACHPATTE: On an integral inequality in two variables 593-597

Wanda SZNITKO: On singular integral equation in Euclidean space E_3 599-604

Antoni SADOWSKI: Concerning some sequences of iterates in R^n-space 605-615

Andrzej ROGOWSKI: Necessary conditions for optimality in problems with hyperbolic partial differential equations 617-632

DO DUC HUNG, Jerzy GAWINECKI: Matrix of fundamental solutions for the system of dynamic equations of classical three-dimensional thermoelasticity theory 633-647

Grażyna KOZŁOWSKA: Some remarks on modular approximation by a filtered family of linear operators 649-657

Roman TABERSKI: Partial exponential approximation 659-676

Aleksei TRALLE: One new existence theorem for the generalized symmetric Riemannian spaces with solvable isometry group 677-683

S.L. SINGH, U.C. GAIROLA: A general coincidence theorem 685-690

Eugeniusz CHRZANOWSKI: On some problems for the system of diffusion equations with overspecified boundary data 691-698

Vladimír JANIŠ: Notes on sums of Boolean algebras and logics 699-708

R.A. RASHWAN: On the convergence of Mann iterates to a common fixed point for a pair of mappings 709-712

Bogdan RZEPECKI: On the hyperbolic equation $z_{xy} = f(x,y,z)$ in Banach spaces. . 713-718

T.D. NARANG, S.K. GARG: Farthest points in ultrametric spaces 719-722

Mariusz STRZEŚNIEWSKI: An example of ambivalent, approximately continuous but not strongly Baire 1 function 723-726

Danuta OZDARSKA: On the relationship between bounded solutions of linear and quasilinear differential equations in Banach spaces 727-738

D. PRZEWORSKA-ROLEWICZ, S. ROLEWICZ: On the spectrum of right inverses 739-751

Eberhard WAGNER: Über ein abzählbares System gewöhnlicher linear Differentialgleichungen von Bandstruktur 753-773

Andrzej BORZYMOWSKI, Zawam A. DALLA: A boundary value problem for a system of hyperbolic differential equations in R^m 775-795

Andrzej STROJNOWSKI, Włodzimierz WOJAS: Some remarks on 0-groups . . . 797-801

Maciej J. MĄCZYŃSKI: A theorem on representation of algebraic systems admitting a full set numerical functionals 803-812

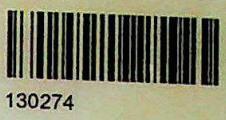
LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Cena zł 12.000,-

Digitized by Arya Samaj Foundation Chennai and eGangotri



Contents

Danuta STACHOWIAK-GNIŁKA: Classes and spaces of Ul'yanov type. II. Topological properties of spaces $\varphi(l)$ 813-827
Khalil I. TABASH: Automorphisms of P_o P-lattices with distinguished chain base . . 829-840
P.K. DAS: H^∞ interpolation into a class of analytic functions 841-845
D.I. ADU: On a characterization of CM-homomorphisms 847-855
Paul ANDREW: Quasi-inverse forms and extended operators 857-862
G.P. KAPOOR, O.P. JUNEJA, J. PATEL: Analytic functions with some univalent Gelfond-Leontiev derivatives 863-877
Eugeniusz JANIEC: Some uniqueness theorems concerning holomorphic mappings . 879-892
Władysław SOSULSKI: On neutral partial functional-differential inclusions of hyperbolic type 893-909
Anatolij DVUREČENSKIJ, František KÔPKA: On representation theorems for observables in weakly complemented posets 911-920
Zbigniew DUDEK: On multiplicative operators in commutative algebras 921-928
Roger YUE CHI MING: Injectivity and annihilators 929-942
Barbara PILECKA-OBORSKA: Une modification de la construction de la fonction de sortie 943-949
H.-G. ROOS, E. PFEIFER: A convergence result for the Tau method 951-959
Jerzy ŻURAWIECKI: Locally reducible iterative systems 961-983
Valeriu POPA: Some properties of almost feebly continuous functions 985-991
Monika KOZAK: An abstract nonlinear temporally inhomogeneous equation 993-1003
Romuald MAŁECKI: Estimates of the solution of the boundary-value problem with a complex parameter for a certain elliptic system 1005-1020
H. DĘBIŃSKI, T. POREDA, A. SZADKOWSKA: On the stability of the generalized linear differential equation of first order Banach spaces 1021-1031
Halina GRABARSKA: The approximation of solutions of the Fourier problem of the first kind for systems of parabolic equations 1033-1042
Zbigniew BINDERMAN: Complex R-shifts for right invertible operators 1043-1053
Abderrahim KALFAT: Remarks on Volterra right inverses for weighted difference operator of high degree 1055-1064
Adam GRABARSKI: Prolongation of the solution of a difference scheme for the equation $-u'' + g(x)u = f(x)$ 1065-1072
Marian MAJCHROWSKI, Jan ROGULSKI: An inverse problem for a parabolic system of partial differential equations 1073-1083
Jacek DĄBROWSKI: Estimation of density function for stationary processes with application to Markov processes 1085-1098
Krzysztof WITCZYŃSKI: On Pappu's theorem in the projective space P_n 1099-1103

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Cena zł 18. 500,–

CC-0. Mumukshu Bhawan Varanasi Collection. Digitized by eGangotri

Digitized by Arya Samaj Foundation Chennai and eGangotri

TRATIO MATHEMATICA

No 1 1990

Jacek Skwierczyński

DUAL VARIATIONAL FORMULATION FOR A HYPERBOLIC PROBLEM

The aim of this work is to give a dual variational and extremal formulation for a second Fourier problem for a linear hyperbolic equation. We use the variational principles of E. Herrera and M.J. Sewell [1] and our result [4],[5] on the variational formulation and the existence of a generalized solution.

Throughout this paper we use symbols and conventions from [4].

1. Variational formulation

Let $G \subset \mathbb{R}^N$, $N \geqslant 2$, be an open and bounded set of class $\mathfrak{A}^{0,1}$, denote by $(n_1,\dots,n_N)$ the outward normal vector to ∂G of length 1.

Denote by $\Gamma = \partial G \times (0,T)$, $G_0 = G \times \{0\}$, $G_s = G \times \{s\}$, $\Gamma_s = \partial G \times (0,s)$, $Q_s = G \times (0,s)$, $s > 0$, $T > 0$, $s \leqslant T$.

Define the following differential problem:

P r o b l e m I. Find $u \in H^2(Q_T)$ such that

$$\Box u := \partial_t^2 u - \partial_i(k_{ij}\partial_j u) + p\partial_t u + qu = f \quad \text{in} \quad L^2(Q_T), \tag{1}$$

$$\operatorname{tr}\big|_{G_0} u = \varphi_0 \quad \text{in} \quad H^1(G), \tag{2}$$

$$\operatorname{tr}\big|_{G_0} \partial_t u = \varphi_1 \quad \text{in} \quad L^2(G), \tag{3}$$

$$\operatorname{tr}\big|_{\Gamma} k_{ij}\partial_j u \circ n_i = 0 \quad \text{in} \quad L^2(\Gamma). \tag{4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We assume that

$$(5)\quad \begin{cases} p \in L^\infty(G),\ p \geqslant 0 \text{ a.e. in } G,\ q \in L^\infty(G),\ q \geqslant 0 \text{ a.e. in } G, \\ f \in L^2(Q_T),\ \varphi_0 \in H^1(G),\ \varphi_1 \in L^2(G), \\ k_{ij} \in L^\infty(G),\ k_{ij} = k_{ji} \text{ a.e. in } G,\ i,j=1,\dots,N,\ N \geqslant 2; \\ \text{there exists } \alpha > 0 \text{ such that} \\ k_{ij} z_i z_j \geqslant \alpha \circ (z_1^2 + \dots + z_N^2),\ z_i \in \mathbb{R},\ i,j=1,\dots,N. \end{cases}$$

Consider besides the following variational problem:

P r o b l e m 2. Find $u \in H^{1,2}(Q_T)$ such that

$$(6)\qquad (Lu,v) = F(v) \quad \forall\, v \in H^{1,2}(Q_T),$$

where $Lu: H^{1,2}(Q_T) \to \mathbb{R}$ is a linear functional given by:

$$(7)\quad (Lu,v) = \int_0^T - (\partial_t u, \partial_t^2 v)_{L^2(Q_s)}\, ds + (\partial_t u, \partial_t v)_{L^2(Q_T)} +$$

$$+ \int_0^T (k_{ij}\partial_j u, \partial_{it}^2 v)_{L^2(Q_s)}\, ds + 2T(k_{ij}\,\mathrm{tr}\big|_{G_0}\partial_j u, \mathrm{tr}\big|_{G_0}\partial_i v)_{L^2(G)} +$$

$$+ \int_0^T (q \circ u, \partial_t v)_{L^2(Q_s)}\, ds + T(q \circ \mathrm{tr}\big|_{G_0} u, \mathrm{tr}\big|_{G_0} v)_{L^2(G)} +$$

$$+ 0{,}5T(\mathrm{tr}\big|_{G_0} u, \mathrm{tr}\big|_{G_0} v)_{L^2(G)} + \int_0^T (p \circ \partial_t u, \partial_t v)_{L^2(Q_s)}\, ds,$$

and $F : H^{1,2}(Q_T) \to \mathbb{R}$ is a linear functional of the form

$$(8)\quad F(v) = \int_0^T (f, \partial_t v)_{L^2(Q_s)}\, ds + T(\varphi_1, \mathrm{tr}\big|_{G_0}\partial_t v)_{L^2(G)} +$$

$$+ 2T(k_{ij}\partial_j\varphi_0, \mathrm{tr}\big|_{G_0}\partial_i v)_{L^2(G)} + T(q \circ \varphi_0, \mathrm{tr}\big|_{G_0} v)_{L^2(G)} +$$

$$+ 0{,}5T(\varphi_0, \mathrm{tr}\big|_{G_0} v)_{L^2(G)}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is easy to verify that, under assumptions (5), problem 2 makes sense.

In [4] we have proved (Theorem 1) the equivalence of these problems for solutions $u \in H^2(Q_T)$ and (Theorem 2) the existence and uniqueness of a generalized solution (from a space denoted V) of problem 2.

Define a mapping $[.,.] : H^{1,2}(Q_T) \times H^{1,2}(Q_T) \longrightarrow \mathbb{R}$ with a formula

$$[u,v] = (u,v)_{L^2(Q_T)} + \sum_{i=1}^{N} (\partial_i u, \partial_i v)_{L^2(Q_T)} +$$

$$+ (\mathrm{tr}|_{G_0} \partial_t u, \mathrm{tr}|_{G_0} \partial_t v)_{L^2(G)} + \sum_{i=1}^{N} (\mathrm{tr}|_{G_0} \partial_i u, \mathrm{tr}|_{G_0} \partial_i v)_{L^2(G)} +$$

$$+ (\partial_t u, \partial_t v)_{L^2(Q_T)} + (\mathrm{tr}|_{G_0} u, \mathrm{tr}|_{G_0} v)_{L^2(G)}.$$

We also define $\|u\|_V = \sqrt{[u,u]}$, $\|\cdot\|_V : H^{1,2}(Q_T) \longrightarrow \mathbb{R}$. It is obvious that $\|\cdot\|_V$ is a norm in $H^{1,2}(Q_T)$, but this space is not complete in this norm.

D e f i n i t i o n 1. Space V is a completion of $H^{1,2}(Q_T)$ in the norm $\|\ \|_V$.

As $\|\ \|_V$ is a norm in $H^{1,2}(Q_T)$ and $[.,.]$ has the properties of a scalar product, V is a Hilbert space with this product. Now we shall give another useful characterization of this space.

Let $I: H^{1,2}(Q_T) \longrightarrow H^1(Q_T) \times H^1(G) \times L^2(G)$ be defined by the formula

$$I(u) = (I_H u, I_G u, I_t u) = (u, \mathrm{tr}|_{G_0} u, \mathrm{tr}|_{G_0} \partial_t u).$$

As $H^{1,2}(Q_T)$ is dense in V and in the norm of the cartesian product $\|Iu\| = \|u\|_V$, we can extend I to an isometry of V into a closed subspace of the cartesian product $H^1(Q_T) \times H^1(G) \times L^2(G)$. This isometry will be also denoted by $I = (I_H, I_G, I_t)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The generalized solution of problem 2 may be characterized as follows ([4], Theorem 3).

An element $\tilde{u} \in V$ is a generalized solution of the problem 2 iff for every $v \in H^{1,2}(Q_T)$ we have $(LI\tilde{u},v) = F(v)$, where

$$(LI\tilde{u},v) = \int_0^T - (\partial_t I_H \tilde{u}, \partial_t^2 v)_{L^2(Q_s)} ds + (\partial_t I_H \tilde{u}, \partial_t v)_{L^2(Q_T)} +$$

$$+ \int_0^T (k_{ij} \partial_j I_H \tilde{u}, \partial_{it}^2 v)_{L^2(Q_s)} ds + 2T(k_{ij} \operatorname{tr}|_{G_0} \partial_j I_G \tilde{u}, \operatorname{tr}|_{G_0} \partial_i v)_{L^2(G)} +$$

$$+ \int_0^T (q \circ I_H \tilde{u}, \partial_t v)_{L^2(Q_s)} ds + T(q \circ I_G \tilde{u}, \operatorname{tr}|_{G_0} v)_{L^2(G)} +$$

$$+ 0{,}5T(I_G \tilde{u}, \operatorname{tr}|_{G_0} v)_{L^2(G)} + \int_0^T (p \circ \partial_t I_H \tilde{u}, \partial_t v)_{L^2(Q_s)} ds.$$

Moreover, $\tilde{u}$ has the following properties: operator $\Box$ is defined on $I_H \tilde{u}$ and $\Box I_H \tilde{u} \in L^2(Q_T)$, $\Box I_H \tilde{u} = f$ in $L^2(Q_T)$, $I_G \tilde{u} = \varphi_0$ in $H^1(G)$.

2. Regularity of the generalized solution.

To apply the variational principles from [1] we have to deal with solutions of problem 2 from the space $H^{1,2}(Q_T)$. In the following theorem we give sufficient conditions for such regularity.

T h e o r e m 1. Let the functions f, p, φ_0, φ_1, q satisfy (5) and assume that:

(9) f is a Lipschitz function in t for a.e. $x \in G$,

(10) φ_0, $\varphi_1 \in H^1(G)$ are the solutions of a homogeneous Neumann problem in G:

$$\begin{cases} \text{find } v \in H^1(G) \text{ such that } \partial_i(k_{i,j} \partial_j v) \in L^2(G), \\ \operatorname{tr}|_{\partial G} k_{ij} \partial_j v \circ n_i = 0 \text{ in } H^{\frac{1}{2}}(\partial G), \ i,j = 1,\dots,N, \end{cases}$$

and satisfy a compatibility condition

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(11)\quad -(\partial_i(k_{ij}\partial_j\varphi_0),w)_{L^2(G)} + (p\varphi_1,w)_{L^2(G)} + (q\varphi_0,w)_{L^2(G)} = (f(x,0),w)_{L^2(G)} \qquad \forall w \in L^2(G).$$

Then $I_H u \in H^{1,2}(Q_T)$, if $u \in V$, is a generalized solution of problem 2.

P r o o f . Take $w(x,t) = \varphi_0(x) + t\circ\varphi_1(x)$, $-T < t < 0$, $x \in G$. If by Q_{-T} we denote the set $G\times(-T,0)$, then from assumptions on φ_1 and φ_0 we get $w \in H^{1,2}(Q_{-T})$, $tr\big|_{G_0} w = \varphi_0$ in $H^1(G)$ and $tr\big|_{G_0}\partial_t w = \varphi_1$ in $L^2(G)$. Moreover, the operator $\Box$ is defined on w and $\Box w \in L^2(Q_{-T})$ and in this case $\Box w = -\partial_i(k_{ij}\partial_j\varphi_0) - t\circ\partial_i(k_{ij}\partial_j\varphi_1) + p\varphi_1 + q\varphi_0 + tq\varphi_1$. As we see $\Box w$ is a function satisfying the Lipschitz condition in t; for arbitrary t_0, $-T < t_0 < 0$, we have the following traces $tr\big|_{G_{t_0}} w = \varphi_0 + t_0\varphi_1 \in H^1(G)$, $tr\big|_{G_{t_0}} w = \varphi_1 \in H^1(G)$.

Let us denote by $Q_{(-T,T)}$ the set $G\times(-T,T)$. From our assumptions we conclude that $tr\big|_{G_0} w = f\big|_{G_0}$ in $L^2(G)$, thus

$$\bar{f} = \begin{cases} f & \text{for } t > 0,\ x \in G, \\ w & \text{for } t < 0,\ x \in G, \end{cases}$$

is a function from $L^2(Q_{(-T,T)})$, satisfying the Lipschitz condition in t for a.e. $x \in G$. As the necessary traces on G_0 are equal, function $\bar{w}(x,t)$ given by the formula

$$\bar{w} = \begin{cases} I_H w & \text{for } t > 0,\ x \in G, \\ w & \text{for } t < 0,\ x \in G, \end{cases}$$

is an element of $H^1(Q_{(-T,T)})$ ([2] Lemma A8).

Let us denote by $\nabla_{ht}\bar{w}(x,t) = \dfrac{\bar{w}(x,t+ht) - \bar{w}(x,t)}{ht}$, $|ht| > 0$, $t + ht \in (-T,T)$. If we consider $\bar{w}$ in Q_T, then, for $-\frac{T}{2} < ht < 0$, $\nabla_{ht}\bar{w}(x,t) \in H^1(Q_T)$ and even $\nabla_{ht}\bar{w} \in V$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now we shall write explicitely the form of $(L\nabla_{ht}\bar{w},v)$, where $v \in H^{1,2}(Q_T)$, and (Lu,v) is given by the formula (7). From the linearity of $(L\circ,v)$ it follows that

$$(L\nabla_{ht}\bar{w},v) = \frac{1}{ht}\left[(L\bar{w}(x,t+ht),v) - (L\bar{w}(x,t),v)\right].$$

We shall begin with transformations of $(L\bar{w}(x,t+ht),v)$; from the properties of $\bar{w}$ we conclude

$$(L\bar{w}(x,t+ht),v) = -((T-t)\partial_t\bar{w}(x,t+ht),\partial_t^2 v)_{L^2(Q_T)} +$$

$$+ (\partial_t\bar{w}(x,t+ht),\partial_t v)_{L^2(Q_T)} + ((T-t)k_{ij}\partial_j\bar{w}(x,t+ht),\partial_{it}^2 v)_{L^2(Q_T)} +$$

$$+ 2T(k_{ij}\partial_j\bar{w}(x,ht),tr|_{G_0}\partial_i v)_{L^2(G)} + ((T-t)q\bar{w}(x,t+ht),\partial_t v)_{L^2(G)} +$$

$$+ 0{,}5T(\bar{w}(x,ht),tr|_{G_0} v)_{L^2(G)} + ((T-t)p\partial_t\bar{w}(x,t+ht),\partial_t v)_{L^2(Q_T)} +$$

$$+ T(q\bar{w}(x,ht),tr|_{G_0} v)_{L^2(G)}.$$

After elementary transformations of the integrals in the above formula (see [5] for details) we get

$$(L\bar{w}(x,t+ht),v) = \int_0^T\int_G \Box\bar{w}(x,t+ht)\circ(T-t)\partial_t v\,dx\,dt +$$

$$+ T(\partial_t\bar{w}(x,ht),tr|_{G_0}\partial_t v)_{L^2(G)} + 2T(k_{ij}\partial_j\bar{w}(x,ht),tr|_{G_0} i^v)_{L^2(G)} +$$

$$+ T(q\bar{w}(x,ht),tr|_{G_0} v)_{L^2(G)} + 0{,}5T(\bar{w}(x,ht),tr|_{G_0} v)_{L^2(G)}.$$

Thus, we can write

$$(L\nabla_{ht}\bar{w},v) = \int_0^T (\nabla_{ht}\bar{f},\partial_t v)_{L^2(Q_s)}\,ds + 2T(k_{ij}\partial_j\varphi_1,tr|_{G_0} i^v)_{L^2(G)} +$$

$$+ T(q\varphi_1,tr|_{G_0} v)_{L^2(G)} + 0{,}5(\varphi_1,tr|_{G_0} v)_{L^2(G)}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Take a sequence $\{v^n\}$ convergent to $\nabla_{ht}\bar{w}$ in the norm $\|\cdot\|_V$. In [4] we have proved that (Lu,u) can be defined for $u \in V$ by taking a sequence of elements of $H^{1,2}(Q_T)$ and passing to the limit; moreover, (Lu,u) is positively definite in the norm $\|\cdot\|_V$. From these facts and the above equality we get $\|\nabla_{ht}\bar{w}\|_V^2 \leqslant C\|\nabla_{ht}\bar{w}\|_V$, where $C = C(\nabla_{ht}\bar{f}, k_{ij}, q, \varphi_1, \varphi_0)$ is a positive constant. Thus, as $\bar{f}$ satisfies the Lipschitz condition in t, we have the estimate $\|\nabla_{ht}I_H\bar{w}\|_{H^1(Q_T)} \leqslant \bar{C}$, with constant $\bar{C} \geqslant 0$.

As V is a Hilbert space, the above equality yields the existence of a subsequence ht^n, $\lim_{n\to\infty} ht^n = 0$, such that $\nabla_{ht^n}\partial_i I_H\bar{w}$ converges weakly to $\partial_i\partial_t I_H\bar{w}$ in $L^2(Q_T)$, thus $\partial_i\partial_t I_H\bar{w} \in L^2(Q_T)$, in a similar way we prove that $\partial_t^2 I_H\bar{w} \in L^2(Q_T)$. As on Q_T we have $I_H\bar{w} = I_H u$, we have proved that $I_H u \in H^{1,2}(Q_T)$.

C o r o l l a r y 1. If $u \in V$ is a generalized solution of problem 2 and $I_H u \in H^{1,2}(Q_T)$, then $I_G u = \mathrm{tr}\big|_{G_0} I_H u$ in $H^1(G)$ and $\varphi_1 = I_t u = \mathrm{tr}\big|_{G_0}\partial_t I_H u$ in $L^2(G)$.

P r o o f . We start with the equality $\varphi_1 = \mathrm{tr}\big|_{G_0}\partial_t I_H u$. From $I_H u \in H^{1,2}(Q_T)$ we have

$$\int_0^T -(\partial_t I_H u, \partial_t^2 v)_{L^2(Q_s)}\,ds + (\partial_t I_H u, \partial_t v)_{L^2(Q_T)} =$$

$$= \int_0^T (\partial_t^2 I_H u, \partial_t v)_{L^2(Q_s)}\,ds + (\mathrm{tr}\big|_{G_0}\partial_t I_H u, \mathrm{tr}\big|_{G_0}\partial_t v)_{L^2(G)}.$$

From the arbitrariness of v (we take v such that $\partial_t v = 0$ in the neighbourhood of Γ) we get our equality.

From the properties of the generalized solution we have $I_G u = \varphi_0$ in $H^1(G)$, but for u such that $I_H u \in H^{1,2}(Q_T)$ we have $I_G u = \mathrm{tr}\big|_{G_0} I_H u$.

To conclude the proof we have to show that $\mathrm{tr}\big|_{G_0}\partial_t I_H u = I_t u$. This follows from the characterisation and uniqueness

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

of the generalized solution, for, if we take as Iu the vector $(I_H u, I_G u, \mathrm{tr}|_{G_0} I_H u)$, then it is an element of I(V). As for $v \in H^{1,2}(Q_T)$ the value of (LIu,v) does not depend explicitely on $I_t u$, it follows from the uniqueness of the generalized solution that $(I_H u, I_G u, \mathrm{tr}|_{G_0} I_H u)$ is its image in I.

C o r o l l a r y 2. If $u \in V$ is a generalized solution of problem 2 and $I_H u \in H^{1,2}(Q_T)$, then for every $v \in H^{1,2}(Q_T)$ we have

$$(Lu,v) = \int_0^T (\Box I_H u, \partial_t v)_{L^2(Q_s)} ds + T(\mathrm{tr}|_{G_0} \partial_t I_H u, \mathrm{tr}|_{G_0} \partial_t v)_{L^2(G)} +$$

$$+ 2T(k_{ij} \mathrm{tr}|_{G_0} \partial_j I_H u, \mathrm{tr}|_{G_0} \partial_i v)_{L^2(G)} + 0{,}5T(\mathrm{tr}|_{G_0} u, \mathrm{tr}|_{G_0} v)_{L^2(G)} +$$

$$+ T(q\,\mathrm{tr}|_{G_0} I_H u, \mathrm{tr}|_{G_0} v)_{L^2(G)}$$

which means that we can represent u by $I_H \in H^{1,2}(Q_T)$.

The proof obviously follows from Corollary 1 and the characterisation of the generalized solution.

It can be easily proved that if $G \in \mathfrak{N}^{1,1}$, all the assumptions of Theorem 1 are satisfied, functions p, q, k_{ij}, $i,j = 1,\ldots,N$, satisfy on $\bar{G}$ the Lipschitz condition in x_i, $i = 1,\ldots,N$, then the generalized solution of problem 2 may be represented as element of $H^2(Q_T)$. For details see [5].

3. Dual variational principle

We follow the theory of dual variational principles presented in [1]. In this section we consider problem 2 under the assumptions of Theorem 1.

I. Herrera and M.J. Sewell have considered the equation $Pu = f$, $P : D \to D^*$, $f \in D^*$, with D an arbitrary vector space. For P they define $P^* : D \to D^*$ by the formula $\langle P^* u, v \rangle = \langle Pv, u \rangle \; \forall u,v \in D$, where $\langle .,. \rangle$ denotes the value of a functional on an element of D. Using the so-called mirror method they consider the pair of equations:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(*) \qquad \begin{cases} Pu_1 = f \\ P^* u_2 = g \end{cases}$$

where $f, g \in D^*$.

If we take $X : D \times D \to \mathbb{R}$ of the form $X(u_1, u_2) = 0{,}5(\langle Pu_1, u_2 \rangle + \langle P^* u_2, u_1 \rangle) - \langle f, u_2 \rangle - \langle g, u_1 \rangle$, then one of the main results of [1] states, that the pair (u_1, u_2) is a solution of the above pair of equations iff it is a critical point of X.

Next, for the so-called nonnegative operator P ($\langle Pu, u \rangle \geqslant 0 \ \forall \ u \in D$) I. Herrera and M.J. Sewell state the following extremum principle (Theorem 4.3 in [1]).

Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha\delta - \beta\gamma \neq 0$, $\alpha\beta < 0$, $\gamma\delta > 0$; define affine subspaces of D×D:

$$D_a = \{(u_1, u_2) : \alpha Pu_1 - \beta P^* u_2 = \alpha f - \beta g\},$$
$$D_b = \{(u_1, u_2) : \gamma Pu_1 - \delta P^* u_2 = \gamma f - \delta g\};$$

then:

a) $(u_1, u_2) \in D \times D$ is a solution of (*) iff $(u_1, u_2) \in D_a \cap D_b$;

b) X is concave on D_a and convex on D_b; for arbitrary $(u_{a1}, u_{a2}) \in D_a$ and $(u_{b1}, u_{b2}) \in D_b$ we have

$$2(X(u_{a1}, u_{a2}) - X(u_{b1}, u_{b2})) = m_1 \| u_{a1} - u_{b1} \|_P^2 + m_2 \| u_{a2} - u_{b2} \|_P^2,$$

where $\langle Pu, u \rangle = \| u \|_P^2$, $m_1 = \dfrac{2\alpha\gamma}{-\beta\gamma + \alpha\delta}$, $m_2 = \dfrac{2\beta\delta}{\beta\gamma - \alpha\delta}$;

c) if $\max\limits_{(w_1, w_2) \in D_a} X(w_1, w_2) = \min\limits_{(w_1, w_2) \in D_b} X(w_1, w_2)$ and these extrema are attained at $(u_{a1}, u_{a2}) \in D_a$ and $(u_{b1}, u_{b2}) \in D_b$, respectively, then $\| u_{a1} - u_{b1} \|_P = \| u_{a2} - u_{b2} \|_P = 0$;

d) if there exists a solution $(u_1, u_2) \in D \times D$ of (*), then for arbitrary $(u_{a1}, u_{a2}) \in D_a$ and $(u_{b1}, u_{b2}) \in D_b$ we have

$$2(X(u_{a1}, u_{a2}) - X(u_{b1}, u_{b2})) = m_1(\| u_{a1} - u_1 \|_P^2 + \| u_{b1} - u_1 \|_P^2) +$$
$$+ m_2(\| u_{a2} - u_2 \|_P^2 + \| u_{b2} - u_2 \|_P^2).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We shall now apply this result in our case. Take $(L.,.): H^{1,2}(Q_T)\times H^{1,2}(Q_T) \to \mathbb{R}$ given by (7). The symmetry of k_{ij} makes the calculation of the dual operator L^*, given by the formula $(Lu,v) = (L^*v,u)$ $\forall u,v \in H^{1,2}(Q_T)$, easy and we get

$$(12)\quad (L^*u,v) = \int_0^T (\partial_t u, \partial_t^2 v)_{L^2(Q_s)}\,ds + T(\mathrm{tr}\big|_{G_0}\partial_t u, \mathrm{tr}\big|_{G_0}\partial_t v)_{L^2(G)} -$$

$$- \int_0^T (k_{ij}\partial_j u, \partial_i \partial_t v)_{L^2(Q_s)}\,ds + (k_{ij}\partial_j u, \partial_i v)_{L^2(Q_T)} +$$

$$+ T(k_{ij}\mathrm{tr}\big|_{G_0}\partial_j u, \mathrm{tr}\big|_{G_0}\partial_i v)_{L^2(G)} + \int_0^T (p\partial_t u, \partial_t v)_{L^2(Q_s)}\,ds +$$

$$- \int_0^T (qu, \partial_t v)_{L^2(Q_s)}\,ds + (qu,v)_{L^2(Q_T)} + 0{,}5T(\mathrm{tr}\big|_{G_0} u, \mathrm{tr}\big|_{G_0} v)_{L^2(G)}.$$

It is easy to see that for $v \in H^{1,2}(Q_T)$ $(L^*u,u) \geq \nu\|u\|_V^2$ and $|(L^*u,v)| \leq C\|u\|_V\|v\|_{H^{1,2}(Q_T)}$ (with constants $\nu > 0$ and $C > 0$).

Thus, if we take $K \in (H^{1,2}(Q_T))^*$ (the dual space of $H^{1,2}(Q_T)$) such that $|K(v)| \leq C_1\|v\|_V$, then the problem (*) has a generalized solution in V. As this problem is only auxiliary, to avoid discussion of the regularity of its solutions we take $K = 0$ and thus $u = 0$ is the unique regular solution of this problem.

Following [1], we take functional $X: H^{1,2}(Q_T)\times H^{1,2}(Q_T) \to \mathbb{R}$ defined by the formula $X(u_1,u_2) = 0{,}5((Lu_1,u_2) + (L^*u_2,u_1)) - F(u_2)$. From (12), (7), (8) we get

$$(13)\quad X(u_1,u_2) = \int_0^T -(\partial_t u_1, \partial_t^2 u_2)_{L^2(Q_s)}\,ds + (\partial_t u_1, \partial_t u_2)_{L^2(Q_T)} +$$

$$+ \int_0^T (k_{ij}\partial_j u_1, \partial_{it}^2 u_2)_{L^2(Q_s)}\,ds + T(q\,\mathrm{tr}\big|_{G_0} u_1, \mathrm{tr}\big|_{G_0} u_2)_{L^2(G)} +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ 2T(k_{ij}\,\mathrm{tr}\big|_{G_0}\partial_j u, \mathrm{tr}\big|_{G_0}\partial_i v)_{L^2(G)} + \int_0^T (q\circ u, \partial_t v)_{L^2(Q_s)}\,ds +$$

$$+ 0{,}5T(\mathrm{tr}\big|_{G_0}u_1, \mathrm{tr}\big|_{G_0}u_2)_{L^2(G)} + \int_0^T (p\partial_t u_1, \partial_t u_2)_{L^2(Q_s)}\,ds +$$

$$- \int_0^T (f, \partial_t u_2)_{L^2(Q_s)}\,ds - T(\varphi_1, \mathrm{tr}\big|_{G_0}\partial_t u_2)_{L^2(G)} +$$

$$+ 2T(k_{ij}\partial_j\varphi_0, \mathrm{tr}\big|_{G_0}\partial_i u_2)_{L^2(G)} - T(q\circ\varphi_0, \mathrm{tr}\big|_{G_0}u_2)_{L^2(G)} +$$

$$- 0{,}5T(\varphi_0, \mathrm{tr}\big|_{G_0}u_2)_{L^2(G)}.$$

The sets D_a and D_b are described as follows

$$(14)\quad D_a = \{(u_1,u_2) \in H^{1,2}(Q_T)\times H^{1,2}(Q_T) : (L\alpha u_1, v) + - (L^*\beta u_2, v) = \alpha F(v) \quad \forall v \in H^{1,2}(Q_T)\},$$

$$(15)\quad D_b = \{(u_1,u_2) \in H^{1,2}(Q_T)\times H^{1,2}(Q_T) : (L\gamma u_1, v) + - (L^*\delta u_2, v) = \gamma F(v) \quad \forall v \in H^{1,2}(Q_T)\},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

By Theorem 1 and the special choice of K, we can state that these sets are nonempty, as the pair $(\tilde{u},0)$, where $\tilde{u}$ is a regular solution of problem 2, belongs to $D_a \cap D_b$.

Define $\|u\|_L$ for $u \in H^{1,2}(Q_T)$ by the formula:

$$\|u\|_L^2 = (Lu,u) = 0{,}5(\partial_t u, \partial_t u)_{L^2(Q_T)} +$$

$$+ 0{,}5(\mathrm{tr}\big|_{G_0}\partial_t u, \mathrm{tr}\big|_{G_0}\partial_t u)_{L^2(G)} + 0{,}5(k_{ij}\partial_j u, \partial_i u)_{L^2(Q_T)} +$$

$$+ 1{,}5T(k_{ij}\,\mathrm{tr}\big|_{G_0}\partial_j u, \mathrm{tr}\big|_{G_0}\partial_i u)_{L^2(G)} + 0{,}5(qu,u)_{L^2(Q_T)} +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ 0,5T(\mathrm{tr}\,|G_0 u, \mathrm{tr}\,|G_0 u)_{L^2(G)} + 0,5T(\mathrm{tr}\,|G_0 u, \mathrm{tr}\,|G_0 u)_{L^2(G)} +$$

$$+ \int_0^T (p\partial_t u, \partial_t u)_{L^2(Q_s)}\, ds.$$

This norm can be defined for $u \in V$ (using the representation given by I); it is obviously equivalent to norm $\|\ \|_V$.

T h e o r e m 2. (Cf. Th.4.3 from [1]). Let $X: H^{1,2}(Q_T) \times H^{1,2}(Q_T) \to \mathbb{R}$ be given by (13); if $\alpha, \beta, \gamma, \delta$ belong to $\mathbb{R}$ and satisfy $\alpha\delta - \beta\gamma \neq 0, \alpha\beta < 0, \gamma\delta > 0$, $m_1 = \frac{2\alpha\gamma}{-\beta\gamma + \alpha\delta}$, $m_2 = \frac{2\beta\delta}{\beta\gamma - \alpha\delta}$, and D_a, D_b be given by (14), (15) then:

a) $u_1 \in H^{1,2}(Q_T)$ is a generalized solution of problem 2 iff its regular representative (also denoted by u_1) is such that $(u_1, 0) \in D_a \cap D_b$;

b) for arbitrary $(u_{a1}, u_{a2}) \in D_a$, $(u_{b1}, u_{b2}) \in D_b$ we have

$$2(X(u_{a1}, u_{a2}) - X(u_{b1}, u_{b2})) = m_1 \|u_{a1} - u_{b1}\|_L^2 + m_2 \|u_{a2} - u_{b2}\|_L^2 =$$

$$= m_1 \left[\|u_{a1} - u_1\|_L^2 + \|u_{b1} - u_1\|_L^2\right] + m_2 \left[\|u_{a2}\|_L^2 + \|u_{b2}\|_L^2\right];$$

c) let $\max_{(w_1, w_2) \in D_a} X(w_1, w_2) = \min_{(w_1, w_2) \in D_b} X(w_1, w_2) = X(u_1, 0) = 0$; then $X(u_{a1}, u_{a2}) = X(u_1, 0) = 0$ for $(u_{a1}, u_{a2}) \in D_a$ iff $\|u_{a1} - u_1\|_L = 0$, $\|u_{a2}\|_L = 0$, and $X(u_{b1}, u_{b2}) = X(u_1, 0) = 0$ for $(u_{b1}, u_{b2}) \in D_b$ iff $\|u_{b1} - u_1\|_L = 0$, $\|u_{b2}\|_L = 0$.

REFERENCES

[1] I. H e r r e r a , M.J. S e w e l l : Dual extremum principles for nonnegative unsymmetric oprators, J. Inst. Maths. Applics., 21 (1978) 95-115.

[2] D. K i n d e r l e h r e r , G. S t a m p a c c i a : An introduction to variational inequalities. New York, 1980.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] J. Nečas : Les méthodes directes en théorie des équations elliptiques. Prague, 1967.

[4] J. Skwierczyński : Variational formulation for a hyperbolic problem, Demonstratio Math. 22 (1989) 887-901.

[5] J. Skwierczyński : Zastosowanie metody Herrery-Bielaka do zagadnień hiperbolicznych, D. Ph. thesis, Warszawa, 1987.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received January 7, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Vu Van Hung

THE STABILITY OF THE DIFFERENCE SCHEMA APPROXIMATING CAUCHY'S PROBLEM FOR THE SECOND ORDER PARABOLIC EQUATION WITH VARIABLE COEFFICIENTS IN L_2

1. Introduction

In [6] a class of difference schemas was presented which allow us to approximate parabolic problem for equation of second order in L_2. Those schemas are open and unconditionally stable. In [6] only the case with constant coefficients in space variables were considered. In the present work we will deal with the case of variable coefficients depending on the space and time parameters. We shall consider the difference schema for solving the stability problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = a(x,t)\dfrac{\partial^2 u}{\partial x^2} + b(x,t)\dfrac{\partial u}{\partial x} + c(x,t)u + f(x,t), \\ (x,t) \in R\times(0,T), \\ U(x,0) = \varphi(x), \quad \varphi \in C(R). \end{cases}$$

We assume that the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $f(x,t)$ are elements of the space $C(R,(0,T))$ and $a(x,t) \geqslant a_0 > 0$. On the rectangular grid

$$R^2_{h\tau} := \left\{ (x,t) : x = mh, \quad t = n\tau, \quad m,n \in Z, \quad N\tau = T, \ h,\tau > 0 \right\}$$

we consider the following difference schema

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$
(1.2)\quad \begin{cases} \frac{2}{\tau}\left[U_{2m+1}^{n+\frac{1}{2}} - U_{2m+1}^{n}\right] = L_{h,2m+1}^{n+\frac{1}{2}}\, U_{2m+1}^{n} + f_{2m+1}^{n}, \\ \frac{2}{\tau}\left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^{n}\right] = L_{h,2m}^{n+\frac{1}{2}}\, U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right], \\ \frac{2}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}}\right] = L_{h,2m}^{n+\frac{1}{2}}\, U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right], \\ \frac{2}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}}\right] = L_{h,2m+1}^{n+\frac{1}{2}}\, U_{2m+1}^{n+1} + f_{2m+1}^{n+1}, \\ U_{m}^{0} = u_{m}, \quad m,n \in Z, \quad 0 \leqslant n \leqslant N-1, \end{cases}
$$

where the difference operator $((L_{h,m}^{n}y)_k$ is defined on the number sequences as follows

$$
(L_{h,m}^{n}y)_k := L_{h,m}^{n}y_k := a_m^n\left[y_{k-1} - 2y_k + y_{k+1}\right] + b_m^n \frac{1}{2h}\left[y_{n+1} - y_{n-1}\right] + c_m^n y_k,
$$

$$
\forall (m,n,k), \quad 2n \in Z, \quad m,n \in Z,
$$

and $U_m^n := u(mh, n\tau)$, Z - the set of integers.

2. The stability of the difference schema (1.2) for the problem (1.1)

To investigate the stability of this problem we prove the following theorem.

T h e o r e m 2.1. The sequence $\{U_m^n\}_{\substack{m,n\in Z \\ 0<n<N}}$ satisfies the system of difference equations (1.2) if and only if this sequence satisfies the relation defined by the following difference schema

$$
(2.1)\quad \begin{cases} \frac{1}{2}\left[U_m^{n+1} - U_m^n\right] = \frac{1}{2}\left[L_{h,m}^{n+\frac{1}{2}}\, U_m^{n+1} + f_m^{n+1}\right] + \frac{1}{2}\left[L_{h,m}^{n+\frac{1}{2}}\, U_m^{n} + f_m^{n}\right], \\ U_m^0 = \varphi_m, \quad m,n \in Z, \quad 0 \leqslant n \leqslant N-1. \end{cases}
$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . First we show that if the sequence $\{U_m^n\}$ satisfies (1.2), then it satisfies (2.1). Summing up the second and third equation of (1.2) and dividing by 2 we obtain

$$\frac{1}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n}\right] = L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right], \tag{2.2}$$

and substracting side by side we obtain

$$U_{2m}^{n+\frac{1}{2}} = \frac{1}{2}\left[U_{2m}^{n+1} + U_{2m}^{n}\right], \quad m \in Z.$$

On both sides of the above equations we act with the operator $L_{h,2m}^{n+1/2}$ and the obtained result $L_{h,2m}^{n+1/2} U_{2m}^{n+1/2}$ we substitute to (2.2), and we obtain

$$\frac{1}{2}\left[U_{2m}^{n+1} - U_{2m}^{n}\right] = \frac{1}{2} L_{h,2m}^{n+\frac{1}{2}}\left[U_{2m}^{n+1} + U_{2m}^{n}\right] + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right] = \tag{2.3}$$

$$= \frac{1}{2}\left[L_{h,2m}^{n+\frac{1}{2}} U_{2m}^{n+1} + f_{2m}^{n+1}\right] + \frac{1}{2}\left[L_{h2m}^{n+\frac{1}{2}} U_{2m}^{n} + f_{2m}^{n}\right].$$

Similarly, adding first and fourth equation of the system (1.2), we obtain

$$\frac{1}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n}\right] = \frac{1}{2} L_{h,2m+1}^{n+\frac{1}{2}}\left[U_{2m+1}^{n+1} + U_{2m+1}^{n}\right] + \frac{1}{2}\left[f_{2m+1}^{n+1} + f_{2m+1}^{n}\right] = \tag{2.4}$$

$$= \frac{1}{2}\left[L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n+1} + f_{2m+1}^{n+1}\right] + \frac{1}{2}\left[L_{h,2m+1}^{n+\frac{1}{2}} U_{2m+1}^{n} + f_{2m+1}^{n}\right].$$

Hence, from (2.3) and (2.4) we obtain

$$\frac{1}{\tau}\left[U_m^{n+1} - U_m^{n}\right] = \frac{1}{2}\left[L_{hm}^{n+\frac{1}{2}} U_m^{n+1} + f_m^{n}\right] + \frac{1}{2}\left[f_m^{n} + L_{hm}^{n+\frac{1}{2}} U_m^{n}\right].$$

This schema together with the condition $U_m^0 = \phi_m$ for $m \in Z$ gives the difference schema (2.1).

To show that the sequence $\{U_m^{m+1/2}\}$ defined by the schema (2.1) satisfies (1.2) we introduce an auxiliary sequence $\{V_m^{n+1/2}\}$ defined as follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.5) $$U_{2m+1}^{n+\frac{1}{2}} = U_{2n+1}^{n} + \frac{\tau}{2}\left[L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n} + f_{2m+1}^{n}\right],$$

(2.6) $$U_{2m}^{n+\frac{1}{2}} = \frac{1}{2}\,(U_{2m}^{n+1} + U_{2m}^{n}), \quad \text{for} \quad m,n \in Z.$$

Using (2.6) and (2.1) we obtain

$$\frac{2}{\tau}\left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^{n}\right] = \frac{1}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n}\right] =$$

$$= \frac{1}{2}\left[L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n} + f_{2m}^{n}\right] + \frac{1}{2}\left[L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n+1} + f_{2m}^{n+1}\right].$$

From the linearity of $L_{h,2m}^{n+1/2}$ and (2.6) we have

(2.7) $$\frac{2}{\tau}\left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^{n}\right] = L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right].$$

Similarly

$$\frac{2}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}}\right] = \frac{1}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n}\right] = \frac{1}{2}\left[L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n} + f_{2m}^{n}\right] +$$

$$+ \frac{1}{2}\left[L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n+1} + f_{2m}^{n+1}\right] = L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right],$$

that is

(2.8) $$\frac{2}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}}\right] = L_{h,2m}^{n+\frac{1}{2}}\ U_{2m}^{n+\frac{1}{2}} + \frac{1}{2}\left[f_{2m}^{n+1} + f_{2m}^{n}\right].$$

Using definitions (2.5), we obtain

$$\frac{2}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}}\right] = \frac{2}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n}\right] - L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n} - f_{2m+1}^{n}.$$

Similarly, using (2.1), we obtain

$$\frac{2}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}}\right] = 2\left[\frac{1}{2}\left(L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n+1} + f_{2m+1}^{n+1}\right) +\right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\frac{1}{2}\left(L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n}+f_{2m+1}^{n}\right)\Big]-L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n}-f_{2m+1}^{n}=$$

$$=L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n+1}+f_{2m+1}^{n+1},$$

that is

$$\frac{2}{\tau}\left[U_{2m+1}^{n+1}-U_{2m+1}^{n+\frac{1}{2}}\right]=L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n+1}+f_{2m+1}^{n+1}. \tag{2.9}$$

From formula (2.5) it follows that

$$\frac{2}{\tau}\left[U_{2m+1}^{n+\frac{1}{2}}-U_{2m+1}^{n}\right]=L_{h,2m+1}^{n+\frac{1}{2}}\ U_{2m+1}^{n}+f_{2m+1}^{n}.$$

Hence Theorem 2.1 has been proved.

This theorem is important, because it reduces the investigations of the stability of schema (1.2) to the stability of schema (2.1).

First, we can write the schema (2.1) in the canonical form. To this aim we denote

$$\hat{U}:=U_m^{n+1},\quad U:=U_m^n,\quad U_t:=\frac{U_m^{n+1}-U_m^n}{\tau},\quad \hat{f}:=f_m^{n+1},$$

$$f:=f_m^n,\ U^0:=U_m^0,\ \varphi:=\varphi_m,\ \bar{t}=tn+\frac{1}{2}\tau,\ \text{for}\ m,n\in Z,\ 0\leqslant n\leqslant N-1.$$

Then (2.1) takes the form

$$U_t=\frac{1}{2}L(\bar{t})(\hat{U}+U)+\frac{1}{2}(\hat{f}+f),\quad U^0=\varphi. \tag{2.10}$$

Using the identity

$$\hat{U}=U+\tau\frac{\hat{U}-U}{\tau}=U+\tau U_t,$$

we can write (2.10) in the form

$$U_t=\frac{1}{2}L(\bar{t})\left[U+\tau U_t+U\right]+\frac{1}{2}(\hat{f}+f),\quad U^0=\varphi,$$

that is

$$(E-\frac{\tau}{2}L(\bar{t}))U_t-L(\bar{t})U=f,\quad U^0=\varphi,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

in other words

$$BU_t + AU = f, \quad U^0 = \varphi, \tag{2.11}$$

where $B := E + \frac{\tau}{2} A$, $A := -L(\bar{t})$.

To investigate the stability of this problem, we apply Theorem 10.14 in [5]. We formulate this fact as the following theorem.

T h e o r e m 2.2. If the coefficients of equation (1.1) and the function $f(x,t)$ are continuous with respect to x and t, and bounded in such a way that

$$\frac{\partial a(x,t)}{\partial t} \in C(R,(0,T)),$$

then there exists τ_*, depending only on the norm of coefficients, such that for $\tau \in (0,\tau_*)$ the schema (1.2) is unconditionally stable.

P r o o f . By Theorem 2.1, the schema (1.2) is equivalent to schema (2.1), whose unconditional stability is proved, e.g. in [5], Th. 10.14.

3. The stability of difference schemes for solving the problem of Cauchy for the parabolic equations of order 2 with many space variables

We consider the following initial problem

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f(x,t), & (x,t) \in R^M \times (0,T), \\ u(x,0) = \varphi(x), & x \in R^M, \end{cases} \tag{3.1}$$

where L is an elliptic operator, $L \in (C^2(R^M), C(R^M))$, defined by

$$Lu := -\sum_{i,j=1}^{M} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{M} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The difference schema approximating (3.1) will be constructed on the parallelepiped grid $R_{h\tau}^M = R_h^M \times \omega_\tau$ with $R_h^M := \{x \in R^M : x = (m_1 h_1, \dots, m_M h_M),\ m_1, \dots, m_M \in Z\}$, $h := \max_{1 \leqslant i \leqslant M} h_i$, $h_1, \dots, h_M \geqslant h_0 > 0$, and $\omega_\tau := \{t = n\tau,\ n = 0, \dots, N,\ N\tau = T\}$, where by ∂_i, $\bar{\partial}_i$ $(i = \overline{1,M})$ we denote the difference operators

$$\partial_i v(x) = \frac{1}{h_i}\left[v(x + e_i h_i) - v(x)\right], \quad \forall x \in R^M,$$

$$\overline{\partial_i} v(x) = \frac{1}{h_i}\left[v(x) - v(x - e_i h_i)\right], \quad \forall x \in R^M,$$

for $i = \overline{1,M}$; $e_1, \dots, e_m$ denote the versors of the respective axes. Let L_h^0, L_h, L_h^1 denote the difference operators defined as follows

130274

$$L_h^0 v(x) := -\frac{1}{2} \sum_{i,j=1}^{M} \left[\partial_i (a_{ij}(x)\, \bar{\partial}_j v(x) + \bar{\partial}_i (a_{ij}(x) \partial_j v(x))\right],$$

$$L_h^1 v(x) := \frac{1}{2} \sum_{i=1}^{M} b_i(x)\left[\partial_i v(x) + \bar{\partial}_i v(x)\right] + c(x) v(x),$$

$$L_h v(x) := L_h^0 v(x) + L_h^1 v(x).$$

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

Let Ω_{1h} and Ω_{2h} denote non-empty domain of R_h^M and let $R_h^M = \Omega_{1h} \cup \Omega_{2h}$, $\Omega_{1h} \cap \Omega_{2h} = \phi$. We determine a sequence $\{U^n(x)\}$ for $x \in R_h^M$, $n \in Z$, such that $U^0(x) = \phi(x)$ for $x \in R_h^N$. The difference problem (3.1) will be approximated by the following schema

$$(3.2)\quad \begin{cases} \frac{2}{\tau}\left[U_{(x)}^{n+\frac{1}{2}} - U_{(x)}^{n}\right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n} = f_{(x)}^{n}, & \text{if } x \in \Omega_{1h}, \\ \frac{2}{\tau}\left[U_{(x)}^{n+\frac{1}{2}} - U_{(x)}^{n}\right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+\frac{1}{2}} = \frac{1}{2}\left[f_{(x)}^{n+1} + f_{(x)}^{n}\right], & \text{if } x \in \Omega_{2h}, \\ \frac{2}{\tau}\left[U_{(x)}^{n+1} - U_{(x)}^{n+\frac{1}{2}}\right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+\frac{1}{2}} = \frac{1}{2}\left[f_{(x)}^{n+1} + f_{(x)}^{n}\right], & \text{if } x \in \Omega_{2h}, \\ \frac{2}{\tau}\left[U_{(x)}^{n+1} - U_{(x)}^{n+\frac{1}{2}}\right] + L_h^{n+\frac{1}{2}} U_{(x)}^{n+1} = f_{(x)}^{n+1}, & \text{if } x \in \Omega_{1h}, \\ U_{(x)}^{0} = \varphi(x), \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $U^n_{(x)} := u(x,n\tau)$. For the difference schema (3.2) there holds the following theorem on equivalence, analogous to Theorem 2.1.

T h e o r e m 3.1. The sequences $\{U^n_{(x)}\}$ for $x \in R^M_h$, $n \in Z$ $(0 \leqslant 2n \leqslant N)$, satisfies the system of equation (3.2) iff the sequence $\{U^n_{(x)}\}$ satisfies the following difference equation

$$(3.3)\quad \begin{cases} \frac{1}{2}\left[U^{n+1}_{(x)} - U^n_{(x)}\right] + \frac{1}{2}\left[L_h^{n+\frac{1}{2}}\, U^{n+1}_{(x)} + L_h^{n+\frac{1}{2}}\, U^n_{(x)}\right] = \frac{1}{2}\left[f^n_{(x)} + f^{n+1}_{(x)}\right], \\ \qquad\qquad \text{for } x \in R^M_h,\ n = \overline{0,N-1} \\ U^0_{(x)} = \varphi(x), \quad \text{for } x \in R^M_h. \end{cases}$$

The proof is similar to that of Theorem 2.1.

T h e o r e m 3.2. If the functions $a_{i,j}$, b_i, c, f $(i,j = \overline{1,M})$ are continuous and bounded and

$$\frac{\partial a_{ij}(x,t)}{\partial t} \in C(R,(0,T)),$$

then there exists a positive number τ_* dependent only on the norms of coefficients and such that for $\tau \in (0,\tau_*)$ the difference schema (3.2) is unconditionally stable and approximates the problem (3.1) with order $O(\tau+h^2)$.

The proof of this theorem can be found in [5].

The schema (3.2) is unconditionally stable but is not always open which is shown in the following theorem.

T h e o r e m 3.3. The schema (3.2) with the condition $\Omega_{1h} \cap \Omega_{2h} = \emptyset$ is open iff the functions a_{ij} $(i \neq j)$ are identically zero and

$$\Omega_{1h} = \left\{ x \in R^M_h;\ x = (m_1h_1,\ldots,m_Mh_M),\ \frac{1}{2}\left[\sum_{j=1}^{M} m_j - 1\right] \in Z \right\},$$

$$\Omega_{2h} = \left\{ x \in R^M_h;\ x = (m_1h_1,\ldots,m_Mh_M),\ \frac{1}{2}\sum_{i=1}^{M} m_j \in Z \right\}.$$

The proof of this theorem is analogous to that of Theorem 3.3 in [6].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

BIBLIOGRAPHY

[1] Р. Рихтмиер, К. Мортон: Разностные методы решения краевых задач. Москва 1972.

[2] В.К. Шаулев: Об одном способе численного интегрирования уравнений диффузий, ДАН СССР 115 (1957).

[3] В.К. Шаулев: Интегрирование уравнений параболического типа методом сеток. Москва 1960.

[4] И. Петровский: Разностно-дифференциальные уравнения. PWN, Warszawa 1955.

[5] M. Dryja, J.M. Jankowscy: A review of numerical methods and algorithms. WNT, Warszawa 1982, part II.

[6] Vu Van Hung: Open, non-conditionally stable differential schemas of solving boundary problems for parabolic equations of second order (Doctoral thesis in Polish), Technical University of Warsaw, Warszawa 1986.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received April 30, 1987; revised version September 23, 1987.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Józef Sidorowicz

ОЦЕНКА ПЕРЕХОДНОЙ ПЛОТНОСТИ ДИФФУЗИОННОГО ПРОЦЕССА В ТОНКОЙ ТРУБКЕ НА ГЛАДКОМ РИМАНОВОМ МНОГООБРАЗИИ

1°. Пусть M - гладкое, ν-мерное риманово многообразие с метрическим тензором $g(x) = \{g_{ij}(x)\}$, $i,j = 1,2,\dots,\nu$, в точке $x \in M$, который определяет оператор Лапласса-Бельтрами

$$\Delta_M = \sum_{i,j=1}^{\nu} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1} b^j(x) \frac{\partial}{\partial x^j}, \tag{1}$$

где

$$b^j(x) = \left[\det g(x)\right]^{-\frac{1}{2}} \sum_{i=1}^{\nu} \frac{\partial}{\partial x^i} \left[g^{ij}(x)(\det g(x))^{\frac{1}{2}}\right],$$

$\{g^{ij}(x)\} = g^{-1}(x)$, $x^i, i=1,2,\dots,\nu$ - локальные координаты точки $x \in M$. Пусть X_s, $s \in [0,t]$ - диффузионный процесс, определяемый стохастическим уравнением

$$dX_s = \sigma(X_s)dw_s + b(X_s)ds,$$

где $\sigma(x)\sigma^*(x) = \{g^{ij}(x)\}$, $b(x) = \{b^j(x)\}$, w_s - винеровский процесс. Пусть также на M определена гладкая класса $C^{2+\alpha}$, $\alpha > 0$, кривая

$$\gamma = \{\gamma_i(s)\} : s \in [0,t] \longrightarrow M, \quad i=1,2,\dots,\nu.$$

Наша цель найти асимптотику переходной плотности $p^\varepsilon(0,t,x,y)$, $\varepsilon > 0$, процесса X_s при $\varepsilon \to 0$ вдоль некоторой ε-трубки

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

кривой $\gamma(s)$, $s \in [0,t]$. Аналогичная задача о переходной плотности винеровского процесса в R^3 представлена в работе С.А. Молчанова и Ю. Сидоровича [1]. Определение ε-трубки начнем следующим образом.

Пусть T_z - касательное пространство к M в точке $z \in \gamma$. $D(s)$, $s \in [0,t]$ - область такая, что $D(s) \subset T_z$, $z \in D(s)$, причем расстояние $\varrho(z, \partial D(s)) = O(\varepsilon)$. Пусть также граница $\partial D(0)$ - гладкая класса не меньше, чем $C^{2+\alpha}$, $\alpha > 0$. Кроме того, предположим, что A_s, $s \in [0,t]$ - гладкое в смысле Фреше семейство диффеоморфизмов класса не меньше $C^{2+\alpha}$, $A_0 = E$. Пусть также

$$D(s) = A_s D(0).$$

Тогда, очевидно, граница $\partial D(s)$, $s \in [0,t]$ тоже класса не меньше, чем $C^{2+\alpha}$. Преобразуем область $D(s)$ по формуле

$$D^\varepsilon(s) = \varepsilon D(s), \quad s \in [0,t].$$

Затем произведем отображение обратное к экспоненциальному пространства T_z на соответствующую карту многообразия M. Пусть в этом отображении области $D^\varepsilon(s) \subset T_z$ будет соответствовать область $\tilde{D}^\varepsilon(s) \subset M$ или же

$$\tilde{D}^\varepsilon(s) = \operatorname{Exp}^{-1}_{\gamma(s)} (D^\varepsilon(s)).$$

Очевидно, что $\partial \tilde{D}^\varepsilon(s)$ класса не меньше $C^{2+\alpha}$, так как $\tilde{D}^\varepsilon(s) = A_s \tilde{D}(0)$. Систему $\tilde{D}^\varepsilon_{[0,t]}$ областей типа $\tilde{D}^\varepsilon(s)$, связанных со всеми точками кривой γ назовем ε-трубкой в M вдоль кривой γ, т.е.

$$\tilde{D}^\varepsilon_{[0,t]} = \bigcup_{s=0}^{t} \left(\{s\} \times \tilde{D}^\varepsilon(s) \right).$$

На произвольном римановом многообразии нет линейной структуры, поэтому преобразование вида $\gamma(s)\delta$, $\delta \to \infty$ здесь не возможно. В связи с этим как и в работе [2], мы определяем трубку как систему областей типа $\tilde{D}^\varepsilon(s)$. Основное отличие от [2] состоит в том, что сечение трубки не шаровое постоянного радиуса, а произвольное.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2°. Согласно теории случайных процессов $p^{\varepsilon}(s,t,x,y)$ - переходная плотность процесса X_s в ε-трубке $\tilde{D}^{\varepsilon}_{[0,t]}$ при $x \in \tilde{D}^{\varepsilon}(s)$, $s \in [0,t]$, $y \in \tilde{D}^{\varepsilon}(t)$ является фундаментальным решением задачи

$$(2)\quad \begin{cases} \dfrac{\partial p^{\varepsilon}}{\partial s} + \dfrac{1}{2}\Delta_M p^{\varepsilon} = 0, \ (s,x) \in \tilde{D}^{\varepsilon}_{[0,t]}, \ x \in \tilde{D}^{\varepsilon}(s), \ y \in \tilde{D}^{\varepsilon}(t), \\ p^{\varepsilon}(s,t,x,y) = 0, \ x \in \partial\tilde{D}^{\varepsilon}(s), \\ p^{\varepsilon}(t,t,x,y) = \delta(x,y). \end{cases}$$

Другими словами, p^{ε} - решение задачи (2) с помощью условной вероятности можно представить следующим образом

$$P\left\{X_\tau \in \tilde{D}^{\varepsilon}_{[s,t]}, \tau \in [s,t], X_t \in y+dy, y \in \tilde{D}^{\varepsilon}(t) \,\middle|\, X_s = x, x \in \tilde{D}^{\varepsilon}(s)\right\} = p^{\varepsilon}(s,t,x,y)dy.$$

Л е м м а 1. (Формула Картана для оператора $\Delta_M^{(s)}$). Пусть Δ_R - оператор Лапласа в евклидовом пространстве R^{ν}, $R_{ikjl}(\gamma(s))$ - римановый тензор кривизны, $\Delta_M^{(s)}$ - оператор Δ_M в точке (s,x).
Тогда

$$(3)\quad \Delta_M^{(s)} = \Delta_R + \sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, x_k \frac{\partial}{\partial x_j} + O(|x|^2),$$

где

$$C_{ijk}(\gamma(s)) = -\frac{1}{3}\left[R_{ikji}(\gamma(s)) + R_{iijk}(\gamma(s)) - \delta_{ij}R_{ikii}(\gamma(s))\right].$$

Д о к а з а т е л ь с т в о. По формуле Картана [3] для ε-трубки $\tilde{D}^{\varepsilon}_{[0,t]}$ имеем

$$(4)\quad g_{ij}(s,x) = \delta_{ij} + \frac{1}{3}\sum_{k,l=1}^{\nu} R_{ikjl}(\gamma(s))x_k x_l + O(|x|^3).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Учитывая формулу (4), получаем следующие результаты для (1):

$$g^{ij}(s,x) = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{\nu} R_{ikjl}(\gamma(s))\, x_k x_l + O(|x|^3),$$

$$\left\{\det\left\{g_{ij}(s,x)\right\}\right\}^{\frac{1}{2}} = 1 + \frac{1}{6}\sum_{i,k,l=1}^{\nu} R_{ikil}(\gamma(s))\, x_k x_l + O(|x|^3),$$

$$\left\{\det\left\{g_{ij}(s,x)\right\}\right\}^{-\frac{1}{2}} = 1 - \frac{1}{6}\sum_{i,k,l=1}^{\nu} R_{ikil}(\gamma(s))\, x_k x_l + O(|x|^3),$$

$$g^{ij}(s,x)\left\{\det\left\{g_{ij}(s,x)\right\}\right\}^{\frac{1}{2}} = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{\nu} R_{ikjl}(\gamma(s))\, x_k x_l +$$

$$+ \frac{1}{6}\,\delta_{ij}\sum_{i,k,l=1}^{\nu} R_{ikil}(\gamma(s))\, x_k x_l + O(|x|^3),$$

$$\frac{\partial}{\partial x^i}\left(g^{ij}(s,x)\left\{\det\left\{g_{ij}(s,x)\right\}\right\}^{\frac{1}{2}}\right) =$$

$$= -\frac{1}{3}\sum_{k=1}^{\nu}\left[R_{ikjl}(\gamma(s)) + R_{iijk}(\gamma(s))\right] x_k +$$

$$+ \frac{1}{6}\,\delta_{ij}\sum_{k=1}^{\nu}\left[R_{ikii}(\gamma(s)) + R_{iiik}(\gamma(s))\right] x_k + O(|x|^2),$$

$$b^j(s,x) = \left[\det\left\{g_{ij}(s,x)\right\}\right]^{-\frac{1}{2}}\sum_{i=1}^{\nu}\frac{\partial}{\partial x^i}\left[g^{ij}(s,x)\left\{\det g(s,x)\right\}^{\frac{1}{2}}\right] =$$

$$= \sum_{i,k=1} -\frac{1}{3}\left[R_{ikji}(\gamma(s)) + R_{iijk}(\gamma(s)) - \delta_{ij}R_{ikii}(\gamma(s))\right] x_k +$$

$$+ O(|x|^2) = \sum_{i,k=1}^{\nu} C_{ijk}(\gamma(s))\, x_k + O(|x|^2).$$

Теперь уже легко получается формула (3).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Л е м м а 2. Пусть λ_0 - младшее собственное значение и $\psi_0(x)$ - соответствующая младшая собственная функция задачи

$$\begin{cases} -\frac{1}{2}\Delta_R(x)\psi(x) = \lambda\psi(x), & x \in D \subset R^\nu, \\ \psi(x) = 0, & x \in \partial D \end{cases}$$

с нормой

$$\|\psi_0\|_L = \int_D \psi_0^2(x)dx = 1.$$

Пусть также λ_0^ε - младшее собственное значение и $\psi_0^\varepsilon(y)$ - соответствующая младшая собственная функция задачи

$$\begin{cases} -\frac{1}{2}\Delta_R(y)\,\psi^\varepsilon(y) = \lambda^\varepsilon\psi^\varepsilon(y), & y = \varepsilon x,\ y \in D^\varepsilon = \varepsilon D, \\ \psi^\varepsilon(y) = 0, & y \in \partial D^\varepsilon \end{cases}$$

с нормой

$$\|\psi_0^\varepsilon\| = \int_{D^\varepsilon} (\psi_0^\varepsilon(y))^2\,dy = 1.$$

Тогда

$$\lambda_0^\varepsilon = \frac{\lambda_0}{\varepsilon^2}, \qquad \psi_0^\varepsilon(y) = \frac{1}{\sqrt{\varepsilon^\nu}}\psi_0\left(\frac{y}{\varepsilon}\right).$$

Доказательство производится прямыми вычислениями.

Л е м м а 3. Младшее собственное значение задачи

$$(5)\quad \begin{cases} -\frac{1}{2}\Delta_M^{(s)}(y)\,\psi^\varepsilon(s,y) = \bar{\lambda}^\varepsilon(s)\,\psi^\varepsilon(s,y), & y = \varepsilon x,\ y \in \tilde{D}^\varepsilon_{(s)}, \\ \psi^\varepsilon(s,y) = 0, & y \in \partial\tilde{D}^\varepsilon_{(s)} \end{cases}$$

имеет вид

$$\bar{\lambda}_0^\varepsilon(s) = \frac{\lambda_0(s)}{\varepsilon^2} + C(s),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

где

$$(6)\quad C(s) = \int_{\tilde{D}_s} \left[\left(\sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, x_k \frac{\partial}{\partial x^j} \psi_0(s,x) \right), \psi_0(s,x) \right] dx,$$

$\lambda_0(s)$ - младшее собственное значение оператора $-\frac{1}{2}\Delta_R$ в области $D(s)$.

Символ $[\cdot,\cdot]$ обозначает скалярное произведение.

Доказательство. Согласно теории возмущений (см. [4]) для оператора

$$\Delta_M^{(s)}(y) = \Delta_R(y) + B^{(s)}(y)$$

собственное значение

$$\bar{\lambda}_0^{\varepsilon}(s) = \lambda_0^{\varepsilon}(s) + \int_{D^{\varepsilon}(s)} \left[B^{(s)}(y)\psi_0^{\varepsilon}(s,y),\ \psi_0^{\varepsilon}(s,y) \right] dy.$$

В нашем случае при $\varepsilon \to 0$

$$B^{(s)}(y) \sim \sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, y_k \frac{\partial}{\partial y^j}$$

и тогда

$$\bar{\lambda}_0^{\varepsilon}(s) \underset{\varepsilon\to 0}{\sim} \frac{\lambda_0(s)}{\varepsilon^2} + \int_{\tilde{D}^{\varepsilon}(s)} \left[\left(\sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, y_k \frac{\partial}{\partial y^j} \psi_0^{\varepsilon}(s,y) \right), \psi_0^{\varepsilon}(s,y) \right] dy =$$

$$= \frac{\lambda_0(s)}{\varepsilon^2} + \int_{\tilde{D}(s)} \left[\left(\sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, x_k \frac{1}{\varepsilon} \frac{\partial}{\partial x^j} \left(\frac{1}{\sqrt{\varepsilon^{\nu}}} \psi_0(s,x) \right) \right), \frac{1}{\sqrt{\varepsilon^{\nu}}} \psi_0(s,x) \right] \varepsilon^{\nu} dx.$$

Отсюда получаем завершение доказательства леммы 3.

Теперь покажем, что решение задачи (2) в области $\tilde{D}^{\varepsilon}(s)$ или $\tilde{D}^{\varepsilon}_{[0,t]}$ вычисляется с помощью собственных функций и соответствующих собственных значений задачи (5).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Произведем аппроксимацию ε-трубки конусами как и в [1]. Здесь (в римановых координатах) оператор $\Delta_M^{(s)}$ заменяется согласно формуле (3) на оператор

$$\Delta_M^{(s)} = \Delta_R + \sum_{i,j,k=1}^{\nu} C_{ijk}(\gamma(s))\, x_k \frac{\partial}{\partial x^j} + O(|x|^2).$$

Повторяя предыдущие рассуждения в [1] и используя лемму 3, получим следующий результат

Т е о р е м а 1. При перечисленных выше условиях гладкости кривой $\gamma(s)$ и семейства областей $D(s)$, $s \in [0,t]$, имеем

$$p^{\varepsilon}(0,t,x,y) \underset{\varepsilon \to 0}{\sim} \psi_0(0,x)\, \psi_0(t,y) \exp\left\{ - \int_0^t \left(\frac{\lambda_0(s)}{\varepsilon^2} + C(s) \right) ds \right\},$$

где $\psi_0(s,\cdot)$ - младшая собственная функция оператора $-\frac{1}{2}\Delta_M$ в области $D(s)$, $C(s)$ - поправка, даваемая формулой (6).

В частном случае, когда $A(s)$ - сдвиг вдоль кривой, мы получим трубку фиксированного (с точностью до величины порядка ε^2) сечения и тогда верна следующая

Т е о р е м а 2. Если исполнены все условия гладкости кривой $\gamma(s)$ и семейства областей $D(s)$, тогда

$$p^{\varepsilon}(0,t,x,y) \underset{\varepsilon \to 0}{\sim} \psi_0(0,x)\, \psi_0(t,y) \exp\left\{ - \frac{\lambda_0(0)}{\varepsilon^2} + C(0) \right\} t.$$

ЛИТЕРАТУРА

[1] С.А. М о л ч а н о в , J. S i d o r o w i c z : Асимптотические формулы для переходной плотности винеровского процесса в тонких трубках, Demonstratio Math., 15 (1982), 579-598.

[2] T. F u j i t a , S. K o t a n i : The Onsager-Machlup funktion for diffusion process, J. Math. Kyoto Univ. 22 (1982), 115-130.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] M. S p i v a k : A comprehensive introduction to differential geometry. Braudeis University, 1970.

[4] К. Ф р и д р и х с : Возмущение спектра операторов в гильбертовом пространстве, Москва 1969.

INSTITUTE OF MATHEMATICS. TECHNICAL UNIVERSITY,
75-620 KOSZALIN, POLAND

Received August 17, 1987.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Ali A. A. Daw

RETRACTS OF HOMOGENEOUS-UNIVERSAL POST ALGEBRAS AND THEIR GENERALIZATIONS

Introduction

The section I of this paper depends on papers [3] and [4]. In this section we introduce the definition of universal Post algebra and homogeneous Post algebra. We apply the theory of both universal and homogeneous Post algebras.

Assuming the generalized continuum hypothesis (GCH) $\alpha^+ = 2^\alpha$, B. Jonsson proved in (2) a general theorem giving sufficient conditions to be satisfied in a class K of similar relational systems in order that for each infinite cardinal α there exist one and, up to isomorphism, only one α^+-homogeneous, α^+-universal system in K. Of six conditions in B. Jonsson's theorem the conditions and V-VI of his theorem clearly hold in the class K of all Boolean algebras. It was also shown that his condition IV′ (the amalgamation property) holds in the class of all Boolean algebra.

Consequently, under the (GCH) for each infinite cardinal α there exists an α^+-homogeneous, α^+-universal Boolean algebra. In this paper we will show that there exists an α^+-homogeneous, α-universal Post algebra. We will denote it by P_{α^+}. Here we prove a homomorphism extension property for retracts of P_{α^+}. Throughout we assume the generalized continuum hypothesis.

In the section 2 of this paper we will introduce the definition of a retract of a generalized Post algebra. Also we will develope in this part the theory that will be used for

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the investigation of properties of retracts generalized Post algebras. The contents of this section depends on paper [1].

1. Homogeneous - universal - Post algebras

1.1. D e f i n i t i o n . Let β denote an infinite cardinal number. A Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B \rangle$ is said to be β-universal if $\bar{\bar{P}} = \beta$ and for each Post algebra $P_1 = \langle (e_{1,0}, e_{1,1}, \ldots, e_{1,n-1}; B) \rangle$ such that $\bar{\bar{P}}_1 \leqslant \beta$ there exists a monomorphism $h: P_1 \longrightarrow P$.

1.2. D e f i n i t i o n . Let β denote an infinite cardinal number. A Post algebras $P = (e_0, e_1, \ldots, e_{n-1}; B)$ is said to be β-homogeneous if each isomorphism $h: P_1 \longrightarrow P_2$ such that $P_i \subset P$ and $\bar{\bar{P}}_i < \beta$ for $i = 1,2$ can be extended to an automorphism of P.

1.3. T h e o r e m . A Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B \rangle$ is β-universal iff B is β-universal.

P r o o f . Let B be β-universal, then $\bar{\bar{B}} = \beta$ and for each Boolean algebra B_1 such that $\bar{\bar{B}}_1 \leqslant \beta$ there exists a monomorphism $f: B_1 \longrightarrow B$ and by Theorem 4.2 and Corollary 4.3 in the paper [4] f can be extended to a monomorphism $h: P_1 \longrightarrow P$ where $\bar{\bar{P}} = \beta$ and $\bar{\bar{P}}_1 \leqslant \beta$. Hence P is β-universal.

The converse is immediate.

1.4. T h e o r e m . A Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B \rangle$ is β-homogeneous iff B is β-homogeneous.

P r o o f . Take any isomorphism $h: P_1 \longrightarrow P_2$, where $P_i \subset P$ for $i = 1,2$ and $\bar{\bar{P}}_i < \beta$ then the restriction $f = h|B_1$ is an isomorphism from B_1 onto B_2. Since B be β-homogeneous, the isomorphism $f: B_1 \longrightarrow B_2$ can be extended to an automorphism $\bar{f}$ on B. From Theorem 4.2 and Corollary 4.3 in the paper [4] $\bar{f}$ is extended to an automorphism $\bar{h}$ of P. $\bar{h}$ is, as well, an extension of h. In fact, we have to prove now that $\bar{h}(a) = h(a)$ for every $a \in P_1$, where $a = \sum_{\alpha=1}^{n-1} a_\alpha e_\alpha$ then

$$h(a) = \sum_{\alpha=1}^{n-1} f(a_\alpha)\, h(e_\alpha) = \sum_{\alpha=1}^{n-1} \bar{f}(a_\alpha)\, h(e_\alpha) = \sum_{\alpha=1}^{n-1} \bar{h}(a_\alpha)\, e_\alpha = \bar{h}(a).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence h is an automorphism of P. Thus P is β-homogeneous. The converse is immediate.

1.5. T h e o r e m . Let P′ be a retract of $P_\alpha^+ = \langle (e_0, e_1, \dots, e_{n-1}); B\rangle$ where B is the α^+-universal homogeneous Boolean algebra. Let $r : B_0 \longrightarrow B_1$ be a monomorphism extending to a monomorphism $f : P_0 \longrightarrow P_1$ where

$$P_0 = \langle (e_{0,1}, e_{0,2}, \dots, e_{0,n-1}); B_0\rangle,$$

$$P_1 = \langle (e_{1,0}, e_{1,1}, \dots, e_{1,n-1}); B_1\rangle,$$

are Post algebras such that

$$\overline{\overline{P}}_0 = \overline{\overline{B}} < \alpha^+, \quad \overline{\overline{P}}_1 = \overline{\overline{B}} < \alpha^+.$$

Then each homomorphism $h_0 : P_0 \longrightarrow P'$ can be extended to a homomorphism $h_1 : P_1 \longrightarrow P'$ such that $h_1 \circ f = h_0$.

P r o o f . Suppose that h_0 is a monomorphism. Since P′ is a rectract of $P_\alpha+$ there exists a homomorphism $h : P_\alpha+ \longrightarrow P'$ and monomorphism $i : P' \longrightarrow P_\alpha+$ such that $h \circ i = i_{P'}$, an identity on P′. Since $\overline{\overline{P}}_1 < \alpha^+$, by the α^+-universality of $P_{\alpha+}$, there exists a monomorphism $g : P_1 \longrightarrow P_{\alpha+}$ and we have $\overline{\overline{gf(P_0)}} < \alpha^+$, $ih_0(P_0) < \alpha^+$. Hence $\varphi = i \circ h_0 \circ f^{-1} \circ g^{-1}$ is an isomorphism between subalgebras of the Post algebras $P_{\alpha+}$. By the α^+-homogeneity of $P_{\alpha+}$, φ can be extended to an automorphism $\sigma : P_{\alpha+} \longrightarrow P_{\alpha+}$ such that $\sigma | gf(P_0) = \varphi$. Now $h_1 = h \circ \sigma \circ g$ is the required homomorphism $h_1 : P_1 \longrightarrow P'$ extending h_0, such that $h_1 \circ f = h_0$.

C o r o l l a r y . A Post algebra P is complete iff it is a retract of $P_{\alpha+}$.

P r o o f . By Theorem 1.5 and Corollary in [3].

2. Retracts of generalized Post algebras

2.1. D e f i n i t i o n . Let P be a distributive lattice with 0 and 1 which are distinct. Let B be the Boolean

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

algebra of complemented elements of P. P is called a generalized Post algebra if the following conditions are satisfied:

I. P has a set of elements $\{e_\alpha : \alpha \in A\}$, where A is a complete lattice such that $e_0 = 0$, $e_1 = 1$ and $\alpha \leqslant \alpha'$ implies $e_\alpha \leqslant e_{\alpha'}$. Moreover, every element of P has unique representation $a = \sum_{\alpha \in A} a_\alpha e_{\alpha'}$ where for every $\alpha \in A$, $a_\alpha \in B$, $a_0 = 1$ and $\alpha \leqslant \alpha'$ implies $a_\alpha \geqslant a_{\alpha'}$.

II. If $\{a_\alpha : \alpha \in A\}$ is a set of elements of B such that $a_0 = 1$ and such that $\alpha \leqslant \alpha'$ implies $a_\alpha \geqslant a'_\alpha$ then $\sum_{\alpha \in A} a_\alpha e_\alpha$ exists.

2.2. D e f i n i t i o n . Suppose $P = \langle B, \{e_\alpha : \alpha \in A\}\rangle$ and $P' = \langle B', \{e'_\alpha : \alpha \in A'\}\rangle$ are generalized Post algebra, where P' is a sublattice of P and A' is a complete regular sublattice of A thus $e'_\alpha = e_\alpha$ for $\alpha \in A'$. A map $h : P \longrightarrow P'$ is said to be a generalized Post retraction if:

I. (i) $h|B$ is a Boolean retraction.

II. h is a lattice homomorphism.

III. There exists a closure operator $g : A \longrightarrow A'$ associated with A' such that $h(e_\alpha) = e'_{g(\alpha)}$ for every $\alpha \in A$.

IV. If a is any element of P' and $a = \sum_{\alpha \in A} a_\alpha e_\alpha$ then

$$h(a) = \sum_{\alpha \in A} h(a_\alpha) e'_{g(\alpha)} = \sum_{\alpha \in A'} a_\alpha e_\alpha = a.$$

2.3. T h e o r e m . Suppose $B' \subset B$ and $P = \langle B, \{e_\alpha : \alpha \in A\}\rangle$ and $P' = \langle B', \{e'_\alpha : \alpha \in A'\}\rangle$ are generalized Post algebras where P' is a subalgebra of P. Let $f : B \longrightarrow B'$ be a Boolean retraction and let $g : A \longrightarrow A'$ be a closure operator associated with A' where A' is complete regular sublattice of A and let $h : P \longrightarrow P'$ be defined as follows: if a is any element of P such that $a = \sum_{\alpha \in A} a_\alpha e_\alpha$ then $h(a) = \sum_{\alpha \in A} f(a_\alpha) e'_{g(\alpha)}$. Then h is a generalized retraction extending the Boolean retraction f.

P r o o f . g is a closure operator hence $g(\alpha) = \alpha$ iff $\alpha \in A'$. Hence $e'_\alpha = e_\alpha$ for all $\alpha \in A'$ and by the definition of the generalized Post algebra if $\alpha' \geqslant \alpha$ then $e'_\alpha \geqslant e_\alpha$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since B' is a retract of B, then $f(a_\alpha) = a_\alpha$ of every $a_\alpha \in B'$ and from above

$$h(a) = \sum_{\alpha \in A} f(a_\alpha) e'_{g(\alpha)} = \sum_{\alpha \in A'} a_\alpha e_\alpha = a$$

for all $a \in P'$ and h is our extension of the retraction f. Hence h is a retraction and P' is retract of P.

2.4. L e m m a [1]. P is m-complete if and only if B is m-complete, where

$$P = \langle B : \{e_\alpha : \alpha \in A\} \rangle .$$

2.5. T h e o r e m . Every retract of a complete generalized Post algebra is complete.

P r o o f . Let $h : P \to P'$ be a retraction from a complete generalized Post algebra P onto a generalized Post algebra P'. Let B be the center of P. Then $h(B) = B'$ is the center of P'. Since B is complete and B' is a retract of B, B' is also complete (every retract of a complete Boolean algebra is complete). Now from Lemma 2.4 it follows that P' is complete. This ends the proof of the theorem.

2.6. T h e o r e m . P is complete if and only if B is a retract of the homogeneous universal Boolean algebra $U_{\alpha+}$, where $P = \langle B : \{e_\alpha : \alpha \in A\} \rangle$.

P r o o f . If P is complete then from Lemma 2.4 B is complete and from Corollary in paper [3] B is a retract of $U_{\alpha+}$. Conversely let B a retract of $U_{\alpha+}$. Then B is complete by corollary in [3]. Hence P is complete by Lemma 2.4.

REFERENCES

[1] Ph. D w i n g e r : Generalized Post algebra, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astronom. Phys. 16 (1968), 559-565.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] B. J o n s s o n : Homogeneous universal relational systems, Notices Amer. Math. Soc. 5 (1958), 776.

[3] M.J. M ą c z y ń s k i : A property of retracts of α^+-homogeneous α^+-universal Boolean Algebras, Bull. Acad. Polon. Sci. Ser. Sci. Math., Astronom. Phys., 14 (1966) 607-608.

[4] T. T r a c z y k : On axioms and some properties of Post Algebras, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astronom. Phys., 10 (1962), 509-512.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received September 15, 1987.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Marian Hotloś

SOME THEOREMS ON DOUBLY WARPED PRODUCTS

1. Introduction

Let (M,g) and (M',g') be Riemannian manifolds whose metrics need not be positive definite.

If $f: M \to (0,\infty)$ and $\varphi: M' \to (0,\infty)$ are smooth functions then the Cartesian product $\overline{M} = M \times M'$ with the metric $\overline{g} = \varphi g \oplus f g'$ (more precisely, $\overline{g} = (\varphi \circ \pi')\pi^* g + (f \circ \pi)(\pi')^* g'$, $\pi(\pi'): \overline{M} \to M(M')$ being the natural projection), is called a doubly warped product [5]. We will use the notation $M_\varphi \times_f M'$ for the manifold $(M \times M', \varphi g \oplus f g')$. If f or φ is constant then we obtain a (singly) warped product [2] (or semi-decomposable space [7]). It is worth noticing that a doubly warped product is the special case of a so-called conformal product ($\overline{g} = hg \oplus kg'$, where h and k are functions defined on $M \times M'$) investigated by Yano [11] and Wong [10]. Lorentzian doubly warped products have been studied by Beem and Powell [1].

In this paper we consider only essentially doubly warped products, i.e., such doubly warped products which are not singly warped.

A Riemannian manifold (M,g) is called locally symmetric if its curvature tensor R is parallel. Analogically, (M,g) is said to be Ricci-symmetric (conformally symmetric) if its Ricci tensor S (the Weyl conformal curvature tensor C) is parallel.

The aim of the present paper is to study doubly warped products satisfying one from the above conditions. In Sec-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

tion 3 we prove that if $M_\varphi \times_f M'$ is Ricci symmetric then manifolds (M,g) and (M',g') are also Ricci-symmetric. Moreover, Theorem 2 gives necessary and sufficient conditions in order that a doubly warped product be Ricci-symmetric.

Section 4 is devoted to locally symmetric manifolds. We prove that if $M_\varphi \times_f M'$ is locally symmetric then (M,g) and (M',g') are also locally symmetric. Next we observe that M and M' are so-called K^*-spaces defined by Walker [9]. As a consequence, we obtain a description of the local structure of locally symmetric doubly warped products. Finally, in Section 5 we prove that every conformally symmetric doubly warped product is necessarily conformally flat.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class C^∞ or analytic. By abuse of notation, concerning Riemannian manifolds we often write M instead of (M,g).

2. Preliminaries

Let $(\bar{M},\bar{g})$ be an n-dimensional $(n \geqslant 3)$ doubly warped product $M_\varphi \times_f M'$ ($\dim M = q$, $1 \leqslant q < n$, $\dim M' = n-q = s$). In a suitable product chart $x^1,\ldots,x^n$ for $\bar{M}$ we have

$$\bar{g}_{ij}dx^i dx^j = \varphi g_{ab}dx^a dx^b + f g'_{\alpha\beta}dx^\alpha dx^\beta,$$

where $i,j,\ldots = 1,\ldots,n$, $a,b,\ldots = 1,\ldots,q$, $\alpha,\beta,\ldots = q+1,\ldots,n$, g_{ab} and f are functions of (x^a) only, $g'_{\alpha\beta}$ and φ are functions of (x^α) only.

We denote by Γ^a_{bc}, R_{abcd}, S_{ab} and K, the components of the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, the Ricci tensor S and the scalar curvature of (M,g), respectively. Moreover, we assume that each object denoted by the bar is formed from the metric $\bar{g}$ of $\bar{M}$ and each object denoted by the prime - from the metric g' of M'.

It is easy to show that the following relations hold:

$$\bar{\Gamma}^c_{ab} = \Gamma^c_{ab}, \quad \bar{\Gamma}^a_{b\gamma} = \frac{1}{2\varphi}\delta^a_b\varphi_\gamma, \quad \bar{\Gamma}^\alpha_{b\gamma} = \frac{1}{2f}\delta^\alpha_\gamma f_b,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma'^{\alpha}_{\beta\gamma}, \quad \bar{\Gamma}^{a}_{\beta\gamma} = -\frac{1}{2\varphi} g^{ad} f_d g'_{\beta\gamma}, \quad \bar{\Gamma}^{\alpha}_{bc} = -\frac{1}{2f} g'^{\alpha\delta} \varphi_\delta g_{bc},$$

where $f_b = \partial_b f$, $\varphi_\alpha = \partial_\alpha \varphi$, $\partial_i = \frac{\partial}{\partial x^i}$.

In the sequel we will use the following notation

$$(1)\quad \begin{cases} G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}, \\ T_{ab} = -\frac{1}{2f}\left(\nabla_b f_a - \frac{1}{2f} f_a f_b\right), \quad T'_{\alpha\beta} = -\frac{1}{2\varphi}\left(\nabla'_\alpha \varphi_\beta - \frac{1}{2\varphi}\varphi_\alpha \varphi_\beta\right), \\ tr(T) = g^{ab}T_{ab}, \quad tr(T') = g'^{\alpha\beta}T'_{\alpha\beta}, \\ P = g^{ab}f_a f_b/4f^2 = \frac{\Delta_1 f}{4f^2}, \quad P' = g'^{\alpha\beta}\varphi_\alpha\varphi_\beta/4\varphi^2 = \frac{\Delta'_1\varphi}{4\varphi^2}, \\ Q = f\left[(s-1)P - tr(T)\right], \quad Q' = \varphi\left[(q-1)P' - tr(T')\right]. \end{cases}$$

An elementary but somewhat lenghty calculation leads to the formulas:

$$\bar{R}_{abcd} = \varphi R_{abcd} + \frac{1}{4f}\Delta'_1\varphi\, G_{abcd},$$

$$\bar{R}_{\alpha\beta\gamma\delta} = fR'_{\alpha\beta\gamma\delta} + \frac{1}{4\varphi}\Delta_1 f\, G'_{\alpha\beta\gamma\delta},$$

$$\bar{R}_{a\beta c\delta} = fT_{ac}g'_{\beta\delta} + \varphi T'_{\beta\delta}g_{ac},$$

$$\bar{R}_{abc\delta} = \frac{\varphi_\delta}{4f}\left(f_b g_{ac} - f_a g_{bc}\right),$$

$$\bar{R}_{\alpha\beta\gamma d} = \frac{f_d}{4\varphi}\left(\varphi_\beta g'_{\alpha\gamma} - \varphi_\alpha g'_{\beta\gamma}\right),$$

$$\bar{R}_{ab\gamma\delta} = 0,$$

$$(2)\quad \begin{cases} \bar{S}_{ab} = S_{ab} - sT_{ab} + \frac{Q'}{f} g_{ab}, \\ \bar{S}_{\alpha\beta} = S'_{\alpha\beta} - qT'_{\alpha\beta} + \frac{Q}{\varphi} g'_{\alpha\beta}, \\ \bar{S}_{a\beta} = -\frac{n-2}{4f\varphi} f_a \varphi_\beta, \end{cases}$$

$$(3)\quad \bar{K} = \frac{K}{\varphi} + \frac{K'}{f} + \frac{s}{\varphi}\left[(s-1)P - 2tr(T)\right] + \frac{q}{f}\left[(q-1)P' - 2tr(T')\right].$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The formula (2) implies that $\bar{M}$ cannot be Einsteinian, so it is never a space of constant curvature.

From the above equations it follows that the only non-zero components of the Weyl conformal curvature tensor $\bar{C}$ are those related to:

$$(4)\quad \bar{C}_{abcd} = \varphi\left(R_{abcd} - \frac{1}{n-2}\left(g_{ad}S_{bc}+g_{bc}S_{ad}-g_{ac}S_{bd}-g_{bd}S_{ac}\right)\right) + (7$$

$$+ \frac{s\varphi}{n-2}\left(g_{ad}T_{bc} + g_{bc}T_{ad} - g_{ac}T_{bd} - g_{bd}T_{ac}\right) + (7$$

$$+ G_{abcd}\left(\frac{\Delta_1'\varphi}{4f} - \frac{2\varphi Q'}{f(n-2)} + \frac{\varphi^2\bar{K}}{(n-1)(n-2)}\right), (8$$

$$(4')\quad \bar{C}_{\alpha\beta\gamma\delta} = f\left(R'_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\left(g'_{\alpha\delta}S'_{\beta\gamma}+g'_{\beta\gamma}S'_{\alpha\delta}-g'_{\alpha\gamma}S'_{\beta\delta}-g'_{\beta\delta}S'_{\alpha\gamma}\right)\right) +$$

$$+ \frac{qf}{n-2}\left(g'_{\alpha\delta}T'_{\beta\gamma} + g'_{\beta\gamma}T'_{\alpha\delta} - g'_{\alpha\gamma}T'_{\beta\delta} - g'_{\beta\delta}T'_{\alpha\gamma}\right) + (8$$

$$+ G'_{\alpha\beta\gamma\delta}\left(\frac{\Delta_1 f}{4\varphi} - \frac{2fQ}{\varphi(n-2)} + \frac{f^2\bar{K}}{(n-1)(n-2)}\right),$$

$$(5)\quad \bar{C}_{a\beta c\delta} = \frac{1}{n-2}\left[\left(S'_{\beta\delta}+(s-2)T'_{\beta\delta}\right)\varphi g_{ac} + \left(S_{ac} + (q-2)T_{ac}\right)fg'_{\beta\delta} + (9)\right.$$

$$\left. + \left(Q + Q' - \frac{f\varphi\bar{K}}{n-1}\right)g_{ac}g'_{\beta\delta}\right].$$

Similarly, by an elementary but lenghty calculation we can easily show that the only non-zero components of $\bar{\nabla}\bar{R}$ are those related to (9'

$$(6)\quad \bar{\nabla}_e\bar{R}_{abcd} = \varphi\nabla_e R_{abcd} - \frac{\Delta_1'\varphi}{8f^2}\left(2f_eG_{abcd} + f_aG_{ebcd} + f_bG_{aecd} +\right.$$

$$\left. + f_cG_{abed} + f_dG_{abce}\right), (10$$

$$(6')\quad \bar{\nabla}_\varepsilon\bar{R}_{\alpha\beta\gamma\delta} = f\nabla'_\varepsilon R'_{\alpha\beta\gamma\delta} - \frac{\Delta_1 f}{8\varphi^2}\left(2\varphi_\varepsilon G'_{\alpha\beta\gamma\delta} + \varphi_\alpha G'_{\varepsilon\beta\gamma\delta} + \varphi_\beta G'_{\alpha\varepsilon\gamma\delta} +\right.$$

$$\left. + \varphi_\gamma G'_{\alpha\beta\varepsilon\delta} + \varphi_\delta G'_{\alpha\beta\gamma\varepsilon}\right), (10$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(7)\quad \bar{\nabla}_\varepsilon \bar{R}_{abcd} = -\varphi_\varepsilon R_{abcd} + \frac{\varphi^2}{f}\,\partial_\varepsilon P' G_{abcd} + \frac{\varphi_\varepsilon}{4f^2}\,(f_a f_d g_{bc} + f_b f_c g_{ad} - f_a f_c g_{bd} - f_b f_d g_{ac}),$$

$$(7')\quad \bar{\nabla}_e \bar{R}_{\alpha\beta\gamma\delta} = -f_e R'_{\alpha\beta\gamma\delta} + \frac{f^2}{\varphi}\,\partial_e P G'_{\alpha\beta\gamma\delta} + \frac{f_e}{4\varphi^2}\,(\varphi_\alpha\varphi_\delta g'_{\beta\gamma} + \varphi_\beta\varphi_\gamma g'_{\alpha\delta} - \varphi_\alpha\varphi_\gamma g'_{\beta\delta} - \varphi_\beta\varphi_\delta g'_{\alpha\gamma}),$$

$$(8)\quad \bar{\nabla}_e \bar{R}_{abcd} = -\frac{\varphi_\delta f_e}{4f^2}\,(f_b g_{ac} - f_a g_{bc}) + \frac{\varphi_\delta}{2}\,(T_{ac} g_{be} - T_{bc} g_{ae} + g_{bc} T_{ae} - g_{ac} T_{be} - R_{abce}) + \frac{\varphi^2}{2f}\,\partial_\delta P' G_{abce},$$

$$(8')\quad \bar{\nabla}_\varepsilon \bar{R}_{\alpha\beta\gamma d} = -\frac{f_d \varphi_\varepsilon}{4\varphi^2}\,(\varphi_\beta g'_{\alpha\gamma} - \varphi_\alpha g'_{\beta\gamma}) + \frac{f_d}{2}\,(T'_{\alpha\gamma} g'_{\beta\varepsilon} - T'_{\beta\gamma} g'_{\alpha\varepsilon} + g'_{\beta\gamma} T'_{\alpha\varepsilon} - g'_{\alpha\gamma} T'_{\beta\varepsilon} - R'_{\alpha\beta\gamma\varepsilon}) + \frac{f^2}{2\varphi}\,\partial_d P G'_{\alpha\beta\gamma\varepsilon},$$

$$(9)\quad \bar{\nabla}_e \bar{R}_{a\beta c\delta} = f\nabla_e T_{ac} g'_{\beta\delta} + \frac{\Delta'_1 \varphi}{8f\varphi}\,(f_a g_{ce} + f_c g_{ae}) g'_{\beta\delta} - \frac{f_e}{f}\,g_{ac}\left(\varphi T'_{\beta\delta} + \frac{\varphi_\beta \varphi_\delta}{4\varphi}\right),$$

$$(9')\quad \bar{\nabla}_\varepsilon \bar{R}_{a\beta c\delta} = \varphi \nabla'_\varepsilon T'_{\beta\delta} g_{ac} + \frac{\Delta_1 f}{8f\varphi}\,(\varphi_\beta g'_{\delta\varepsilon} + \varphi_\delta g'_{\beta\varepsilon}) g_{ac} - \frac{\varphi_\varepsilon}{\varphi}\,g'_{\beta\delta}\left(f T_{ac} + \frac{f_a f_c}{4f}\right),$$

$$(10)\quad \bar{\nabla}_\varepsilon \bar{R}_{abc\delta} = \frac{\varphi}{f}\,T'_{\varepsilon\delta}(f_a g_{bc} - f_b g_{ac}) + \frac{1}{2}\,g'_{\varepsilon\delta}(f_a T_{bc} - f_b T_{ac}) + \frac{1}{8f\varphi}\,(f_a g_{bc} - f_b g_{ac})(\Delta'_1 \varphi g'_{\delta\varepsilon} + 2\varphi_\varepsilon \varphi_\delta) + \frac{1}{2}\,f^d R_{abcd} g'_{\delta\varepsilon},$$

$$(10')\quad \bar{\nabla}_e \bar{R}_{\alpha\beta\gamma d} = \frac{f}{\varphi}\,T_{ed}(\varphi_\alpha g'_{\beta\gamma} - \varphi_\beta g'_{\alpha\gamma}) + \frac{1}{2}\,g_{ed}(\varphi_\alpha T'_{\beta\gamma} - \varphi_\beta T'_{\alpha\gamma}) + \frac{1}{8f\varphi}\,(\varphi_\alpha g'_{\beta\gamma} - \varphi_\beta g'_{\alpha\gamma})(\Delta_1 f g_{ed} + 2f_e f_d) + \frac{1}{2}\,\varphi^\delta R'_{\alpha\beta\gamma\delta} g_{ed}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From the above formulas the following relations follow:

$$(11)\quad \bar{\nabla}_c \bar{S}_{ab} = \nabla_c S_{ab} - s\nabla_c T_{ab} - \frac{f_c}{f^2} Q' g_{ab} - \frac{n-2}{8f^2\varphi} \Delta_1' \varphi (f_b g_{ac} + f_a g_{bc}),$$

$$(11')\quad \bar{\nabla}_\gamma \bar{S}_{\alpha\beta} = \nabla_\gamma' S'_{\alpha\beta} - q\nabla_\gamma' T'_{\alpha\beta} - \frac{\varphi_\gamma}{\varphi^2} Q g'_{\alpha\beta} - \frac{n-2}{8f\varphi^2} \Delta_1 f (\varphi_\beta g'_{\alpha\gamma} + \varphi_\alpha g'_{\beta\gamma}),$$

$$(12)\quad \bar{\nabla}_c \bar{S}_{a\beta} = \frac{n-2}{4f^2\varphi} f_a f_c \varphi_\beta + \frac{n-2+s}{2\varphi} T_{ac}\varphi_\beta - \frac{1}{2\varphi} S_{ac}\varphi_\beta + (\varphi^\gamma S'_{\gamma\beta} - q\varphi^\gamma T'_{\gamma\beta} + \frac{Q-Q'}{\varphi}\varphi_\beta) \frac{1}{2f} g_{ac},$$

$$(12')\quad \bar{\nabla}_\gamma \bar{S}_{\alpha b} = \frac{n-2}{4f\varphi^2} \varphi_\alpha \varphi_\gamma f_b + \frac{n-2+q}{2f} T'_{\alpha\gamma} f_b - \frac{1}{2f} S'_{\alpha\gamma} f_b + (f^c S_{cb} - sf^c T_{cb} + \frac{Q'-Q}{f} f_b) \frac{1}{2\varphi} g'_{\alpha\gamma},$$

$$(13)\quad \bar{\nabla}_\gamma \bar{S}_{ab} = \frac{\partial_\gamma Q'}{f} g_{ab} - \frac{\varphi_\gamma}{\varphi}(S_{ab} - sT_{ab} + \frac{Q'}{f} g_{ab}) + \frac{n-2}{4f^2\varphi} f_a f_b \varphi_\gamma,$$

$$(13')\quad \bar{\nabla}_c \bar{S}_{\alpha\beta} = \frac{\partial_c Q}{\varphi} g'_{\alpha\beta} - \frac{f_c}{f}(S'_{\alpha\beta} - qT'_{\alpha\beta} + \frac{Q}{\varphi} g'_{\alpha\beta}) + \frac{n-2}{4f\varphi^2} \varphi_\alpha \varphi_\beta f_c.$$

L e m m a 1. Let $M_\varphi \times_f M'$ be an n-dimensional ($n = q+s \geq 4$) doubly warped product.

(i) If $\bar{C}_{abcd} = 0$, then

$$(14)\quad s(s-1)f[K + 2(q-1)tr(T) + q(q-1)P] = -\varphi q(q-1)[K' + 2(s-1)tr(T') + s(s-1)P'],$$

$$(15)\quad S_{ad} + (q-2)T_{ad} = g_{ad}[K + (q-2)tr(T)]/q$$

and for $q > 1$

$$(16)\quad (q-2)R_{abcd} = g_{ad}S_{bc} + g_{bc}S_{ad} - g_{ac}S_{bd} - g_{bd}S_{ac} - \frac{K}{q-1} G_{abcd}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) If $\bar{C}_{\alpha\beta\gamma\delta} = 0$, then the equation (14) and the relations

(15′) $$S'_{\alpha\delta} + (s-2)T'_{\alpha\delta} = g'_{\alpha\delta}[K' + (s-2)\mathrm{tr}(T')]/s,$$

and for $s > 1$

(16′) $$(s-2)R'_{\alpha\beta\gamma\delta} = g'_{\alpha\delta}S'_{\beta\gamma} + g'_{\beta\gamma}S'_{\alpha\delta} - g'_{\alpha\gamma}S'_{\beta\delta} - g'_{\beta\delta}S_{\alpha\gamma} - \frac{K'}{s-1}G'_{\alpha\beta\gamma\delta}$$

hold.

(iii) If $\bar{C}_{abcd} = 0$ and $\bar{C}_{\alpha\beta\gamma\delta} = 0$, then also $\bar{C}_{a\beta c\delta} = 0$. Consequently $\bar{M}$ is conformally flat.

P r o o f . Using our assumption and (4), we find

(17) $$R_{abcd} = \frac{1}{n-2}(g_{ad}S_{bc} + g_{bc}S_{ad} - g_{ac}S_{bd} - g_{bd}S_{ac}) -$$
$$- \frac{s}{n-2}(g_{ad}T_{bc} + g_{bc}T_{ad} - g_{ac}T_{bd} - g_{bd}T_{ac}) -$$
$$- G_{abcd}\left(\frac{\Delta'_1\varphi}{4f\varphi} - \frac{2Q'}{(n-2)f} + \frac{\varphi\bar{K}}{(n-1)(n-2)}\right).$$

Contracting (17) with g^{bc} and applying (3), we obtain

(18) $$s[S_{ad} + (q-2)T_{ad}] = \frac{g_{ad}}{n-1}[sK + s(q-s-1)\mathrm{tr}(T) - s(s-1)(q-1)P -$$
$$- \frac{\varphi}{f}(q-1)(K' + 2(s-1)\mathrm{tr}(T') + s(s-1)P')],$$

which, by contraction with g^{ad}, implies (14).

Substituting (14) into (18), we have (15). Using now (15) and (17), we obtain (16).

The proof of (ii) is the same. Finally, substituting (15), (15′) and (14) into (5), we easily obtain the last assertion. This completes the proof.

3. Ricci-symmetric doubly warped products

In this section we assume that $\bar{M} = M \underset{\varphi}{\times}_f M'$ ($\dim \bar{M} = n \geqslant 3$) is an essentially doubly warped product (more precisely, we assume that $f_a \neq 0$ and $\varphi_\alpha \neq 0$ everywhere) whose Ricci tensor satisfies the relation

(19) $$\bar{\nabla}\bar{S} = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now we study some consequences of these assumptions. Without loss of generality we can suppose that $q > 1$. Using (19) and (13), we obtain

$$(20)\qquad \frac{\partial_\gamma Q'}{f} g_{ab} = \frac{\varphi_\gamma}{\varphi}(S_{ab} - sT_{ab} + \frac{Q'}{f} g_{ab}) - \frac{n-2}{4f^2\varphi} f_a f_b \varphi_\gamma .$$

Multiplying (20) by φ_δ and alternating the resulting equality in δ and γ, we have $(\partial_\gamma Q')\varphi_\delta = (\partial_\delta Q')\varphi_\gamma$. Therefore

$$(21)\qquad \partial_\gamma Q' = \chi(x^\alpha)\varphi_\gamma .$$

Now the formula (20) can be rewritten in the form

$$(22)\qquad (\chi\varphi - Q')g_{ab} = f(S_{ab} - sT_{ab}) - \frac{n-2}{4f} f_a f_b ,$$

which immediately implies $\chi\varphi - Q' = \text{constant}$, and, in virtue of (21), $\chi = \text{constant}$. Differentiating (22) covariantly, we obtain

$$0 = f_c(S_{ab} - sT_{ab}) + f(\nabla_c S_{ab} - s\nabla_c T_{ab}) + \frac{n-2}{4f^2} f_a f_b f_c -$$

$$- \frac{n-2}{4f}(f_b\nabla_c f_a + f_a\nabla_c f_b).$$

Since (see (22))

$$(23)\qquad S_{ab} - sT_{ab} = \frac{\chi\varphi - Q'}{f} g_{ab} + \frac{n-2}{4f^2} f_a f_b ,$$

so substituting this equation into the preceding one, we get

$$(24)\qquad \nabla_c S_{ab} - s\nabla_c T_{ab} = \frac{n-2}{4f^2}(f_b\nabla_c f_a + f_a\nabla_c f_b) -$$

$$- \frac{n-2}{2f^3} f_a f_b f_c - \frac{\chi\varphi - Q'}{f^2} f_c g_{ab} .$$

On the other hand, from (19) and (11) follows the equality

$$(25)\qquad \nabla_c S_{ab} - s\nabla_c T_{ab} = \frac{f_c}{f^2} Q' g_{ab} + \frac{n-2}{8f^2\varphi} \Delta'_1\varphi (f_b g_{ac} + f_a g_{bc}).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Comparing (24) and (25), we have

$$(26)\quad 2(f_b\nabla_c f_a + f_a\nabla_c f_b) - \frac{4}{f} f_a f_b f_c - \frac{8\chi\varphi}{n-2} f_c g_{ab} = \frac{\Delta_1'\varphi}{\varphi}(f_b g_{ac} + f_a g_{bc}).$$

Alternating (26) in b and c, we obtain

$$2(f_b\nabla_c f_a - f_c\nabla_b f_a) = \left(\frac{\Delta_1'\varphi}{\varphi} - \frac{8\chi\varphi}{n-2}\right)(f_b g_{ac} - f_c g_{ab})$$

which immediately implies $(q > 1)$

$$(27')\qquad \frac{\Delta_1'\varphi}{\varphi} - \frac{8\chi\varphi}{n-2} = \text{constant}.$$

Hence easily follows the formula

$$(28')\qquad (\nabla_\gamma'\varphi_\delta)\varphi^\delta = \left(\frac{\Delta_1'\varphi}{2\varphi} + \frac{4\chi\varphi}{n-2}\right)\varphi_\gamma .$$

In the same way, using (13') and (11'), we can obtain:

$$\partial_c Q = hf_c, \quad Hh = \text{constant},$$

$$(23')\qquad S'_{\alpha\beta} - qT'_{\alpha\beta} = \frac{fh - Q}{\varphi} g'_{\alpha\beta} + \frac{n-2}{4\varphi^2}\varphi_\alpha\varphi_\beta ,$$

$$(24')\qquad \nabla'_\gamma S'_{\alpha\beta} - q\nabla'_\gamma T'_{\alpha\beta} = \frac{n-2}{4\varphi^2}(\varphi_\beta\nabla'_\gamma\varphi_\alpha + \varphi_\alpha\nabla'_\gamma\varphi_\beta) - \frac{n-2}{2\varphi^2}\varphi_\alpha\varphi_\beta\varphi_\gamma - \frac{fh-Q}{\varphi^2}\varphi_\gamma g'_{\alpha\beta},$$

$$(25')\qquad \nabla'_\gamma S'_{\alpha\beta} - q\nabla'_\gamma T'_{\alpha\beta} = \frac{\varphi_\gamma}{\varphi^2} Q g'_{\alpha\beta} + \frac{n-2}{8f\varphi^2}\Delta_1 f(\varphi_\beta g'_{\alpha\gamma} + \varphi_\alpha g'_{\beta\gamma}),$$

$$(26')\qquad 2(\varphi_\beta\nabla'_\gamma\varphi_\alpha + \varphi_\alpha\nabla'_\gamma\varphi_\beta) - \frac{4}{\varphi}\varphi_\alpha\varphi_\beta\varphi_\gamma - \frac{8hf}{n-2}\varphi_\gamma g'_{\alpha\beta} = \frac{\Delta_1 f}{f}(\varphi_\beta g'_{\alpha\gamma} + \varphi_\alpha g'_{\beta\gamma}),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and, for $s > 1$, also

$$\frac{\Delta_1 f}{f} - \frac{8hf}{n-2} = \text{constant}, \tag{27}$$

$$(\nabla_c f_d) f^d = \left(\frac{\Delta_1 f}{2f} - \frac{4hf}{n-2}\right) f_c. \tag{28}$$

On the other hand, if $s = 1$ then (26′) implies

$$\frac{\Delta_1 f}{f} + \frac{4hf}{n-2} = \text{constant},$$

which easily leads to

$$(\nabla_c f_d) f^d = \left(\frac{\Delta_1 f}{2f} - \frac{2hf}{n-2}\right) f_c. \tag{29}$$

Contracting (26) with g^{bc}, we get

$$2(f^d \nabla_d f_a + f_a \Delta_2 f) - \frac{4}{f} f_a \Delta_1 f - \frac{8\chi\varphi}{n-2} f_a = \frac{\Delta_1' \varphi}{\varphi} (q+1) f_a,$$

where $\Delta_2 f = g^{ab} \nabla_b f_a$. Taking into account (28) or (29), we obtain

$$2\Delta_2 f - \frac{3}{f} \Delta_1 f + \frac{8hf}{n-2} = \frac{8\chi\varphi}{n-2} + (q+1) \frac{\Delta_1' \varphi}{\varphi} \quad \text{or} \tag{30}$$

$$2\Delta_2 f - \frac{3}{f} \Delta_1 f - \frac{4hf}{n-2} = \frac{8\chi\varphi}{n-2} + (q+1) \frac{\Delta_1' \varphi}{\varphi},$$

respectively, which immediately implies

$$(q+1) \frac{\Delta_1' \varphi}{\varphi} + \frac{8\chi\varphi}{n-2} = \text{constant}.$$

But comparing this result with (27′), we have

$$\frac{\Delta_1' \varphi}{\varphi} = \text{constant} \tag{31′}$$

and $\frac{8\chi\varphi}{n-2} = \text{constant}$, which gives

$$\chi = 0. \tag{32′}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proceeding analogously in the case $s > 1$, we can obtain

$$\frac{\Delta_1 f}{f} = \text{constant}, \tag{31}$$

$$h = 0. \tag{32}$$

We assert that (31) and (32) hold also in the case $s = 1$. Making use of (19), (12) and (23′), we easily obtain

$$f\left[S_{ac} - (n-2+s)T_{ac}\right] = \frac{n-2}{2f} f_a f_c + g_{ac}\left(hf + \frac{n-2}{4\varphi}\Delta_1'\varphi - Q'\right).$$

On the other hand, using (23) and (32′), we have

$$f(S_{ac} - sT_{ac}) = \frac{n-2}{4f} f_a f_c - Q'g_{ac}. \tag{33}$$

Subtracting up two last equations, we get

$$-f(n-2)T_{ac} = \frac{n-2}{4f} f_a f_c + g_{ac}\left(hf + \frac{n-2}{4\varphi}\Delta_1'\varphi\right) \tag{34}$$

which, by contraction with g^{ac}, implies

$$f(n-2)tr(T) + \frac{n-2}{4f}\Delta_1 f + ghf = -\frac{n-2}{4} q\Delta_1'\varphi.$$

Therefore

$$(n-2)\frac{\Delta_1 f}{4f} + ftr(T) + qhf = \text{constant}. \tag{35}$$

Using (19), (11′), (23′) and (32′), we obtain

$$\frac{n-2}{2\varphi}\varphi_\alpha\varphi_\gamma + (n-2+q)\varphi T'_{\alpha\gamma} - \varphi S'_{\alpha\gamma} = g'_{\alpha\gamma}\left(Q - \frac{n-2}{4f}\Delta_1 f\right).$$

So, the expression into the last bracket must be constant. Assuming now $s = 1$, and taking into account the fact $Q = -ftr(T)$ for $s = 1$, we have

$$(n-2)\frac{\Delta_1 f}{4f} + ftr(T) = \text{constant}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Combining this result, (35) and $\frac{\Delta_1 f}{f} + \frac{4hf}{n-2} = \text{constant}$, we easily obtain (31) and (32). Thus (31) and (32) hold for all s. Contracting (26) with g^{ab}, we have

$$2f^d \nabla_c f_d - \frac{2}{f}\Delta_1 f\, f_c = \frac{\Delta_1'\varphi}{\varphi} f_c$$

which, in virtue of $f^d \nabla_c f_d = \frac{\Delta_1 f}{2f} f_c$ (see (28) and (29)), implies

$$-\frac{\Delta_1 f}{f} = \frac{\Delta_1'\varphi}{\varphi}. \tag{36}$$

The formulas (30), in view of (32), (32′) and (36), take the form

$$2\Delta_2 f + (q-2)\frac{\Delta_1 f}{f} = 0. \tag{37}$$

In the same way we can obtain

$$2\Delta_2'\varphi + (s-2)\frac{\Delta_1'\varphi}{\varphi} = 0. \tag{37′}$$

Since $\mathrm{tr}(T) = -\frac{1}{2f}\Delta_2 f + \frac{\Delta_1 f}{4f^2}$, so making use of (37), we get

$$\mathrm{tr}(T) = (q-1)P, \quad Q = f(s-q)P \tag{38}$$

and, similarly,

$$\mathrm{tr}(T') = (s-1)P', \quad Q' = \varphi(q-s)P'. \tag{38′}$$

The above equations, by (36), imply

$$Q = Q'. \tag{39}$$

The relations (33) and (34), in view of (38), (39), (36) and (32), take the form

$$S_{ac} - sT_{ac} = \frac{n-2}{4f^2} f_a f_c + (q-s)Pg_{ac},$$

$$T_{ac} = Pg_{ac} - \frac{f_a f_c}{4f^2}. \tag{40}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Substituting this relation into the preceding one, we obtain

$$S_{ac} = qPg_{ac} + \frac{q-2}{4f^2} f_a f_c. \tag{41}$$

Using (40) and (1), we have

$$\nabla_a f_c = \frac{f_a f_c}{f} - 2fPg_{ac}. \tag{42}$$

Differentiating (41) covariantly, in virtue of (42) and $\partial_b P = -\frac{f_b}{f} P$ which is an obvious consequence of $fP = \text{const.}$, we get

$$\nabla_b S_{ac} = -\frac{P}{2f}\left[(q-2)(f_a g_{bc} + f_c g_{ab}) + 2qf_b g_{ac}\right]. \tag{43}$$

Similarly, differentiating (40), we obtain

$$\nabla_b T_{ac} = \frac{P}{2f}(f_a g_{bc} + f_c g_{ab} - 2f_b g_{ac}). \tag{44}$$

Analogously, we can get corresponding formulas for M'.

Setting $k = \ln f$ and using (42), we have

$$\nabla_b \nabla_a k = -2Pg_{ab}.$$

We assert that $P = 0$ (since $fP = \text{constant}$, so P is equal to zero either everywhere or nowhere).

Suppose that $P \neq 0$. We need the following result due to Kručkovič ([8], p.116).

T h e o r e m . A Riemannian space V^q admits a solution $k \neq \text{const.}$ of the equation

$$\nabla_b \nabla_a k = pg_{ab}$$

if and only if V is semi-decomposable with a one-dimensional main part, i.e., in some coordinate system

$$g_{ab}dx^a dx^b = g_{11}(x^1)(dx^1)^2 + \frac{1}{g_{11}(x^1)} a_{\mu\nu} dx^\mu dx^\nu, \tag{45}$$

where $\mu,\nu = 2,\ldots,q$, $a_{\mu\nu}$ are components of a metric of a $(q-1)$-dimensional space.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We will also use the fact that in this coordinate system $k = x^1$ and $\nabla_a k = \delta^1_a$. Consequently, $f = d\exp(x^1)$, d being constant. Using (31), we have $g_{11} = c\exp(x^1)$, where c = const. The equation (41) yields

$$S_{\mu\nu} = qP \frac{1}{g_{11}} a_{\mu\nu}.$$

On the other hand, it is easy to see (cf. [6]) that for the metric (45) the following equality holds

$$S_{\mu\nu} = S^*_{\mu\nu} + La_{\mu\nu},$$

where S^* is the Ricci tensor of the metric a and

$$L = \frac{1}{2}\left(\partial_{11}l + \frac{q-3}{2l}(\partial_1 l)^2\right)g^{11}, \quad l = \frac{1}{g_{11}}.$$

Comparing these formulas, we obtain

$$S^*_{\mu\nu} = a_{\mu\nu}\left[\frac{qP}{g_{11}} - L\right].$$

Consequently, $\frac{qP}{g_{11}} - L = \text{const}$. On the other hand, an elementary calculation shows that in our coordinate system this expression is equal to $\exp(-2x^1)/(4c^2)$, a contradiction. Thus our hypothesis $P \neq 0$ is false.

(R e m a r k . We cannot use Kručkovič's theorem for P = 0, because when p = 0, then the vector field gradk must be non-null).

So we have P = 0 and, by (38), (38′) and (39),

(46) $$P = Q = \mathrm{tr}(T) = \mathrm{tr}(T') = Q' = P' = 0.$$

In particular,

(47) $$\Delta_1 f = 0, \quad \Delta'_1 \varphi = 0.$$

This implies that $q > 1$, $s > 1$, the metrics g and g′ cannot be definite, the vector fields grad(lnf) and grad(lnφ) are parallel and null. Moreover, (43) and corresponding formula for g′ yield

$$\nabla_b S_{ac} = 0 \quad \text{and} \quad \nabla'_\beta S'_{\alpha\gamma} = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now (41), in view of (46), takes the form

$$S_{ac} = \frac{q-2}{4f^2} f_a f_c. \tag{48}$$

Similarly, we have

$$S'_{\alpha\gamma} = \frac{s-2}{4\varphi^2} \varphi_\alpha \varphi_\gamma. \tag{48'}$$

This, in view of (47), immediately implies $K = 0$ and $K' = 0$. Taking into account this fact, (46) and (3), we get $\bar{K} = 0$. From (48) it also follows that if $q = 2$ then M is flat. Thus we have proved the following

T h e o r e m 1. Assume that an n-dimensional ($n \geqslant 3$) doubly warped product $\bar{M} = M \times_f M'$ is Ricci-symmetric. Then $\dim M > 1$, $\dim M' > 1$ and the manifolds M and M' are also Ricci-symmetric. Moreover, scalar curvatures of $\bar{M}$, M and M' are equal to zero.

T h e o r e m 2. A doubly warped product $M \times_f M'$ is Ricci-symmetric if and only if

(i) the vector fields grad(lnf) and grad(lnφ) are parallel and null,

(ii) the relations (48) and (48') hold.

P r o o f . We must only show that these conditions are sufficient. The condition (i) implies $P = 0$ and $\nabla_b f_a = \frac{1}{f} f_b f_a$. This yields $T_{ab} = -\frac{f_a f_b}{f^2}$, $\operatorname{tr}(T) = 0$, $Q = 0$ and $\nabla_c T_{ab} = 0$. In virtue of (i) and (ii), we immediately obtain $\nabla_b S_{ac} = 0$. Corresponding equations we also have for M'. Substituting these equalities into relations (11)-(13'), we easily obtain $\bar{\nabla}\bar{S} = 0$. This completes the proof.

4. Locally symmetric doubly warped products

In this section we consider essentially doubly warped products which are locally symmetric, i.e., satisfy the condition

$$\bar{\nabla}\bar{R} = 0. \tag{49}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Clearly, this condition implies $\bar{\nabla}\bar{S} = 0$ and we can make use of the results contained in Section 3. In particular, by (47), (6) and (6′), we obtain $\nabla R = 0$ and $\nabla' R' = 0$. Thus, in virtue of Theorem 1, we have

T h e o r e m 3. If a doubly warped product $\bar{M} = M \underset{\varphi}{\times}_{f} M'$ $(\dim \bar{M} \geqslant 3)$ is locally symmetric then M and M′ are also locally symmetric. Moreover, $\dim M > 1$, $\dim M' > 1$ and scalar curvatures of $\bar{M}$, M and M′ are equal to zero.

T h e o r e m 4. Every locally symmetric doubly warped product is conformally flat.

P r o o f . Let $\bar{M} = M \underset{\varphi}{\times}_{f} M'$ satisfy our assumptions. The relation (7), by (46) and (49), takes the form

$$R_{abcd} = \frac{1}{4f^2}\left(f_a f_d g_{bc} + f_b f_c g_{ad} - f_a f_c g_{bd} - f_b f_d g_{ac}\right). \tag{50}$$

Similarly, the equation (40) can now be written as

$$T_{ac} = -\frac{f_a f_c}{4f^2}.$$

Using (4), (48), (46), the above equality and $\bar{K} = 0$, we obtain $\bar{C}_{abcd} = 0$. In the same way we can get $\bar{C}_{\alpha\beta\gamma\delta} = 0$. The assertion follows now from Lemma 1. This completes the proof.

T h e o r e m 5. (i) Let $\bar{M} = M \underset{\varphi}{\times}_{f} M'$ be a locally symmetric doubly warped product. Then there exists a coordinate system $x^1,\ldots,x^n$ on a neighbourhood of an arbitrary point $x \in \bar{M}$ such that the metric g of M takes the form

$$g_{ab}dx^a dx^b = A(dx^1)^2 + k_{\mu\nu}dx^\mu dx^\nu + 2dx^1 dx^q, \tag{51}$$

$$A = k_{\mu\nu}x^\mu x^\nu + p_\mu x^\mu,$$

where the indices μ,ν run over the set $\{2,\ldots,q-1\}$ (empty for $q = 2$), $[k_{\mu\nu}]$ is a symmetric and non-singular matrix and p_μ are functions of x^1 only. The metric g′ of M′ takes the corresponding form and

$$f = c\exp(2x^1), \quad \varphi = c'\exp(2x^{q+1}), \tag{52}$$

where c and c′ are arbitrary constants.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Let M and M′ denote the Euclidean q-space ($q \geqslant 2$) and s-space ($s \geqslant 2$) endowed with the metric (51) and the metric of the same type, respectively. If f and φ satisfy (52), then the doubly warped product $M_\varphi \times_f M'$ is locally symmetric.

p r o o f . We study the first factor M (considerations for M′ would be the same). By Theorem 2, the vector field u with components $u_a = \frac{1}{2f} f_a$ is parallel and null. The relation (50) can be rewritten in the form

$$R_{abcd} = u_a u_d g_{bc} + u_b u_c g_{ad} - u_a u_c g_{bd} - u_b u_d g_{ac}, \tag{53}$$

which immediately yields

$$u_e R_{abcd} + u_c R_{abde} + u_d R_{abec} = 0. \tag{54}$$

This, together with Theorem 3, denotes that M is a K^*-space in the sense of Walker [9] (V is a K^*-space if it is either recurrent or locally symmetric and the equality (54) holds for some non-zero vector field). Thus, for $q > 2$, coordinates can be chosen so that the metric g takes the form

$$g_{ab}dx^a dx^b = A(dx^1)^2 + k_{\mu\nu}dx^\mu dx^\nu + 2dx^1 dx^q,$$

$$A = a_{\mu\nu}x^\mu x^\nu + p_\mu x^\mu,$$

where $\mu, \nu = 2, \ldots, q-1$, $[a_{\mu\nu}]$ and $[k_{\mu\nu}]$ are non-zero symmetric matrices, $|k_{\mu\nu}| \neq 0$, p_μ are functions of x^1 only.

Moreover, in this coordinate system the vector field u satisfying (54) has components $u_a = \delta^1_a$. Consequently, $f = c \exp(2x^1)$, c being constant. The only components of R not identically zero are those related to

$$R_{1\mu\nu 1} = a_{\mu\nu}.$$

On the other hand, the equality (53) yields $R_{1\mu\nu 1} = k_{\mu\nu}$. Hence we have $a_{\mu\nu} = k_{\mu\nu}$. Since for $q = 2$ the metric g is flat and has index one, we see that in this case it is also of the above form with $A = 0$ and μ, ν running over the empty set.

Conversely, suppose that the assumptions of (ii) hold. It is obvious that the vector fields grad(ln$f$) and grad(lnφ)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

are parallel and null. Hence $\Delta_1 f = 0 = P$, $\Delta_1' \varphi = 0 = P'$, $T_{ab} + \frac{f_a f_b}{4f^2} = 0$, $T'_{\alpha\beta} + \frac{\varphi_\alpha \varphi_\beta}{4\varphi^2} = 0$. The relation (50) and corresponding relation for M' are satisfied. Substituting all these equalities to the formulas (6)-(10'), we easily obtain (49). This completes the proof.

5. Conformally symmetric doubly warped products

T h e o r e m 6. Every conformally symmetric doubly warped product is conformally flat.

P r o o f . Let $\bar{M} = M \underset{\varphi}{\times} \underset{f}{M'}$ be a conformally symmetric doubly warped product.

If $\bar{\nabla}\bar{R} = 0$ everywhere, then, by Theorem 4, we have $\bar{C} = 0$. Consider thus the case $\bar{\nabla}\bar{R} \neq 0$ and suppose that $\bar{C} \neq 0$. Then $\bar{M}$ is an essentially conformally symmetric manifold [3]. In the sequel we need the following properties of essentially conformally symmetric manifolds (see [3], [4]):

for each $x \in \bar{M}$ there exists a neighbourhood W and a vector field w such that $w(y) \neq 0$, $y \in W$ and

$$w_l \bar{C}_{hijk} + w_j \bar{C}_{hikl} + w_k \bar{C}_{hilj} = 0, \tag{55}$$

$$\bar{S}_{ml} \bar{C}_{hijk} + \bar{S}_{mj} \bar{C}_{hikl} + \bar{S}_{mk} \bar{C}_{hilj} = 0, \tag{56}$$

$$F \bar{C}_{hijk} = \bar{S}_{hk} \bar{S}_{ij} - \bar{S}_{hj} \bar{S}_{ik} \text{ for some function } F \text{ on } \bar{M}, \tag{57}$$

$$\bar{S}_{ir} \bar{C}^r{}_{jkl} = 0. \tag{58}$$

The condition (55) implies

$$w_\alpha \bar{C}_{abcd} = 0 \quad \text{and} \quad w_a \bar{C}_{\alpha\beta\gamma\delta} = 0.$$

Since $\bar{C}_{abcd} = 0$ and $\bar{C}_{\alpha\beta\gamma\delta} = 0$, by Lemma 1, imply $\bar{C} = 0$, we can assume that $\bar{C}_{\alpha\beta\gamma\delta} \neq 0$. Thus $w_a = 0$, $w_\alpha \neq 0$ and $\bar{C}_{abcd} = 0$. Henceforth we restrict our considerations to some open subset U of $\bar{M}$ such that $f_a \neq 0$, $\varphi_\alpha \neq 0$, $\bar{C}_{\alpha\beta\gamma\delta} \neq 0$, $\bar{C}_{abcd} = 0$ on U. The last equality, by Lemma 1, implies

$$S_{ad} + (q-2)T_{ad} = g_{ad}(K + (q-2)\mathrm{tr}(T))/q. \tag{59}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Using (56), we have

$$\bar{S}_{ab}\bar{C}_{\alpha\beta\gamma\delta} = 0,$$

which yields $\bar{S}_{ab} = 0$. Applying this relation to (57), in view of (2), we obtain

$$F\,\bar{C}_{a\beta c\delta} = \bar{S}_{a\delta}\bar{S}_{c\beta} = \left(\frac{n-2}{4f\varphi}\right)^2 f_a f_c \varphi_\beta \varphi_\delta \neq 0.$$

Hence, in particular

$$\bar{C}_{a\beta c\delta} \neq 0. \tag{60}$$

Using again (56), we get

$$\bar{S}_{\alpha b}\bar{C}_{c\delta e\gamma} = \bar{S}_{\alpha e}\bar{C}_{c\delta b\gamma},$$

which implies

$$f_b\bar{C}_{c\delta e\gamma} = f_e\bar{C}_{c\delta b\gamma}. \tag{61}$$

The relation (58) yields $f^d\bar{C}_{d\beta c\delta} = 0$ which, in virtue of (60) and by transvection of (61) with f^b, implies

$$\Delta_1 f = 0. \tag{62}$$

Substituting (59) into (5), we obtain

$$\bar{C}_{a\beta c\delta} = g_{ac}\Big[\varphi(S'_{\beta\delta} + (s-2)T'_{\beta\delta} + \frac{f}{q}\,(K + (q-2)\mathrm{tr}(T))g'_{\beta\delta} + \\ + (Q + Q' - \frac{f_\varphi \bar{K}}{n-1})\,g'_{\beta\delta}\Big]\Big/(n-2),$$

thus $\bar{C}_{a\beta c\delta} = g_{ac}H_{\beta\delta}$. This turns (61) into

$$H_{\delta\gamma}(f_b g_{ec} - f_e g_{bc}) = 0.$$

If $H_{\delta\gamma}$ did not vanish identically, then we would have

$$f_b g_{ec} - f_e g_{bc} = 0$$

and, after transvection with f^e, in view of (62), $f_b = 0$, a contradiction. Thus we obtain $H_{\delta\gamma} = 0$ which implies $\bar{C}_{a\beta c\delta} = 0$, a contradiction. This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] J.K. Beem, T.G. Powell: Geodesic completness and maximality in Lorentzian warped products, Tensor, N.S., 39 (1982), 31-36.

[2] R.L. Bishop, B.O'Neill: Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969) 1-49.

[3] A. Derdziński, W. Roter: Some theorems on conformally symmetric manifolds, Tensor, N.S., 32 (1978), 11-23.

[4] A. Derdziński, W. Roter: Some properties of conformally symmetric manifolds which are not Ricci-recurrent, Tensor, N.S., 34 (1980), 11-20.

[5] P.E. Ehrlich: Metric deformations of Ricci and sectional curvature on compact Riemannian manifolds, Ph.D. dissertation, SUNY, Stony Brook, New York, 1974.

[6] A. Krawczyk: Some theorems on semi-decomposable conformally symmetric spaces, Prace Nauk. Inst. Mat. Politech. Wrocław., 16 (1976), 3-10.

[7] G.I. Kručkovič: On semi-reducible Riemannian spaces (in Russian), Dokl. Akad. Nauk SSSR, 115 (1957), 862-865.

[8] G.I. Kručkovič: On some class of Riemannian spaces (in Russian), Trudy Sem. Vekt. Tens. Anal., 11 (1961), 103-128.

[9] A.G. Walker: On Ruse's spaces of recurrent curvature, Proc. Lond. Math. Soc., 52 (1950), 36-64.

[10] Y.C. Wong: Some Einstein spaces with conformally separable fundamental tensors, Trans. Amer. Math. Soc., 53 (1943), 157-194.

[11] K. Yano: Conformally separable quadratic differential forms, Proc. Imp. Acad. Tokyo, 16 (1940), 83-86.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY IN WROCŁAW,
50-376 WROCŁAW, POLAND
Received October 14, 1987

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Mieczysław Król

SOME PROPERTIES OF A QUASI-DIFFUSION VECTOR-VALUED PROCESS

130274

Let $X_t : \Omega \times T \longrightarrow R^m$ be a m-dimensional stochastic process, where $\langle \Omega, \mathcal{F}, P \rangle$ is a probability space and $t \in [t_0, T_0] \subset R$. Further, let $t_0 \leqslant t_1 < \ldots < t_n < t < \tau < T_0$, $\vec{t}_n = \langle t_1, \ldots, t_n \rangle$, $\vec{x}_n = \langle x_1, \ldots, x_n \rangle \in R^{nm}$, where $x_k = \langle x_k^1, \ldots, x_k^m \rangle \in R^m$ for $k = 1, \ldots, n$, $x = \langle x^1, \ldots, x^m \rangle \in R^m$, $y = \langle y^1, \ldots, y^m \rangle \in R^m$ and $A, B \in \mathcal{B}^m$, where $\mathcal{B}^m$ is a σ-algebra of Borel sets in R^m.

In papers [1], [3], [4] there are given definitions and some properties of a quasi-diffusion process in the one-dimensional case, and in [2] the m-dimensional generalization of these processes. In this paper, using results of [2], we describe some properties of a quasi-diffusion vector-valued process, connected with its conditional moments and so we generalize the results of [1].

Let $P_{\vec{t}_n, \vec{x}_n}(t, x; \tau, A)$ be a probability of the incident that the process X_t is in the state belonging to the Borel set A in the moment τ, while it was in $x_1, \ldots, x_n, x$ states in earlier moments $t_1, \ldots, t_n, t$. It means that in general we have

$$P_{\vec{t}_n, X_{t_1}, \ldots, X_{t_n}}(t, X_t; \tau, A) = P(X_\tau \in A \,|\, X_{t_1}, \ldots, X_{t_n}, X_t)$$

(with probability to 1).

Further, let $V_\varepsilon(x) = \{ y \in R^m ; |y - x| < \varepsilon \}$, $V'_\varepsilon(x) = R^m - V_\varepsilon(x)$ and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\int_{V_\varepsilon(x)} (y-x) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) =$$

$$= \Big\langle \int_{V_\varepsilon(x)} (y^1-x^1) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy), \ldots, \int_{V_\varepsilon(x)} (y^m-x^m) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) \Big\rangle,$$

$$\int_{V_\varepsilon(x)} (y-x,z)^2 P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{V_\varepsilon(x)} (y^i-x^i)(y^j-x^j) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$$

for every $z = \langle z^1,\ldots,z^m\rangle \in R^m$.

The m-dimensional process X_t with values in R^m is called the quasi-diffusion, if for $n \geqslant 1$ and arbitrarily chosen $\varepsilon > 0, \vec{t}_n, t, \tau, \vec{x}_n, x$ there exist a vector-valued function

$$A_{\vec{t}_n,\vec{x}_n}(t,x) = \big\langle A^1_{\vec{t}_n,\vec{x}_n}(t,x),\ldots,A^m_{\vec{t}_n,\vec{x}_n}(t,x)\big\rangle$$

and positively defined matrix

$$B_{\vec{t}_n,\vec{x}_n}(t,x) = \Big[b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) \Big]_{i,j=1,\ldots,m}$$

such that the following conditions are fulfilled ([2], [4])

(1) $$P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,V'_\varepsilon(x)) = o(\tau-t),$$

(2) $$\int_{V_\varepsilon(x)} (y-x) P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) = A_{\vec{t}_n,\vec{x}_n}(t,x)(\tau-t) + o(\tau-t),$$

(3) $$\int_{V_\varepsilon(x)} (y-x,z)^2 P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy) = (B_{\vec{t}_n,\vec{x}_n}(t,x)z,z)(\tau-t) + o(\tau-t).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $(\cdot,\cdot)$ be the scalar product. For example, the following scalar products

$$(B_{\vec{t}_n,\vec{x}_n}(t,x)\nabla,\nabla) = \sum_{i=1}^{m}\sum_{j=1}^{m} b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\,\frac{\partial^2}{\partial x^i \partial x^j},$$

$$(A_{\vec{t}_n,\vec{x}_n}(t,x),\nabla) = \sum_{i=1}^{m} A^{i}_{\vec{t}_n,\vec{x}_n}(t,x)\,\frac{\partial}{\partial x^i},$$

define differential operators.

Obviously, for every $i,j = 1,\ldots,m$, we have

$$A^{i}_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{\tau\to t}\frac{1}{\tau-t}\int_{V_\varepsilon(x)}(y^i-x^i)P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),$$

$$b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{\tau\to t}\frac{1}{\tau-t}\int_{V_\varepsilon(x)}(y^i-x^i)(y^j-x^j)P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy).$$

T h e o r e m 1. If the function $g:R^m \to R$ is from the class φ^2 and is integrable with respect to the probability distribution $P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$, conditions (1)-(3) are fulfilled and $A_{\vec{t}_n,\vec{x}_n}(t,x)$, $B_{\vec{t}_n,\vec{x}_n}(t,x)$ are continuous, then

$$\lim_{\substack{s\nearrow t\\ \tau\searrow t}}\frac{1}{\tau-s}\left[\int_{R^m} g(y)P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - g(x)\right] =$$

$$= (A_{\vec{t}_n,\vec{x}_n}(t,x),\nabla)g(x) + \frac{1}{2}(B_{\vec{t}_n,\vec{x}_n}(t,x)\nabla,\nabla)g(x).$$

P r o o f . Expanding the function $g(y)$ into the Taylor power series in the neighbourhood $V_\varepsilon(x)$ of the point $x\in R^m$, we have

$$g(y) = g(x) + (y-x,\nabla)g(x) + \frac{1}{2}(y-x,\nabla)^2 g(x) + r(y,x), \tag{1.1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where

$$(1.2)\quad |r(y,x)| \leqslant \alpha_\varepsilon |y-x|^2, \quad \alpha_\varepsilon = \sup_{\substack{1\leqslant i,j\leqslant m \\ y\in V_\varepsilon(x)}} \left| \frac{\partial^2 g(x)}{\partial x^i \partial x^j} - \frac{\partial^2 g(y)}{\partial y^i \partial y^j} \right|,$$

$$(1.3)\quad \lim_{\varepsilon \to 0} \alpha_\varepsilon = 0.$$

Let $\{U_k\}_{k=1}^{\infty}$ be any family of open and bounded sets such that $U_1 \subset U_2 \subset \ldots \subset U_k \subset \ldots$, $R^m = \bigcup_{k=1}^{\infty} U_k$, $V_\varepsilon(x) \subset U_k$ for every $k = 1,2,\ldots$. The closure $G_k = \overline{U}_k$ is a compact set in R^m. For $k = 1,2,\ldots$ we define the function $g_k(y) = g(y)$ for $y \in G_k$ and $g_k(y) = 0$ for $y \in R^m - G_k$. Obviously, for every k the function $g_k(y)$ is bounded. Therefore, for $k = 1,2,\ldots$ with $t_0 \leqslant t_1 < \ldots < t_n < s \leqslant t \leqslant \tau$, we have

$$\frac{1}{\tau - s}\left[\int_{G_k} g(y) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - g(x)\right] =$$

$$= \frac{1}{\tau - s}\left[\int_{V_\varepsilon(x)} g(y) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - g(x)\right] + \frac{o(\tau - s)}{\tau - s}.$$

Using (1.1), we get

$$(1.4)\quad \frac{1}{\tau - s}\left[\int_{G_k} g(y) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - g(x)\right] =$$

$$= \frac{1}{\tau - s}\left[\int_{V_\varepsilon(x)} (y-x,\nabla) g(x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) + \right.$$

$$+ \frac{1}{2}\int_{V_\varepsilon(x)} (y-x,\nabla)^2 g(x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) -$$

$$\left. - g(x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,V_\varepsilon'(x))\right] +$$

$$+ \frac{1}{\tau - s}\int_{V_\varepsilon(x)} r(y,x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) + \frac{o(\tau - s)}{\tau - s}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since, from (1.2), we have the equality

$$\left| \int_{V_\varepsilon(x)} r(y,x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) \right| \leqslant \alpha_\varepsilon \int_{V_\varepsilon(x)} |y-x|^2 P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy)$$

which, by (3), implies

$$\left| \lim_{\substack{s\nearrow t \\ \tau\searrow t}} \frac{1}{\tau-s} \int_{V_\varepsilon(x)} r(y,x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) \right| \leqslant$$

$$\leqslant \alpha_\varepsilon \left(B_{\vec{t}_n,\vec{x}_n}(t,x) \frac{y-x}{|y-x|}, \frac{y-x}{|y-x|} \right),$$

and, as ε is arbitrarily chosen, so from (1.3) we have

$$\int_{V_\varepsilon(x)} r(y,x) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) = o(\tau-s). \tag{1.5}$$

Using now the relations (1)-(3) and (1.5), from (1.4) we find

$$\lim_{\substack{s\nearrow t \\ \tau\searrow t}} \frac{1}{\tau-s} \left[\int_{G_k} g(y) P_{\vec{t}_n,\vec{x}_n}(s,x;\tau,dy) - g(x) \right] = \tag{1.6}$$

$$= (A_{\vec{t}_n,\vec{x}_n}(t,x),\nabla)g(x) + \frac{1}{2}(B_{\vec{t}_n,\vec{x}_n}(t,x)\nabla,\nabla)g(x) =$$

$$= \sum_{i=1}^{m} A^i_{\vec{t}_n,\vec{x}_n}(t,x) \frac{\partial g(x)}{\partial x^i} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) \frac{\partial^2 g(x)}{\partial x^i \partial x^j}.$$

Since k is arbitrary and $R^m \subset \bigcup_{k=1}^{\infty} G_k$, we obtain from (1.6) the thesis of our theorem.

The conditional moments for the process X_t are denoted by

$$\mu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i) = \int_{R^m} y^i P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),$$

$$\nu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i,j) = \int_{R^m} y^i y^j P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\varkappa_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i,j) =$$

$$= \int_{R^m} (y^i - \mu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i))(y^j - \mu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,j))P_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy),$$

for $i,j=1,\ldots,m$.

$$\varkappa_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i,i) = \sigma^2_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i).$$

Moreover, let

$$E_{\vec{t}_n,\vec{x}_n}(t,x;\tau) = \langle \mu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,1),\ldots,\mu_{\vec{t}_n,\vec{x}_n}(t,x;\tau,m)\rangle =$$

$$= \int_{R^m} yP_{\vec{t}_n,\vec{x}_n}(t,x;\tau,dy)$$

be a vector of a conditional expectation value, and let

$$K_{t_n,x_n}(t,x;\tau) = \left[\varkappa_{\vec{t}_n,\vec{x}_n}(t,x;\tau,i,j)\right]_{i,j=1,\ldots,m}$$

be a matrix of conditional covariances of vector-valued process X_t. Infinitesimal moments depend on conditional moments. In fact, we have the following result.

T h e o r e m 2. If there exist conditional moments for the process X_t and $A_{\vec{t}_n,\vec{x}_n}(t,x)$, $B_{\vec{t}_n,\vec{x}_n}(t,x)$ are continuous, then

$$(4)\qquad A_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{h\to 0}\frac{1}{h}\left[E_{\vec{t}_n,\vec{x}_n}(t,x;t+h) - x\right],$$

$$(5)\qquad B_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{h\to 0}\frac{1}{h}K_{\vec{t}_n,\vec{x}_n}(t,x;t+h)$$

and, in particular,

$$(6)\qquad b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) = \lim_{h\to 0}\frac{1}{h}\left[\nu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) - \right.$$

$$\left. - x^i\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,j) - x^j\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i) + x^ix^j\right],$$

$i,j=1,\ldots,m.$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . For the functions $g_i(y) = y^i$, $g_{i,j}(y) = y^i y^j$, $i,j = 1,\dots,m$, the conditions of Theorem 1 are fulfilled. Therefore, if we put $t_0 \leqslant t_1 < \dots < t_n < s = t < \tau = t+h$ for $h > 0$ (or $t_0 \leqslant t_1 < \dots < t_n < s = t+h < \tau = t$ for $h < 0$), then in the case of $h > 0$, we have

$$(2.1)\quad \lim_{h\to 0} \frac{1}{h}\left[\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i) - x^i\right] = \sum_{j=1}^{m} A^j_{\vec{t}_n,\vec{x}_n}(t,x)\,\frac{\partial x^i}{\partial x^j} +$$

$$+ \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} b^{j,k}_{\vec{t}_n,\vec{x}_n}(t,x)\,\frac{\partial^2 x^i}{\partial x^j \partial x^k} = A^i_{\vec{t}_n,\vec{x}_n}(t,x), \quad i=1,\dots,m,$$

which proves (4). Similarly

$$(2.2)\qquad \lim_{h\to 0} \frac{1}{h}\left[\vartheta_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) - x^i x^j\right] =$$

$$= x^i A^j_{\vec{t}_n,\vec{x}_n}(t,x) + x^j A^i_{\vec{t}_n,\vec{x}_n}(t,x) + b^{i,j}_{\vec{t}_n,\vec{x}_n},$$

since $b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x) = b^{j,i}_{\vec{t}_n,\vec{x}_n}(t,x)$ for $i,j = 1,\dots,m$. The relations (2.1), (2.2) imply (6). Besides from (2.1), (2.2) we have

$$\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i) = x^i + A^i_{\vec{t}_n,\vec{x}_n}(t,x)h + o(h),$$

$$\vartheta_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) = x^i x^j + \left[x^i A^j_{\vec{t}_n,\vec{x}_n}(t,x) + \right.$$

$$\left. + x^j A^i_{\vec{t}_n,\vec{x}_n}(t,x) + b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)\right] h + o(h)$$

and, moreover,

$$\chi_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) = \vartheta_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) -$$

$$-\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i)\mu_{\vec{t}_n,\vec{x}_n}(t,x;t+h,j),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

so we obtain the relation

$$\varkappa_{\vec{t}_n,\vec{x}_n}(t,x;t+h,i,j) = b^{i,j}_{\vec{t}_n,\vec{x}_n}(t,x)h + o(h), \qquad i,j=1,\ldots,m,$$

which justifies (4)-(6). In the case of $h<0$, the proof is analogical.

If probability distributions of the quasi-diffusion vector-valued process X_t are absolutely continuous with respect to the Lebesque measure, i.e. if for $t_0 \leqslant t_1 < \ldots < t_n < t$ we have

$$P\{X_{t_1} \in A_1,\ldots,X_{t_n} \in A_n, X_t \in A\} = \int\limits_{A_1\times\ldots\times A} f(\vec{t}_n,\vec{x}_n,t,x)\,d\vec{x}_n dx,$$

$$P_{\vec{t}_{n-1},\vec{x}_{n-1}}(t_n,x_n;t,A) = \int\limits_A f_{\vec{t}_n,\vec{x}_n}(t,x)\,dx,$$

where $A_1,\ldots,A_n,A \in \mathfrak{B}^m$, then the knowledge of $A_{\vec{t}_n,\vec{x}_n}(t,x)$, $B_{\vec{t}_n,\vec{x}_n}(t,x)$ and $f(t_0,x_0)$ suffices to finding the function $f_{\vec{t}_n,\vec{x}_n}(t,x)$; so it suffices to finding the function $f(\vec{t}_n,\vec{x}_n,t,x)$, since $f(\vec{t}_n,\vec{x}_n,t,x) = f(\vec{t}_{n-1},\vec{x}_{n-1},t_n,x_n) f_{\vec{t}_n,\vec{x}_n}(t,x)$ for any n. For that, we will simply use results contained in [2].

Theorem 2 gives the possibility of finding $A_{\vec{t}_n,\vec{x}_n}(t,x)$ and $B_{\vec{t}_n,\vec{x}_n}(t,x)$ basing on the knowledge of the density function $f_{\vec{t}_n,\vec{x}_n}(t,x)$ (cf. [1]).

REFERENCES

[1] A. K o s y k : Some properties of diffusion processes in the wide sense, Demonstratio Math. 7 (1974) 495-505.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] M. K r ó l : On some retrospective and prospective equations for a vector-valued quasi-diffusion process, Demonstratio Math. 22 (1989) 903-916.

[3] A. P l u c i ń s k a : Some transformations of multi-dimensional quasi-diffusion process, Rev. Roum. Math. Pure and Appl. 23 (1978)

[4] A. P l u c i ń s k a : Remarks on prospective equations, Teor. Wierojatn. i Prim. 25 (1980) 350-358.

INSTITUT OF MATHEMATICS AND PHYSICS, TECHNICAL UNIVERSITY IN RZESZÓW, 35-959 RZESZÓW, POLAND

Received January 11, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Stanisław Midura

SUR LES SOLUTIONS DES ÉQUATIONS FONCTIONNELLES QUI DÉTERMINENT CERTAINS SOUS-DEMI-GROUPES À DEUX PARAMÈTRES DU GROUPE L_4^1

Soit $R_0 = R\setminus\{0\}$, où R est l'ensemble des nombres réels. Dans $R_0 \times R^3$ introduisons l'opération suivante:

$$(1.1)\quad \langle \beta_1,\beta_2,\beta_3,\beta_4\rangle\cdot\langle \alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle = \langle \beta_1\alpha_1, \beta_1\alpha_2+\beta_2\alpha_1^2,\ \beta_1\alpha_3 + 3\beta_2\alpha_2\alpha_1 + \beta_3\alpha_1^3,\ \beta_1\alpha_4 + 4\beta_2\alpha_1\alpha_3 + 6\beta_3\alpha_1^2\alpha_2 + 3\beta_2\alpha_2^2 + \beta_4\alpha_1^4\rangle.$$

L'ensemble $R_0 \times R^3$ avec l'opération (1.1) est un groupe que nous désignerons par L_4^1 (v. [1] p.23-24 ou [3]). Le groupe L_4^1 est à quatre paramètres.

Dans ce travail nous nous proposons de déterminer les sous-demi-groupes du groupe L_4^1 dans lesquels deux paramètres sont des fonctions des deux autres. Dans [2]-[7] et [10] on a déterminé des sous-demi-groupes dans lesquels l'un des paramètres est une fonction des autres.

1. Soient des fonctions $f_2, g_4: R_0 \times R \longrightarrow R$ et l'ensemble

$$(1.2)\quad P_{24} = \{\langle \alpha_1, f_2(\alpha_1,\alpha_3), \alpha_3, g_4(\alpha_1,\alpha_3)\rangle : \alpha_1 \in R_0,\ \alpha_3 \in R\}$$

qui est fermé par rapport à l'opération (1.1) si et seulement si pour β_1, β_3 et α_1, α_3 quelconques on a

$$\langle \beta_1, f_2(\beta_1,\beta_3), \beta_3, g_4(\beta_1,\beta_3)\rangle\cdot\langle \alpha_1, f_2(\alpha_1,\alpha_3), \alpha_3, g_4(\alpha_1,\alpha_3)\rangle \in P_{24};$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

il s'ensuit

$$\langle \beta_1\alpha_1, \beta_1 f_2(\alpha_1,\alpha_3) + f_2(\beta_1,\beta_3)\alpha_1^2,\ \beta_1\alpha_3 + 3f_2(\beta_1,\beta_3) f_2(\alpha_1,\alpha_3)\alpha_1 +$$

$$+ \beta_3\alpha_1^3,\ \beta_1 g_4(\alpha_1,\alpha_3) + 4f_2(\beta_1,\beta_3)\alpha_1\alpha_3 + 6\beta_3\alpha_1^2 f_2(\alpha_1,\alpha_3) +$$

$$+ 3f_2^2(\alpha_1,\alpha_3) f_2(\beta_1,\beta_3) + g_4(\beta_1,\beta_3)\alpha_1^4 \rangle \in P_{24}.$$

Donc, vu (1.2), les fonctions f_2, g_4 satisfont au système d'équations fonctionnelles

$$(1.3)\qquad f_2\left[\beta_1\alpha_1,\ \beta_1\alpha_3 + 3\alpha_1 f_2(\beta_1,\beta_3)\, f_2(\alpha_1,\alpha_3) + \beta_3\alpha_1^3\right] =$$

$$= \beta_1 f_2(\alpha_1,\alpha_3) + \alpha_1^2 f_2(\beta_1,\beta_3),$$

$$(1.4)\qquad g_4\left[\beta_1\alpha_1,\ \beta_1\alpha_3 + 3\alpha_1 f_2(\beta_1,\beta_3)\, f_2(\alpha_1,\alpha_3) + \beta_3\alpha_1^3\right] =$$

$$= \beta_1 g_4(\alpha_1,\alpha_3) + \alpha_1^4 g_4(\beta_1,\beta_3) + 4\alpha_1\alpha_3 f_2(\beta_1,\beta_3) +$$

$$+ 6\beta_3\alpha_1^2 f_2(\alpha_1,\alpha_3) + 3f_2^2(\alpha_1,\alpha_3) f_2(\beta_1,\beta_3).$$

Le problème consistant à déterminer tous les sous-demi-groupes du groupe L_4^1 de la forme (1.2) se trouve ainsi ramené à résoudre le système (1.3), (1.4). On a montré dans [7] que la solution générale de (1.3) est une famille de fonctions

$$(1.5)\qquad f_2(\beta_1,\beta_3) = p(\beta_1^2 - \beta_1),$$

où p est un nombre réel quelconque. L'équation (1.4) prend donc la forme

$$(1.6)\qquad g_4\left[\beta_1\alpha_1,\ \beta_1\alpha_3 + 3p^2(\beta_1^2-\beta_1)(\alpha_1^2-\alpha_1)\alpha_1 + \beta_3\alpha_1^3\right] =$$

$$= g_4(\alpha_1,\alpha_3)\beta_1 + g_4(\beta_1,\beta_3)\alpha_1^4 + 4p(\beta_1^2-\beta_1)\alpha_1\alpha_3 +$$

$$+ 6\beta_3\alpha_1^2\, p(\alpha_1^2-\alpha_1) + 3p^3(\alpha_1^2-\alpha_1)^2(\beta_1^2-\beta_1).$$

Si on pose $\alpha_1 = \beta_1 = 1$ dans (1.6), on obtient $g_4(1, \alpha_3+\beta_3) = g_4(1,\alpha_3) + g_4(1,\beta_3)$. La fonction $\alpha_3 \longrightarrow g_4(1,\alpha_3)$ est donc additive. En posant $\alpha_1 = 1$ et $\beta_3 = 0$ dans (1.6), on obtient

$$(1.7)\qquad g_4(\beta_1,\beta_1\alpha_3) = g_4(1,\alpha_3)\beta_1 + g_4(\beta_1,0) + 4p(\beta_1^2-\beta_1)\alpha_3$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

et ensuite pour $\alpha_3 = \frac{\beta}{\beta_1}$ on a

$$(1.8)\qquad g_4(\beta_1,\beta) = g_4\left(1, \frac{\beta}{\beta_1}\right)\beta_1 + g_4(\beta_1,0) + 4p(\beta_1^2-\beta_1)\frac{\beta}{\beta_1}.$$

Posant $\beta_1 = 1$ et $\alpha_3 = 0$ dans (1.6), on obtient

$$(1.9)\qquad g_4(\alpha_1,\beta_3\alpha_1^3) = g_4(\alpha_1,0) + g_4(1,\beta_3)\alpha_1^4 + 6\beta_3\alpha_1^2 p(\alpha_1^2-\alpha_1)$$

et pour $\beta_3 = \frac{\beta}{\alpha_1^3}$ on trouve

$$(1.10)\qquad g_4(\alpha_1,\beta) = g_4(\alpha_1,0) + \alpha_1^4 g_4\left(1, \frac{\beta}{\alpha_1^3}\right) + 6\frac{\beta}{\alpha_1}p(\alpha_1^2-\alpha_1).$$

Les premiers membres de (1.8) et (1.10) sont égaux pour $\beta_1 = \alpha_1$, il s'ensuit donc

$$(1.11)\qquad \alpha_1^4 g_4\left(1, \frac{\beta}{\alpha_1^3}\right) + 6\beta p(\alpha_1-1) = g_4\left(1, \frac{\beta}{\alpha_1}\right)\alpha_1 + 4\beta p(\alpha_1-1).$$

La fonction $g_4(1,\cdot)$ est additive, donc pour α_1 rationnel (1.11) devient

$$g_4(1,\beta)\alpha_1 + 6\beta p(\alpha_1-1) = g_4(1,\beta) + 4\beta p(\alpha_1-1),$$

d'où, fixant arbitrairement $\alpha_1 \neq 1$, on obtient

$$(1.12)\qquad g_4(1,\beta) = -2p\beta.$$

Posant $\beta_1 = -1$ dans (1.7) et tenant compte de (1.12), on obtient

$$(1.13)\qquad g_4(-1,-\alpha_3) = 10p\alpha_3 + g_4(-1,0).$$

Posons maintenant $\beta_1 = \frac{1}{\alpha_1}$ et $\alpha_3 = \beta_3 = 0$ dans (1.6). On a alors

$$g_4(1,-3p^2(\alpha_1-1)^2) = g_4(\alpha_1,0)\frac{1}{\alpha_1} + g_4\left(\frac{1}{\alpha_1},0\right)\alpha_1^4 - 3p^3(\alpha_1-1)^3$$

et, vu (1.12), on trouve

$$(1.14)\qquad g_4\left(\frac{1}{\alpha_1},0\right) = 6p^3(\alpha_1-1)^2\frac{1}{\alpha_1^4} + 3p^3(\alpha_1-1)^3\frac{1}{\alpha_1^4} - g_4(\alpha_1,0)\frac{1}{\alpha_1^5}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

En posant $\alpha_1 = \frac{1}{\beta_1}$ et $\alpha_3 = \beta_3 = 0$ dans (1.6), on obtient

$$g_4\left(1, -3p^2(\beta_1-1)^2\frac{1}{\beta_1^2}\right) = g_4\left(\frac{1}{\beta_1}, 0\right)\beta_1 + g_4(\beta_1,0)\frac{1}{\beta_1^4} - 3p^3(\beta_1-1)^3\frac{1}{\beta_1^3},$$

d'où il vient, compte tenu de (1.12),

$$(1.15)\quad g_4\left(\frac{1}{\beta_1}, 0\right) = 6p^3(\beta_1-1)^2\frac{1}{\beta_1^3} + 3p^3(\beta_1-1)^3\frac{1}{\beta_1^4} - g_4(\beta_1,0)\frac{1}{\beta_1^5}.$$

Les premiers membres de (1.14) et (1.15) sont égaux pour $\beta_1 = \alpha_1$, donc

$$6p^3(\alpha_1-1)^2\frac{1}{\alpha_1^4} = 6p^3(\alpha_1-1)^2\frac{1}{\alpha_1^3},$$

ce qui donne

$$(1.16)\qquad p = 0$$

pour $\alpha_1 \neq 1$. Par conséquent, l'équation (1.6) prend la forme

$$(1.17)\quad g_4(\beta_1\alpha_1, \beta_1\alpha_3 + \beta_3\alpha_1^3) = g_4(\alpha_1,\alpha_3)\beta_1 + g_4(\beta_1,\beta_3)\alpha_1^4.$$

Il résulte des formules (1.12), (1.13), (1.16) que

$$(1.18)\qquad g_4(1,\beta) = 0, \quad g_4(-1,\beta) = g_4(-1,0).$$

Si on pose $\alpha_3 = \beta_3 = 0$ dans (1.17), on obtient

$$(1.19)\qquad g_4(\beta_1\alpha_1, 0) = g_4(\alpha_1,0)\beta_1 + g_4(\alpha_1,0)\alpha_1^4.$$

On a démontré dans [3] (théorème 1) que la solution générale de l'équation (1.19) est une famille de fonctions de la forme

$$(1.20)\qquad g_4(\alpha_1,0) = l(\alpha_1^4 - \alpha_1),$$

l étant un nombre réel quelconque. Si on pose $\alpha_1 = 1$ et $\beta_3 = -\beta_1\alpha_3$ dans (1.17), on obtient

$$g_4(\beta_1,0) = g_4(1,\alpha_3)\beta_1 + g_4(\beta_1, -\beta_1\alpha_3).$$

Vu (1.20) et (1.18), il s'ensuit $l(\beta_1^4 - \beta_1) = g_4(\beta_1, -\beta_1\alpha_3)$ avec $\beta_1 \neq 0$ et α_3 réel quelconque; on obtient

$$(1.21)\qquad g_4(\beta_1,\beta_3) = l(\beta_1^4 - \beta_1).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On vérifie aisément que toute fonction de la forme (1.21), où l est un nombre réel quelconque, satisfait à l'équation (1.6). Nous avons ainsi établi les théorèmes suivants.

T h é o r è m e 1. La solution générale de l'équation (1.6) est un ensemble de fonctions de la forme (1.21), où l est un nombre réel quelconque.

T h é o r è m e 2. La solution générale du système d'équations fonctionnelles (1.3), (1.4) est une famille de couples de fonctions (1.5), (1.21), où p, l sont des nombres réels quelconques.

T h é o r è m e 2 . Les seuls sous-demi-groupes de la forme (1.2) du groupe L_4^1 sont des ensembles de la famille

$$\left\{\left\{\langle \alpha_1,\ p(\alpha_1^2-\alpha_1),\ \alpha_3,\ l(\alpha_1^4-\alpha_1)\rangle : \alpha_1 \in R_0,\ \alpha_3 \in R\right\}: p,l \in R\right\}.$$

2. Soient des fonctions $f_1: R\times R \longrightarrow R_0$, $g_4: R\times R \longrightarrow R$ et l'ensemble

$$P_{14} = \left\{\langle f_1(\alpha_2,\alpha_3),\ \alpha_2,\ \alpha_3,\ g_4(\alpha_2,\alpha_3)\rangle: \alpha_2,\alpha_3 \in R\right\}. \tag{2.1}$$

De même que dans le cas de l'ensemble P_{24} on peut démontrer que P_{14} est un sous-demi-groupe du groupe L_4^1 si et seulement si les fonctions f_1, g_4 satisfont au système d'équations fonctionnelles

$$f_1\left[\alpha_2 f_1(\beta_2,\beta_3) + \beta_2 f_1^2(\alpha_2,\alpha_3),\ \alpha_3 f_1(\beta_2,\beta_3) + \right. \tag{2.2}$$
$$\left. + 3\beta_2\alpha_2\, f_1(\alpha_2,\alpha_3) + \beta_3 f_1^3(\alpha_2,\alpha_3)\right] = f_1(\beta_2,\beta_3)\, f_1(\alpha_2,\alpha_3),$$

$$g_4\left[\alpha_2 f_1(\beta_2,\beta_3) + \beta_2 f_1^2(\alpha_2,\alpha_3),\ \alpha_3 f_1(\beta_2,\beta_3) + 3\beta_2\alpha_2 f_1(\alpha_2,\alpha_3) + \right. \tag{2.3}$$
$$\left. + \beta_3 f_1^3(\alpha_2,\alpha_3)\right] = g_4(\alpha_2,\alpha_3)\, f_1(\beta_2,\beta_3) + g_4(\beta_2,\beta_3) f_1^4(\alpha_2,\alpha_3) +$$
$$+ 4\beta_2\alpha_3 f_1(\alpha_2,\alpha_3) + 6\beta_3\alpha_2 f_1^2(\alpha_2,\alpha_3) + 3\beta_2\alpha_2^2.$$

Nous allons résoudre (2.2), (2.3) dans la classe des fonctions f_1 telles que

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.4) la fonction $f_1(\cdot,0)$ est continue,

(2.5) la fonction $f_1(0,\cdot)$ est continue.

Si on pose $\alpha_2 = \beta_2 = 0$ dans (2.2), on obtient

$$(2.6)\qquad f_1(0,\alpha_3 f_1(0,\beta_3) + \beta_3 f_1^3(0,\alpha_3)) = f_1(0,\beta_3)\, f_1(0,\alpha_3).$$

Avec

$$(2.7)\qquad F(\alpha_3) := f_1(0,\alpha_3),$$

l'égalité (2.6) prend la forme

$$(2.8)\qquad F(\alpha_3 F(\beta_3) + \beta_3 F^3(\alpha_3)) = F(\beta_3)\, F(\alpha_3).$$

On a démontré dans [9] que la solution unique de (2.8), sous l'hypothèses (2.5), est une fonction constante et que $F(\alpha_3)=1$ ou bien, vu (2.7),

$$(2.9)\qquad f_1(0,\alpha_3) = 1.$$

Si on pose $\beta_2 = 0$ et $\alpha_3 = 0$ dans (2.2), on obtient, compte tenu de (2.9),

$$(2.10)\qquad f_1(\alpha_2,\ \beta_3 f_1^3(\alpha_2,0)) = f_1(\alpha_2,0)$$

ce qui entraîne

$$(2.11)\qquad f_1(\alpha_2,\alpha_3) = f_1(\alpha_2,0),\quad \alpha_3 \in R,$$

puisque $f_1(\alpha_2,0) \neq 0$ et β_3 peut être un nombre réel quelconque. Il en résulte que l'équation (2.2) prend la forme

$$(2.12)\qquad f_1(\alpha_2 f_1(\beta_2,0) + \beta_2 f_1^2(\alpha_2,0),\ 0) = f_1(\beta_2,0)\, f_1(\alpha_2,0).$$

En vertu d'un théorème établi dans [9], l'équation (2.12) admet, une seule solution constante $f_1(\alpha_2,0) = 1$ dans la classe des fonctions qui satisfont à l'hypothèse (2.4). Il en résulte, vu (2.11), que

$$(2.13)\qquad f_1(\alpha_2,\alpha_3) = 1$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

pour α_2, α_3 réels quelconques. On voit aisément que (2.13) satisfait à (2.2). Ainsi se trouve démontré le théorème suivant.

T h é o r è m e 3. La solution unique de l'équation (2.2), dans la classe des fonctions qui satisfont aux hypothèses (2.4), (2.5), est la fonction constante (2.13).

Il s'ensuit du théorème 3 que l'équation (2.3) se réduit à

$$(2.14)\quad g_4(\alpha_2+\beta_2,\ \alpha_3+3\beta_2\alpha_2+\beta_3) = g_4(\alpha_2,\alpha_3) + g_4(\beta_2,\beta_3) + \\ + 4\beta_2\alpha_3 + 6\beta_3\alpha_2 + 3\beta_2\alpha_2^2.$$

Notons que le premier membre de (2.14) est symétrique par rapport à (α_2,α_3) et (β_2,β_3), d'où il résulte que le second l'est aussi, donc

$$4\beta_2\alpha_3 + 6\beta_3\alpha_2 + 3\beta_2\alpha_2^2 = 4\beta_3\alpha_2 + 6\beta_2\alpha_3 + 3\alpha_2\beta_2^2.$$

LIBRARY Gurukul Kangri Vishwavidyalaya HARDWAR

Il s'ensuit que $\alpha_3 = \beta_3$ pour $\alpha_2 = \beta_2 = 1$ et on aboutit à une contradiction. Nous avons ainsi prouvé que l'équation (2.14) n'admet pas de solutions. On a donc le résultat suivant.

T h é o r è m e 4. Le système d'équations fonctionnelles (2.2), (2.3) n'admet pas de solutions.

T h é o r è m e 4'. Le groupe L_4^1 n'admet pas de sous-demi-groupe de la forme (2.1) dans la classe des fonctions satisfaisant aux hypothèses (2.4), (2.5).

3. Considérons des fonctions $F_1: R\times R \longrightarrow R_0$, $g_3: R\times R \longrightarrow R$ et l'ensemble

$$P_{13} = \left\{ \langle F_1(\alpha_2,\alpha_4),\ \alpha_2,\ g_3(\alpha_2,\alpha_4),\ \alpha_4\rangle : \alpha_2,\alpha_4 \in R \right\},$$

fermé par rapport à l'opération (1.1) si et seulement si les fonctions F_1 et g_3 satisfont au système

$$(3.1)\quad F_1\big[\alpha_2F_1(\beta_2,\beta_4) + \beta_2F_1^2(\alpha_2,\alpha_4),\ \alpha_4F_1(\beta_2,\beta_4) + \beta_4F_1^4(\alpha_2,\alpha_4) + \\ + 4\beta_2F_1(\alpha_2,\alpha_4)\ g_3(\alpha_2,\alpha_4) + 6\alpha_2F_1^2(\alpha_2,\alpha_4)\ g_3(\beta_2,\beta_4) + \\ + 3\beta_2\alpha_2^2\big] = F_1(\alpha_2,\alpha_4)\ F_1(\beta_2,\beta_4),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.2) $g_3\big[\alpha_2 F_1(\beta_2,\beta_4) + \beta_2 F_1^2(\alpha_2,\alpha_4),\ \alpha_4 F_1(\beta_2,\beta_4) + \beta_4 F_1^4(\alpha_2,\alpha_4) +$
$+ 4\beta_2 F_1(\alpha_2,\alpha_4)\, g_3(\alpha_2,\alpha_4) + 6\alpha_2 F_1^2(\alpha_2,\alpha_4)\, g_3(\beta_2,\beta_4) +$
$+ 3\beta_2\alpha_2^2\big] = F_1(\beta_2,\beta_4)\, g_3(\alpha_2,\alpha_4) + 3\alpha_2\beta_2 F_1(\alpha_2,\alpha_4) +$
$+ g_3(\beta_2,\beta_4)\, F_1^3(\alpha_2,\alpha_4).$

Admettons que

(3.3) la fonction $F_1(0,\cdot)$ est continue,

(3.4) la fonction $F_1(\cdot,0)$ est continue,

(3.5) la fonction $g_3(0,\cdot)$ est continue en un point.

Si on pose $\alpha_2 = \beta_2 = 0$ dans (3.1), on obtient

(3.6) $F_1(0,\ \alpha_4 F_1(0,\beta_4) + \beta_4 F_1^4(0,\alpha_4)) = F_1(0,\alpha_4)\, F_1(0,\beta_4).$

Il en résulte, en vertu de l'hypothèse (3.3) et d'un théorème établi dans [9], que

(3.7) $$F_1(0,\alpha_4) = 1.$$

Si on pose $\alpha_2 = \beta_2 = 0$ dans (3.2), on obtient, en vertu de (3.7), $g_3(0,\alpha_4+\beta_4) = g_3(0,\alpha_4) + g_3(0,\beta_4)$. La fonction $g_3(0,\cdot)$ étant donc additive, il existe une constante c telle que

(3.8) $$g_3(0,\alpha_4) = c\,\alpha_4.$$

Si on pose $\alpha_2 = 0$ et ensuite $\beta_2 = 0$ dans (3.1), on obtient, en vertu de (3.7), (3.8),

(3.9a) $$F_1\big[\beta_2,\ \alpha_4(F_1(\beta_2,\beta_4) + 4\beta_2 c\big] + \beta_4) = F_1(\beta_2,\beta_4),$$

et

(3.9b) $$F_1\big[\alpha_2,\ \alpha_4+\beta_4(F_1^4(\alpha_2,\alpha_4) + 6c\alpha_2\, F_1^2(\alpha_2,\alpha_4)\big] = F_1(\alpha_2,\alpha_4),$$

respectivement. On constante donc sans peine que $F_1(\beta_2,\beta_4) + 4\beta_2 c \neq 0$ ou bien $F_1^4(\beta_2,\beta_4) + 6c\beta_2\, F_1^2(\beta_2,\beta_4) \neq 0$, respectivement, pour tout couple (β_2,β_4). On en déduit que l'ensemble

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

des valeurs de l'une des fonctions $\alpha_4 \longrightarrow \alpha_4(F_1(\beta_2,\beta_4)+4\beta_2 c)$, $\beta_4 \longrightarrow \beta_4(F_1^4(\alpha_2,\alpha_4) + 6c\alpha_2 F^2(\alpha_2,\alpha_4))$ est un ensemble des nombres réels. Il en résulte, vu (3.9a) ou bien (3.9b), que

$$F_1(\beta_2,\beta_4) = F_1(\beta_2,0). \tag{3.10}$$

Si on pose $\beta_4 = \alpha_4 = 0$ dans (3.1), on obtient, compte tenu de (3.10), la relation

$$F_1\left[\alpha_2 F_1(\beta_2,0) + \beta_2 F_1^2(\alpha_2,0),\ 0\right] = F_1(\alpha_2,0)\ F_1(\beta_2,0) \tag{3.11}$$

qui nous ramène, en vertu de (3.4) et d'un théorème établi dans [9], à $F_1(\alpha_2,0) = 1$. Cette égalité avec (3.10) donne

$$F_1(\alpha_2,\alpha_4) = 1. \tag{3.12}$$

En tenant compte de (3.12), on déduit de (3.2) que

$$g_3\left[\alpha_2+\beta_2,\alpha_4+\beta_4+4\beta_2 g_3(\alpha_2,\alpha_4) + 6\alpha_2 g_3(\beta_2,\beta_4) + 3\beta_2\alpha_2^2\right] = \tag{3.13}$$

$$= g_3(\alpha_2,\alpha_4) + 3\alpha_2\beta_2 + g_3(\beta_2,\beta_4).$$

Si on y pose $\alpha_2 = 0$ et puis $\beta_2 = 0$, on obtient, en vertu de (3.8),

$$g_3(\beta_2,\alpha_4+\beta_3+4\beta_2 c\alpha_4) = c\alpha_4 + g_3(\beta_2,\beta_4) \tag{3.14}$$

et

$$g_3(\alpha_2,\alpha_4+\beta_4+6\alpha_2 c\beta_4) = g_3(\alpha_2,\alpha_4) + c\beta_4, \tag{3.15}$$

respectivement. Si $c = 0$, l'égalité (3.14) entraîne

$$g_3(\beta_2,\beta_4) = g_3(\beta_2,0). \tag{3.16}$$

Supposons que $c \neq 0$. En posant

$$\alpha_4 = -\frac{\beta_4}{1 + 4\beta_2 c} \quad \text{pour} \quad \beta_2 \neq -\frac{1}{4c} \tag{3.17}$$

et

$$\beta_4 = -\frac{\alpha_4}{1 + 6c\alpha_2} \quad \text{pour} \quad \alpha_2 \neq -\frac{1}{6c} \tag{3.18}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

dans (3.14) et (3.15), on obtient

$$g_3(\beta_2,0) = \frac{-c\beta_4}{1 + 4\beta_2 c} + g_3(\beta_2,\beta_4) \tag{3.19}$$

et

$$g_3(\alpha_2,0) = -\frac{c\alpha_4}{1 + 6c\alpha_2} + g_3(\alpha_2,\alpha_4), \tag{3.20}$$

respectivement. Notons que, si l'on pose $\alpha_2 = \beta_2$ et $\alpha_4 = \beta_4$ dans (3.20), on a de (3.19), (3.20)

$$\frac{1}{1 + 4\beta_2 c} = \frac{1}{1 + 6\beta_2 c}$$

et on aboutit ainsi à une contradiction. L'hypothèse $c \neq 0$ est donc fausse, tandis que l'égalité (3.16) est vraie et, de plus, (3.8) entraîne $g_3(0,\alpha_4) = 0$. Il résulte de (3.13) et (3.16) que

$$g_3(\alpha_2+\beta_2,0) = g_3(\alpha_2,0) + 3\alpha_2\beta_2 + g_3(\beta_2,0). \tag{3.21}$$

En vertu du théorème 6 de [3], la solution générale de l'équation fonctionnelle (3.21) est une famille de fonctions de la forme

$$g_3(\alpha_2,0) = \varphi_2(\alpha_2) + \frac{3}{2}\alpha_2^2,$$

où φ est une fonction additive quelconque. On en déduit, vu (3.16), que

$$g_3(\alpha_2,\alpha_4) = \varphi_2(\alpha_2) + \frac{3}{2}\alpha_2^2. \tag{3.22}$$

On vérifie aisément que tout couple de fonctions (3.12), (3.22) satisfait au système (3.1), (3.2). Nous avons ainsi démontré les théorèmes suivants.

T h é o r è m e 5. La solution générale du système d'équations fonctionnelles (3.1), (3.2) dans la classe des fonctions qui satisfont aux hypothèses (3.3)-(3.5) est une famille de couples de fonctions (3.12), (3.22), où φ est une fonction quelconque.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

T h é o r è m e 5'. Les sous-demi-groupes du groupe L_4^1 de la forme P_{13} avec les fonctions F_1, g_3 satisfaisant aux hypothèses (3.3)-(3.5) sont les ensembles

$$\left\{\langle 1, \alpha_2, \varphi(\alpha_2) + \tfrac{3}{2}\alpha_2^2 , \alpha_4\rangle : \alpha_2, \alpha_4 \in R\right\},$$

où φ est une fonction additive quelconque.

4. Soient des fonctions $F_2, g_3 : R_0 \times R \longrightarrow R$ et l'ensemble

$$P_{23} = \left\{\langle \alpha_1, F_2(\alpha_1,\alpha_4), g_3(\alpha_1,\alpha_4), \alpha_4\rangle : \alpha_1 \in R_0 \text{ et } \alpha_4 \in R\right\},$$

fermé par rapport à l'opération (1.1) si et seulement si les fonctions F_2, g_3 satisfont au système d'équations

$$(4.1)\quad F_2\big[\beta_1\alpha_1,\ \alpha_4\beta_1 + \beta_4\alpha_1^4 + 4\alpha_1 F_2(\beta_1,\beta_4)\, g_3(\alpha_1,\alpha_4) + 6\alpha_1^2\, g_3(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4) + 3F_2(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4)\big] = \beta_1 F_2(\alpha_1,\alpha_4) + \alpha_1^2\, F_2(\beta_1,\beta_4),$$

$$(4.2)\quad g_3\big[\beta_1\alpha_1,\ \alpha_4\beta_1 + \beta_4\alpha_1^4 + 4\alpha_1 F_2(\beta_1,\beta_4)\, g_3(\alpha_1,\alpha_4) + 6\alpha_1^2 g_3(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4) + 3F_2(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4)\big] = \beta_1 g_3(\alpha_1,\alpha_4) + 3\alpha_1 F_2(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4) + \alpha_1^3\, g_3(\beta_1,\beta_4).$$

Nous allons résoudre ce système dans le cas où $F_2 = g_3$. Les premiers membres de (4.1), (4.2) sont alors égaux, d'où

$$(4.3)\quad F_2(\beta_1,\beta_4)(\alpha_1^3 - \alpha_1^2) = -3F_2(\beta_1,\beta_4)\, F_2(\alpha_1,\alpha_4)\alpha_1.$$

Il s'ensuit donc que pour tout couple (β_1,β_4) on a

$$(4.4)\quad F_2(\beta_1,\beta_4) = 0$$

ou bien il existe un couple $(\bar\beta_1,\bar\beta_4)$ tel que $F_2(\bar\beta_1,\bar\beta_4) \neq 0$. Dans ce dernier cas il résulte de (4.3) que

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(4.5) $$F_2(\alpha_1,\alpha_4) = \frac{1}{3}(\alpha_1 - \alpha_1^2).$$

On voit aisément que pour $g_3 = F_2$, où F_2 prend la forme (4.4) ou (4.5), le couple de fonctions F_2, g_3 satisfait au système (4.1), (4.2).

Nous n'avons pas réussi à résoudre le système (4.1), (4.2) dans le cas général. Le problème suivant reste ainsi ouvert: quelle est la solution générale du système (4.1), (4.2)?

Les sous-demi-groupes de la forme P_{34} du groupe L_4^1 dans lesquels deux derniers paramètres sont des fonctions de deux premiers ont été déterminés dans [8].

TRAVAUX CITÉS

[1] J. Aczél, S. Gołąb: Funktionalgleichungen der Theorie der geometrischen Objekte, Monografie Mat. 39, Warszawa 1960.

[2] J. Aczél, S. Gołąb: Remarks on one parameter subsemigroups of the affine group and their homo- and isomorphism, Aequationes Math. 4 (1970) 1-11.

[3] S. Midura: Sur la détermination de certains sous-groupes du groupe L_s^1 à l'aide d'équations fonctionnelles, Dissertationes Math. 105 (1973) 1-34.

[4] S. Midura: Sur la détermination de certains sous-groupes du groupe L_3^1 à l'aide d'équations fonctionnelles, Aequationes Math. 11 (1974) 295-296.

[5] S. Midura: O rozwiązaniach równań funkcyjnych wyznaczających pewne podpółgrupy grupy L_4^1, Rocznik Nauk.-Dydakt. WSP w Rzeszowie 4/32 (1977) 48-58.

[6] S. Midura: Sur la détermination de certains sous-demi-groupes du groupe L_4^1 à l'aide d'équations fonctionnelles, Aequationes Math. 15 (1977) 285-286.

[7] S. Midura: O rozwiązaniach równania funkcyjnego $f(\beta_1\alpha_1, \beta_2\alpha_1^3 + 3\alpha_1 f(\alpha_1,\alpha_2)\, f(\beta_1,\beta_2) + \beta_1\alpha_2) = \alpha_1^2 f(\beta_1,\beta_2) + \beta_1 f(\alpha_1,\alpha_2)$, Rocznik Nauk.-Dydakt. WSP w Rzeszowie 6/50 (1982), 85-92.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[8] W. K ę d z i e r s k i : O rozwiązaniach układu równań funkcyjnych wyznaczających pewne podpółgrupy grupy L_4^1 i L_5^1, Rocznik Nauk.-Dydakt. WSP w Rzeszowie 6 (1982) 51-62.

[9] M. S a b l i k , P. U r b a n : On solutions of the equation $f(xf(y)^k + yf(x)^l) = f(x)f(y)$, Demonstratio Math. 18 (1985) 863-867.

[10] N. B r i l l o u e t , J. D h o m b r e s : Équations fonctionnelles et recherche de sous-groupes, Aequationes Math. 31 (1986) 253-293.

[11] S. M i d u r a : Sur les solutions des équations fonctionnelles qui déterminent certains sous-demi-groupes à deux paramètres du groupe L_4^1, 26 th International Symposium on Functional Equations 24th April - 3rd May, 1988 - Sant Feliu de Guixols (Spain).

[12] S. M i d u r a : Sur les solutions des équations fonctionnelles qui déterminent certains sous-demi-groupes à deux paramètres du groupe L_4^1, Aequationes Math. 39 (1990) 288-289. Reports of meetings Twenty-sixth International Symposium on Functional Equations, April 24 - May 3, 1988, Sant Feliu de Guixols, Catalonia, Spain.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
35-310 RZESZÓW, POLAND

Received March 3, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

RE

J ⊂ γ

We s

that

meas

= su

an e

of o

noni

F : I

for

F(R)

with

(1)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Grażyna Kozłowska

REARRANGEMENTS OF FUNCTIONS IN SOME FUNCTION SPACES

Let $(D, \mathfrak{M}, \mu)$ be a measure space ([1] p.36). Denote by $J \subset \mathfrak{M}$ the family with the properties:

1) J is simply ordered with respect to inclusion (for every $A, B \in J$, $A \neq B$, $\mu(A) \neq \mu(B)$).

2) For every $B \in \mathfrak{M}$ there is an $A \in J$ such that $\mu(B) = \mu(A)$.

3) For $A \in J$, $A = \bigcap \{B : B \in J, \mu(B) > \mu(A)\}$.

Of course, by property 1 the A in property 2 is unique. We shall denote it by M(B) ([2]).

Let f be a nonnegative, measurable function on D, such that for each $\lambda > 0$ the set $E_f(\lambda) = \{x \in D : f(x) \geqslant \lambda\}$ has finite measure. We define its rearrangement Sf by $(Sf)(x) = \sup\{\lambda : x \in M(E_f(\lambda))\}$.

It is easily seen that $E_{Sf}(\lambda) = M(E_f(\lambda))$, so that Sf is an equimeasurable rearrangement of f ([2]).

If $D = \langle 0, +\infty)$, μ - Lebesgue measure, and J is the family of closed intervals, with the left end at zero, then Sf is nonincreasing and right-continuous.

The following theorem was proved in [2] (Th. 3): Let $F : R^2 \longrightarrow R$ be continuous, such that $F(0,0) = 0$ and $\mathbb{F}(\mathbb{R}) \geqslant 0$ for every $\mathbb{R}$, where $\mathbb{R} = \{(x,y) \in R^2 : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$ and $\mathbb{F}(\mathbb{R}) = F(b,d) + F(a,c) - F(a,d) - F(b,c)$.

Let f and g be nonnegative, measurable functions on $G \subset R$, with $\mu(E_f(\lambda))$ and $\mu(E_g(\lambda))$ finite for all $\lambda > 0$. Then

(1) $$\int_G F(f,g)\, d\mu \leqslant \int_G F(Sf,Sg)\, d\mu.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

There was also presented the following remark: If φ is convex and $\varphi(0) = 0$, then for $F(x,y) = -\varphi(x-y)$ by (1) we obtain

$$\int_G \varphi(Sf-Sg)\, d\mu \leqslant \int_G \varphi(f-g)\, d\mu. \tag{2}$$

In particular, if $\varphi(x) = |x|^p$ and $1 \leqslant p \leqslant +\infty$, we get $\left\{\int_G |Sf-Sg|^p\, d\mu\right\}^{1/p} \left\{\int_G |f-g|^p\, d\mu\right\}^{1/p}$ and so

$$\|Sf - Sg\|_p \leqslant \|f - g\|_p \quad \text{on} \quad L^p \quad \text{for} \quad p \geqslant 1. \tag{3}$$

Now, we prove that the inequality (3) holds on other spaces, too.

It is not difficult to show that we can take $F(x,y) = \Phi(x-y)$, where Φ is a concave φ-function ([5] p.83).

From this by (1) $\int_G \Phi(f - g)\, d\mu \leqslant \int_G \Phi(Sf - Sg)\, d\mu$.

In particular, if $\Phi(x) = |x|^p$ and $0 < p < 1$, we get

$$\int_G |Sf - Sg|^p\, d\mu \geqslant \int_G |f - g|^p\, d\mu$$

for F-norm on L^p, $0 < p < 1$ ([4] p.385).

Now, suppose that $D = R^+ \times R^+$, μ - Lebesgue measure, J is a family of closed quadrants of circles with center at the origin.

Let $G = \langle a,b;c,d\rangle \subset R^2$ (may be infinite $<0,\infty;0,\infty>$).

Let $f = f(x,y)$, $g = g(x,y)$ be nonnegative, measurable functions on G with $\mu(E_f(\lambda))$ and $\mu(E_g(\lambda))$ finite for all $\lambda > 0$. Moreover, $\left\{\int_a^b |f(\cdot,y) - g(\cdot,y)|^{p_1}\, dx\right\}^{1/p_1} \in L^{p_2}(c,d)$ for almost every $y \in \langle c,d\rangle$.

Then by (3) we get

$$\left\{\int_a^b |Sf(\cdot,y) - Sg(\cdot,y)|^{p_1}\, dx\right\}^{1/p_1} \leqslant \left\{\int_a^b |f(\cdot,y) - g(\cdot,y)|^{p_1}\, dx\right\}^{1/p_1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From it

$$\int_c^d \left\{ \int_a^b \left| Sf(x,y) - Sg(x,y) \right|^{p_1} dx \right\}^{p_2/p_1} dy \leqslant$$

$$\leqslant \int_c^d \left\{ \int_a^b \left| f(x,y) - g(x,y) \right|^{p_1} dx \right\}^{p_2/p_1} dy$$

and finally

$$\left\| Sf - Sg \right\|_{L^P} \leqslant \left\| f - g \right\|_{L^P} \quad \text{on } L^P, \quad P = (p_1, p_2), \quad 1 \leqslant p_1, p_2 < \infty .$$

Now, we denote ([3]):

φ, ψ - convex φ-functions

$L^{*\varphi}(a,b)$ - the class of the functions $f(\cdot,y)$ defined and measurable on $\langle a,b \rangle$, such that

$$\int_a^b \varphi(|f(x,y)|)\, dx < \infty \text{ for almost every } y \in \langle c,d \rangle$$

$$\left\| f(\cdot,y) \right\|_\varphi = \inf \left\{ \varepsilon > 0;\ \int_a^b \varphi\left(\frac{|f(x,y)|}{\varepsilon} \right) dx \leqslant 1 \right\}. \tag{4}$$

$L^{*\psi}(c,d)$ - the class of the functions $f(x,\cdot)$ defined and measurable on $\langle c,d \rangle$, such that $\int_c^d \psi(|f(x,y)|)\,dy < \infty$ for almost every $x \in \langle a,b \rangle$

$$\left\| f(x,\cdot) \right\|_\psi = \inf \left\{ k > 0 : \int_c^d \psi\left(\frac{|f(x,y)|}{k} \right) dy \leqslant 1 \right\}.$$

$L^{*(\varphi,\psi)}(G)$ - Orlicz space with mixed norm $\left\| f(x,y) \right\|_{(\varphi,\psi)} = \left\| \left\| f(\cdot,y) \right\|_\varphi \right\|_\psi$. The space of the functions $f = f(x,y)$, defined and measurable on $G = \langle a,b;c,d \rangle$, such that $kf(\cdot,y) \in L^{*\varphi}(a,b)$ for almost every $y \in \langle c,d \rangle$ and some $k > 0$, and $\left\| f(\cdot,y) \right\|_\varphi \in L^{*\psi}(c,d)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $f = f(x,y)$, $g = g(x,y)$ be nonnegative, measurable functions on G with $\mu(E_f(\lambda))$ and $\mu(E_g(\lambda))$ finite for all $\lambda > 0$.

Moreover, $\int_a^b \varphi(|f(\cdot,y) - g(\cdot,y)|)\,dx \in L^{*\psi}(c,d)$ for almost every $y \in \langle c,d \rangle$.

From it, by (2) we get

$$\int_a^b \varphi(|Sf(\cdot,y) - Sg(\cdot,y)|)\,dx \leqslant \int_a^b \varphi(|f(\cdot,y) - g(\cdot,y)|)\,dx .$$

Hence, by (4) $\|Sf(\cdot,y) - Sg(\cdot,y)\|_\varphi \leqslant \|f(\cdot,y) - g(\cdot,y)\|_\varphi$ and finally

$$\|Sf - Sg\|_{(\varphi,\psi)} \leqslant \|f - g\|_{(\varphi,\psi)} \quad \text{on} \quad L^{*(\varphi,\psi)}(G).$$

I am grateful to Professor Julian Musielak for his kind remarks to this paper.

REFERENCES

[1] A. Alexiewicz : Analiza funkcjonalna. PWN, Warszawa 1969.

[2] J.A. Crowe, J.A. Zweibel : Rearrangements of Functions, J. Funkt. Anal. 66 (1986) 432-438.

[3] G. Kozłowska : Aproksymacja funkcjami całkowitymi dwóch zmiennych w przestrzeniach Orlicza z normami mieszanymi, Zeszyty Nauk. Politech. Śl., Ser. Mat.-Fiz. 30 (1978) 187-191.

[4] G. Kozłowska : Direct and indirect Jackson's type theorems in some Fréchet functions spaces, Constructive Function Theory'81, Bulg. Acad. Sc., Sofia (1983) 385-388.

[5] J. Musielak : Wstęp do analizy funkcjonalnej. PWN, Warszawa 1976.

INSTITUTE OF MATHEMATICS, SILESIAN TECHNICAL UNIVERSITY,
44-100 GLIWICE, POLAND
Received May 6, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Jerzy Motyl

EXISTENCE AND UNIQUENESS OF SOLUTION OF NEUTRAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS VIA DISSIPATIVITY

I. Introduction

The theory of nonlinear dissipative operators has been the subject of many works in the last thirty years. During 60-70's years it was applicated to the theory of ordinary differential inclusions (see e.g. [2], [3], [7]). On the other hand, in the past years there has been an increasing interest in neutral functional-differential equations (NFDE). They were investigated by Das and Parhi [4], Driver [6], Kisielewicz [8], [9], Melvin [10] and others. Their existence theorems most often required the assumptions of Lipschitz type.

In the present paper we deal with some neutral functional--differential inclusion (NFDI), combining the methods used in the theory of dissipative operators with that of classical NFDI. In Section III we investigate existence and uniqueness problem for some functional inclusion and then, in Section IV, we show how obtained results can be applicable to NFDI.

II. Definitions and notation

We denote by $\mathcal{L}^p[a,b]$, $1 \leqslant p < \infty$, the linear space of functions $x: [a,b] \longrightarrow \mathbb{R}^n$ such that $|x|^p$ is integrable on $[a,b]$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$; $L^p[a,b]$ is a Banach space of equivalence classes of functions from $\mathcal{L}^p[a,b]$ with the usual norm $\|\cdot\|$; $C[a,b]$ is a Banach space of continuous functions acting from $[a,b]$ to $\mathbb{R}^n$ with the supremum norm.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By $W^p[a,b]$ we denote the space of all absolutely continuous functions x such that $\dot{x}$ belongs to $L^p[a,b]$, with the norm $\|\|x\|\| = \sup_{t\in[a,b]} |x(t)| + \int_a^b |\dot{x}(t)|^p dt$. We do not discuss here in details the problem of distinguish between functions in $\mathcal{L}^p$ and classes of functions in L^p. For example, the symbol $\sup x(t)$ for $x \in L^p[a,b]$ we mean as the $\inf\{\sup \bar{x}(t) : \bar{x}\in x\}$, where the infimum is taking over all representants $\bar{x}$ from the class x.

Let $a,r,T \in \mathbb{R}$, $r,T \geq 0$, and $\mathcal{X} := [a,a+T] \times C[-r,0] \times L^p[-r,0]$. Let $F : \mathcal{X} \to \mathbb{R}^n$ and $\Phi \in W^p[-r,0]$. By Neutral Functional-Differential Inclusion (NFDI) we mean the problem

$$(A) \qquad \begin{cases} \dot{x}(t) \in F(t,x_t,\dot{x}_t), \\ x_a = \Phi, \end{cases}$$

where $x_t(s) = x(t+s)$ for $t \in [a,a+T]$, $s\in[-r,0]$.

We call x to be a global solution of problem (A) with initial condition Φ, if x belongs to $W^p[a-r,a+T]$, $\dot{x}(t)\in$ $\in F(t,x_t,\dot{x}_t)$ for a.e. $t \in [a,a+T]$ and $x_a(t) = \Phi(t)$ for $t \in [-r,0]$.

D e f i n i t i o n 1. Let Y be a Banach space and

$$F(x) := \left\{ x^* \in Y^* : x^*(x) = \|x\|^2 = \|x^*\|^2 \right\}.$$

The set-valued function F acting from Y to subsets of Y is called the duality mapping, while $x^* \in F(x)$ is called the dual functional.

A set-valued function $A : Y \to Y$ is called dissipative, if for every $x,y \in$ Dom A, $\lambda > 0$, $u \in A(x)$, $v \in A(y)$ the following inequality holds

$$\|x - y\| \leq \|x - y - \lambda(u - v)\|.$$

A dissipative set-valued function A is called m-dissipative, if Rang$(I - A) = Y$, where I denotes the identity operator in Y ([7]).

For further convenience we collect together some known properties of dissipative set-valued functions (see e.g. [1]).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1. For $x,y \in Y$ the following properties are equivalent:
(a) $\|x\| \leqslant \|x - \lambda y\|$ for every $\lambda > 0$,
(b) there exists a functional $F_x \in F(x)$ such that $F_x(y) \leqslant 0$.

2. Let A be m-dissipative. Then the graph of A is closed in the sense that, if $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in A(x_n)$, then $y \in A(x)$. If moreover Y is uniformly convex, then the graph of A is strong-weak closed. It means that, if $x_n \longrightarrow x$, $y_n \longrightarrow y$ weakly and $y_n \in A(x_n)$, then $y \in A(x)$.

3. A dissipative set-valued function A is m-dissipative, if and only if for every (equivalently for some) $\lambda > 0$ we have $\mathrm{Rang}(I - \lambda A) = Y$.

4. Let A be m-dissipative, $J_\lambda(x) = (I - \lambda A)^{-1}(x)$ and $A_\lambda(x) = \frac{1}{\lambda}(J_\lambda - I)(x)$ be, respectively, the resolventa and Yosida approximation of A. Then we have:
(a) J_λ and A_λ are well defined operators in Y,
(b) $\|J_\lambda(x) - J_\lambda(y)\| \leqslant \|x - y\|$ for every $x,y \in Y$,
(c) A_λ is dissipative and Lipschitz operator with a Lipschitz constant $\frac{2}{\lambda}$,
(d) $A_\lambda(x) \in A(J_\lambda(x))$ for every $x \in Y$,
(e) $\|A_\lambda(x)\| \leqslant \inf\{\|y\| : y \in A(x)\}$ for every $x \in Y$.

III. Existence and uniqueness

Let $A : L^p[a,b] \longrightarrow L^p[a,b]$, $1 \leqslant p < \infty$, be set-valued. Let us consider the functional problem

(B) $$\dot{x} \in A(x), \quad x(a) = x_o \quad (x_o \in \mathbb{R}^n).$$

By the solution of (B) we mean the function $x \in W^p[a,b]$ such that $\dot{x} \in A(x)$ and $x(a) = x_o$.

T h e o r e m 1. Let A be a m-dissipative set-valued function acting from $L^p[a,b]$ to subsets of $L^p[a,b]$, $1 < p < \infty$, and such that

(i) there exists a constant M such that for every $x \in \mathrm{Dom}\, A$ $\mathrm{dist}(0,A(x)) = \inf\{\|z\| : z \in A(x)\} \leqslant M$,

(ii) for every $x,y \in \mathrm{Dom}\, A$, $\lambda > 0$, $t \in [a,b]$, $c \in [a,t]$, $u \in A(x)$, $v \in A(y)$ and for some nonincreasing, continuous function $K: [0,b-a] \longrightarrow \mathbb{R}$ such that $K(0) = 0$ there occurs

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left[\int_0^t |x(s)-y(s)|^p ds\right]^{\frac{1}{p}} \leqslant K(t-c) \sup_{c\leqslant s\leqslant t} \left|x(s) - y(s) - \lambda[u(s)- v(s)]\right|.$$

Then for every x_0 the problem (B) admits a solution.

P r o o f . Let A_λ denote the Yosida approximation of A for every $\lambda>0$. By the property 4.(a), A_λ is a well defined operator in $L^p[a,b]$, so by the property 3, we obtain Dom $A_\lambda = L^p[a,b]$.

Let us consider the problem

(C) $$\dot{x}_\lambda(t) = A_\lambda(x_\lambda)(t),\ x_\lambda(a) = x_0 \quad \text{and} \quad t \in [a,b].$$

By the definition of A_λ, we obtain

$$\left[\int_a^t |A_\lambda(x)(s)-A_\lambda(y)(s)|^p ds\right]^{\frac{1}{p}} \leqslant \frac{1}{\lambda}(t-a)^{\frac{1}{p}} \sup_{a\leqslant s\leqslant t} |x(s)-y(s)| +$$

$$+ \frac{1}{\lambda}\left[\int_a^t |J_\lambda(x)(s)-J_\lambda(y)(s)|^p ds\right]^{\frac{1}{p}}.$$

Denote $z_1 = J_\lambda(x)$, $z_2 = J_\lambda(y)$. Let $f \in A(z_1)$ and $g \in A(z_2)$ be such that $x = z_1 - \lambda f$ and $y = z_2 - \lambda g$, respectively. By (ii), we obtain

$$\left[\int_a^t |z_1(s)-z_2(s)|^p ds\right]^{\frac{1}{p}} \leqslant K(t-a) \sup_{a\leqslant s\leqslant t} |z_1(s)-z_2(s)-\lambda(f(s)-g(s))|.$$

Therefore, we have the inequality

$$\left[\int_a^t |A_\lambda(x)(s)-A_\lambda(y)(s)|^p ds\right]^{\frac{1}{p}} \leqslant$$

$$\leqslant \frac{1}{\lambda}\left[(t-a)^{\frac{1}{p}} + K(t-a)\right] \sup_{a\leqslant s\leqslant t} |x(s)-y(s)|$$

which implies that A_λ has Volterra's property.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let us select a number c from the interval $[a,b]$ such that $(c-a)^{\frac{1}{q}} \cdot \left[(c-a)^{\frac{1}{p}} + K(c-a)\right] < \lambda$, $\frac{1}{p} + \frac{1}{q} = 1$. It is possible, because $K(c-a) \to 0$ as $c \to a^+$.

Now we can define an operator $H_\lambda^1 : C[a,b] \to C[a,b]$ by the formula

$$(1) \qquad H_\lambda^1(x)(t) = \begin{cases} x_0 + \int_a^t A_\lambda(x)(s)ds & \text{for } t \in [a,c], \\ x_0 + \int_a^c A_\lambda(x)(s)ds & \text{for } t \in [c,b]. \end{cases}$$

For every $x,y \in W^p[a,b]$ and $t \in [a,c]$ we obtain

$$\left|H_\lambda^1(x)(t) - H_\lambda^1(y)(t)\right| \leqslant \int_a^t \left|A_\lambda(x)(s) - A_\lambda(y)(s)\right| ds \leqslant$$

$$\leqslant (t-a)^{\frac{1}{q}} \cdot \frac{1}{\lambda} \cdot \left[(t-a)^{\frac{1}{p}} + K(t-a)\right] \sup_{a \leqslant s \leqslant t} |x(s)-y(s)| \leqslant \gamma \sup_{a \leqslant s \leqslant b} |x(s)-y(s)|$$

for some $\gamma \in (0,1)$.

The same inequality holds for $t \in [c,b]$, because $H_\lambda^1(s)$ is a constant function here, so H_λ^1 is a contraction on $C[a,b]$. Let $x_1 \in C[a,b]$ be its fixed point. We have $x_1(a) = H_\lambda^1(x_1)(a) = x_0$. Therefore, x_1 is a solution of the problem (C) on $[a,c]$.

Let H_λ^2 be defined by the formula

$$(2) \qquad H_\lambda^2(x)(t) = \begin{cases} x_1(t) & \text{for } t \in [a,c], \\ x_1(c) + \int_a^t A_\lambda(x)(s)ds & \text{for } t \in [c,2c-a], \\ x_1(c) + \int_a^c A_\lambda(x)(s)ds & \text{for } t \in [2c-a,b]. \end{cases}$$

Similarly as above, we obtain that H_λ^2 is a contraction on $C[a,b]$, so there exists a function $x_2 \in C[a,b]$ such that $x_2 = H_\lambda^2(x_2)$, x_2 being a solution of (C) on $[a,2c-a]$, because, by Volterra's property, we have $A_\lambda(x_2)(t) = A_\lambda(x_1)(t)$ for $t \in [a,c]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Continuing this process we obtain a function x_λ which is a solution of (C) on the whole interval $[a,b]$. We remark that our construction is finished after n steps, where n is such that $c+n(c-a) \geq b$. Taking λ tending to zero, we obtain a sequence (x_λ) of solutions of (C). We show that it contains a convergent subsequence (x_{λ_n}) such that its limit is a solution of problem (B).

Let $t_1, t_2 \in [a,b]$ be arbitrary fixed. By the property 4.(e), we obtain

$$|x_\lambda(t_1)-x_\lambda(t_2)| \leq \int_{t_1}^{t_2} |A_\lambda(x_\lambda)(s)|\,ds \leq (t_2-t_1)^{\frac{1}{q}} \operatorname{dist}(0,A(x_\lambda)) \leq$$

$$\leq (t_2-t_1)^{\frac{1}{q}} M.$$

Then x_λ are uniformly continuous. They are also uniformly bounded, because $x_\lambda(t) \leq |x_o| + M\cdot(b-a)^{\frac{1}{q}}$. By the Arzela-Ascoli theorem, there is a subsequence (x_{λ_n}) of (x_n) convergent uniformly to some x on $[a,b]$. Of course $x \in W^p[a,b]$ and $x_{\lambda_n} \to x$ in L^p norm. Since $\|\dot{x}_{\lambda_n}\| \leq M$, we can select another subsequence denoted again by $\dot{x}_{\lambda_n}$ weakly convergent in L^p. But x_{λ_n} are absolutely continuous, so the weak limit of $\dot{x}_{\lambda_n}$ must be equal to $\dot{x}$. For this subsequence we have the following inequality (see property 4)

$$\left\| J_{\lambda_n}(x_{\lambda_n}) - x \right\| \leq \lambda_n \cdot M + \left\| x_{\lambda_n} - x \right\|.$$

Therefore, $J_{\lambda_n}(x_{\lambda_n}) \to x$ in $L^p[a,b]$. Moreover, $\dot{x}_{\lambda_n} \to \dot{x}$ weakly in L^p and $\dot{x}_{\lambda_n} = A_{\lambda_n}(x_{\lambda_n}) \in A(J_{\lambda_n}(x_{\lambda_n}))$. By this, 4.(d) and 2.(b), we obtain that $\dot{x} \in A(x)$.

For every $\lambda > 0$ we have $x_\lambda(a) = x_o$ and therefore $x(a) = x_o$. This finishes the proof of Theorem 1.

T h e o r e m 2. If A, acting from $L^p[a,b]$ to subsets of $L^p[a,b]$, $1 < p < \infty$, satisfies the following conditions:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) there exists a constant M such that for every $x \in \mathrm{Dom}\, A$ $\mathrm{dist}(0, A(x)) \leqslant M$,

(ii) $\mathrm{Rang}(I-A) = L^p[a,b]$,

(iii) for every $x,y \in \mathrm{Dom}\, A$, $\lambda > 0$, $t \in [a,b]$, $u \in A(x)$, $v \in A(y)$ there occurs

$$\int_a^t |x(s)-y(s)|^p ds \leqslant \int_a^t |x(s)-y(s) - \lambda(u(s)-v(s))|^p ds,$$

then for every x_0 there exists a solution of problem (B).

Condition (iii) means that A is dissipative in $L^p[a,t]$ for every $t \in [a,b]$.

P r o o f . The proof is the same as that of Theorem 1. We must only construct more carefully the solution of (C). By the property 4.(c), we obtain

$$\left[\int_a^t |A_\lambda(x)(s)-A_\lambda(y)(s)|^p ds\right]^{\frac{1}{p}} \leqslant \frac{2}{\lambda}(t-a)^{\frac{1}{p}} \sup_{a\leqslant s\leqslant t} |x(s)-y(s)|.$$

Therefore, A has Volterra's property. Let us take such a number $c \in [a,b]$ that $c-a < \frac{1}{2}\lambda$. We define the function H^1_λ as in (1) and for $t \in [a,c]$ we obtain

$$|H^1_\lambda(x)(t) - H^1_\lambda(y)(t)| \leqslant \frac{2}{\lambda}(t-a) \sup_{a\leqslant s\leqslant t} |x(s)-y(s)|.$$

Then $x_1 \in C[a,b]$, a fixed point of H^1_λ, is a solution of (C) on $[a,c]$.

Now, let Z be a set of all continuous functions $z:[a,b] \to R^n$ and such that $z(t) = x_1(t)$ for $t \in [a,c]$. We introduce in Z the metric d induced from $C[a,b]$. Then (Z,d) is a complete metric space. Consider H^2_λ, defined by (2) as an operator in Z. For every $z \in Z$ we have $H^2_\lambda \in Z$. Let us take arbitrary $x,y \in Z$ and $t \in [c,2c-a]$. We obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left|H_\lambda^2(x)(t) - H_\lambda^2(y)(t)\right| \leqslant (t-c)^{\frac{1}{q}} \left[\int_c^t \left|A_\lambda(x)(s) - A_\lambda(y)(s)\right|^p ds\right]^{\frac{1}{p}} \leqslant$$

$$\leqslant (t-c)^{\frac{1}{q}} \cdot \frac{2}{\lambda} \left[\int_c^t \left|x(s)-y(s)\right|^p ds\right]^{\frac{1}{p}} < \gamma \sup_{c \leqslant s \leqslant t} \left|x(s)-y(s)\right| \leqslant \gamma \left|x-y\right|,$$

where $\gamma = \frac{2}{\lambda}(c-a) < 1$, and $|\cdot|$ is a norm in $C[a,b]$.

It is clear that for $t \in [2c-a,b]$ and for $t \in [a,c]$ also $\left|H_\lambda^2(x)(t) - H_\lambda^2(y)(t)\right| < \gamma |x-y|$ and, therefore, there exists a function $x_2 \in Z \subseteq C[a,b]$, a fixed point of H_λ^2. Further, we proceed as in the proof of Theorem 1.

R e m a r k 1. Let A, acting from $L^1[a,b]$ to subsets of $L^1[a,b]$, satisfy assumptions of Theorem 1. Moreover, let graph of A be strong-weak closed. Then for every x_o there exists a solution of problem (B).

P r o o f . The proof of Remark is identical as of Theorem 1. We only claim that the existence of sequence $\dot{x}_{\lambda_n}$ converging weakly to $\dot{x}$ follows now by Lemma 2 of [9], because the argument of reflexivity of L^p cannot be used.

T h e o r e m 3. If A is a set-valued function acting from $L^p[a,b]$ to subsets of $L^p[a,b]$, $1 < p < \infty$, and is dissipative for every $t \in [a,b]$ in $L^p[a,t]$ (in the sense of Theorem 2 (iii)), then there exists at most one solution of problem (B).

P r o o f . We prove the theorem in the case of real-valued functions. The case of R^n is similar.

Let x_1, x_2 be two different solutions of problem (B). Then there exists $t_o \in (a,b)$ such that $x_1(t_o) \neq x_2(t_o)$. We have $\dot{x}_1 \in A(x_1)$ and $\dot{x}_2 \in A(x_2)$. Therefore, for every $\lambda > 0$ we obtain

$$\left[\int_a^{t_o} \left|x_1(s)-x_2(s)\right|^p ds\right]^{\frac{1}{p}} \leqslant \left[\int_a^{t_o} \left|x_1(s)-x_2(s)-\lambda[\dot{x}_1(s)-\dot{x}_2(s)]\right|^p ds\right]^{\frac{1}{p}},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By the property 1.(b), there exists a functional $\ell \in (L^p[a,t_o])^*$ assigned to the point $x_1-x_2\big|_{[a,t_o]}$ and such that $\ell(x_1-x_2\big|_{[a,t_o]}) \leqslant 0$. The symbol $x_1-x_2\big|_{[a,t_o]}$ denotes the function x_1-x_2 reduced to the interval $[a,t_o]$. The dual functional ℓ is represented by an element $f \in L^q[a,t_o]$ of the form

$$f(t) = \|x_1-x_2\|^{2-p} \cdot |x_1(t)-x_2(t)|^{p-2} \cdot (x_1(t)-x_2(t)) \qquad \text{(see [5])}.$$

Therefore, we obtain

$$\int_a^{t_o} (\dot{x}_1(s)-\dot{x}_2(s)) \cdot |x_1(s)-x_2(s)|^{p-2} \cdot (x_1(s)-x_2(s)))ds \leqslant 0.$$

Functions x_i are absolutely continuous, so $x_i(\cdot)$ and $|x_i(\cdot)|$ are almost everywhere differentiable. Then for almost all $s \in [a,t_o]$ we have the equality

$$\left[\frac{d}{ds}[x_1(s)-x_2(s)]\right] \cdot [x_1(s)-x_2(s)] \cdot |x_1(s)-x_2(s)|^{p-2} =$$

$$= \frac{1}{p}\frac{d}{ds}\left[|x_1(s)-x_2(s)|^p\right],$$

which implies

$$\int_a^{t_o} \frac{1}{p}\frac{d}{ds}\left[|x_1(s)-x_2(s)|^p\right]ds \leqslant 0.$$

Since $x_1(a) - x_2(a) = 0$, we get the inequality

$$\frac{1}{p}|x_1(t_o)-x_2(t_o)| \leqslant 0,$$

being a contradiction to the assumption $x_1(t_o) \neq x_2(t_o)$.

IV. Main result

D e f i n i t i o n 2. Let X be a Banach space, $a,b \in \mathbb{R}$, $a < b$. Denote by $H_X^p[a,b]$ the family of all set-valued functions F acting from $[a,b] \times X$ to subsets of $\mathbb{R}^n$ and such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $F(\cdot,x)$ is measurable for every $x \in X$,

(ii) the family $\{F(\cdot,x)\}_{x\in X}$ is uniformly p-integral bounded (it means that there exists a function $m \in L^p[a,b]$ such that $h(F(t,x),0) < m(t)$ for a.e. $t \in [a,b]$ and every $x \in X$, h denotes here the Hausdorff distance) ([9]).

For every fixed x, symbol $F(\cdot,x)$ denotes the set-valued function acting from $[a,b]$ to subsets of $\mathbb{R}^n$.

Let $\mathcal{F}(F) : X \to L^p[a,b]$ be defined by the formula

$$\mathcal{F}(F)(x) = \{ f \in L^p[a,b] : f(t) \in F(t,x) \text{ for a.e. } t \in [a,b] \}$$

and called the subtrajectory integral of F.

Let $a,r,T \in \mathbb{R}$, $r,T \geq 0$. Define the sets

$$C_a[a,a+T] = \{ x \in C[a,a+T] : x(a) = 0 \},$$

$$W^p_a[a,a+T] = \{ x \in W^p_a[a,a+T] : x(a) = 0 \}.$$

For every $x \in C_a[a,a+T]$, $z \in L^p[a,a+T]$ and $\Phi \in W^p[-r,0]$ we define $\Phi \oplus x$ and $\dot{\Phi} \boxplus z$ by setting

$$(\Phi \oplus x)(t) = \begin{cases} \Phi(t-a) & \text{for } t \in [a-r,a], \\ \Phi(0) + x(t) & \text{for } t \in [a,a+T], \end{cases}$$

$$(\dot{\Phi} \boxplus z)(t) = \begin{cases} \dot{\Phi}(t-a) & \text{for } t \in [a-r,a], \\ z(t) & \text{for } t \in [a,a+T]. \end{cases}$$

Let G^F be a set-valued function associated to F and defined by the formula:

$$G^F(t,x,z) = F(t,(\Phi \oplus x)_t,(\dot{\Phi} \boxplus z)_t)$$

for $x \in C_a[a,a+T]$, $z \in L^p[a,a+T]$.

Let us consider the problem

(D) $$\dot{x}(t) \in G^F(t,x,\dot{x}), \quad x(a) = 0.$$

By solution of (D) we mean the function $x \in W^p_a[a,a+T]$ such that $\dot{x}(t) \in G^F(t,x,\dot{x})$ for a.e. $t \in [a,a+T]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

L e m m a 1. If a function $x \in W^p[a-r,a+T]$ is a solution of problem (A), then a function $y = x|_{[a,a+T]} - \Phi(0)$ is a solution of (D). Conversely, if y is a solution of (D), then $\Phi \oplus y$ is a solution of (A).

P r o o f . Since $y(a) = x_a(0) - \Phi(0) = 0$, we have $y \in W^p_a[a,a+T]$. Moreover, for every $t \in [a-r.a+T]$, we have $x(t) = (\Phi \oplus y)(t)$ and $\dot{x}(t) = (\dot{\Phi} \boxplus \dot{y})(t)$. Then for every $t \in [a,a+T]$ we obtain $\dot{y}(t) = \frac{d}{dt}(y(t)+\Phi(0)) = \frac{d}{dt}(\Phi \oplus y)(t)|_{[a,a+T]} = \dot{x}(t)|_{[a,a+T]} \in F(t,(\Phi \oplus y)_t,(\dot{\Phi} \boxplus \dot{y})_t) = G^F(t,y,\dot{y})$. Therefore, y is a solution of (D).

If $y \in W^p_a[a,a+T]$, then $(\Phi \oplus y) \in W^p_a[a-r,a+T]$ and we have $\frac{d}{dt}[(\Phi \oplus y)(t)] \in F(t,(\Phi \oplus y)_t,(\dot{\Phi} \boxplus \dot{y})_t)$ for a.e. $t \in [a,a+T]$. Moreover, $(\Phi \oplus y)_a(s) = \Phi(s)$ for $s \in [-r,0]$, so $\Phi \oplus y$ is a solution of (A).

T h e o r e m 4. If $F : \mathcal{X} \to \mathbb{R}^n$ and $p > 1$ satisfy the following conditions

(i) $F(\cdot,u,\dot{u}) \in H^p_X[a,a+T]$ for every $u \in W^p[-r,0]$, $X = C[-r,0] \times L^p[-r,0]$,

(ii) for every $y \in L^p[a,a+T]$ there exist $x \in W^p[a-r,a+T]$ and $f \in L^p[a,a+T]$ such that $y = x|_{[a,a+T]} - f$ and $f(t) \in F(t,x_t,\dot{x}_t)$ for a.e. $t \in [a,a+T]$,

(iii) for every $x,y \in W^p[a-r,a+T]$, $\lambda > 0$ and $u,v \in L^p[a,a+T]$ the following implication holds: if $u(t) \in F(t,x_t,\dot{x}_t)$, $v(t) \in F(t,y_t,\dot{y}_t)$ for a.e. $t \in [a,a+T]$, then

$$\left[\int_a^{a+T} |x(s)-y(s)|^p ds\right]^{\frac{1}{p}} \leqslant \left[\int_a^{a+T} |x(s)-y(s) - \lambda(u(s)-v(s))|^p ds\right]^{\frac{1}{p}},$$

(iv) for every $x,y \in W^p[a-r,a+T]$, $\lambda > 0$ and $u,v \in L^p[a,a+T]$, $t \in [a,a+T]$ and $c \in [a,t]$ the following implication holds: if $u(t) \in F(t,x_t,\dot{x}_t)$, $v(t) \in F(t,y_t,\dot{y}_t)$ for a.e. $t \in [a,a+T]$, then

$$\left[\int_c^t |x(s)-y(s)|^p ds\right]^{\frac{1}{p}} \leqslant K(t-c) \sup_{c \leqslant s \leqslant t} |x(s)-y(s)-\lambda(u(s)-v(s))|,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$K : [0,b-a] \to \mathbb{R}$, $K(0) = 0$ being some nonincreasing continuous function,
then, for every $\Phi \in W^p[-r,0]$, the problem (A) admits a solution.

P r o o f . Let G^F be a set-valued function associated to F and $\mathcal{F}(G^F)$ be its subtrajectory integral. For every $x \in W^p[a,a+T]$ let $\mathcal{G}(x) := \mathcal{F}(G^F) \circ \Delta x$, where $\Delta x := (x,\dot{x})$. By the assumption (i), we obtain that $\mathrm{Dom}\ \mathcal{G} = W^p_a[a,a+T]$. We show that $\mathcal{G}$ satisfies assumptions of Theorem 1. For every $x \in W^p[a,a+T]$ we have

$$\mathrm{dist}(0,\mathcal{G}(x)) = \inf\{\|y\| : y \in \mathcal{F}(G^F)(x)\} \leqslant \left[\int_a^{a+T} h^p(G^F(s,x,\dot{x}),0)\,ds\right]^{\frac{1}{p}}$$

$\leqslant \|m\|$, where m is p-integral boundedness of F.

Assumption (b) means that $\mathrm{Rang}(I-\mathcal{G}) = L^p[a,a+T]$, while (c) denotes that $\mathcal{G}$ is dissipative in $L^p[a,a+T]$. Taking it together with (d), we obtain, by Theorem 1, that there exists a function $x \in W^p[a,a+T]$ such that $\dot{x} \in \mathcal{G}(x)$ and $x(a) = 0$. Therefore, $\dot{x}(t) \in G^F(t,x,\dot{x})$ for a.e. $t \in [a,a+T]$ and, by Lemma 1, we deduce that the function $\Phi \oplus x$ is a solution of problem (A).

T h e o r e m 5. Let $F : \mathfrak{X} \to \mathbb{R}^n$ satisfy assumptions (i), (ii) of Theorem 4 and besides

(iii′) for every $x,y \in W^p[a-r,a+T]$, $\lambda > 0$, $t \in [a,a+T]$ and u, v such that $u(t) \in F(t,x_t,\dot{x}_t)$, $v(t) \in F(t,\dot{y}_t,y_t)$ the following inequality hold

$$\int_a^t |x(s)-y(s)|^p\,ds \leqslant \int_a^t |x(s)-y(s) - \lambda(u(s)-v(s))|^p\,ds.$$

Then, for every $\Phi \in W^p[-r,0]$, there exists a solution of problem (A).

P r o o f . The proof follows immediately from Theorem 2.

R e m a r k 2. Theorems 4 and 5 hold in the case $L^1[-r,0]$, if we assume additionally that the set $\{x \times B_x\}_{x \in W^1[a,a+T]}$ is strong-weak closed in the space $L^1[a,a+T] \times L^1[a,a+T]$, where $B_x := \{y \in L^1[a,b] : y(t) \in F(t,x_t,\dot{x}_t)$ for a.e. $t \in [a,a+T]\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 6. If $F : \mathfrak{X} \to \mathbb{R}^n$ and $p > 1$ satisfy the assumption (iii') of Theorem 5, then the problem (A) admits at most one solution.

proof. In virtue of Theorem 3, proof follows by the same subtrajectory integral technique as above.

Acknowledgment. I am very grateful to Prof. M. Kisielewicz for his valuable discussions and suggestions in the preparation of this paper.

REFERENCES

[1] V. Barbu : Nonlinear semigroups and differential equations in Banach spaces, Leyden 1976.

[2] M.G. Crandall, A. Pazy : Semigroups of nonlinear contractions and dissipative sets, J. Funct. Anal., 3 (1969) 376-418.

[3] M.G. Crandall, A. Pazy : On accretive sets in Banach spaces, J. Funct. Anal., 5 (1970) 204-217.

[4] P.C. Das, N. Parhi : On a functional differential equations of neutral type, J. Differential Equations, 35 (1971) 67-82.

[5] K. Deimling : Ordinary differential equations in Banach spaces, Berlin-Heidelberg-New York, Springer Verlag 596, 1977.

[6] R.D. Driver : Existence and continuous dependence of solutions of a neutral functional-differential equation, Arch. Ratio. Mech., 19 (1965) 149-166.

[7] T. Kato : Nonlinear semigroups and evolution equations, J. Math. Soc. Japan., 19 (1967) 508-520.

[8] M. Kisielewicz : Existence theorem for generalized functional differential equations of neutral type, J. Math. Anal. Appl., 78 (1980) 173-182.

[9] M. Kisielewicz : Subtrajectory integrals of set-valued functions and neutral functional-differential equations, (to appear).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[10] W. M e l v i n : A class of neutral functional-differential equations, J. Differential Equations, 12 (1972) 524-534.

DEPARTMENT OF MATHEMATIC, HIGHER COLLEGE OF ENGINEERING,
65-246 ZIELONA GÓRA, POLAND

Received May 9, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Andrzej Rychlewicz

ON THE ALGEBRAIC EMBEDDABILITY IN THE CLASS OF FUNCTIONS POSSESSING CONNECTED GRAPHS

In this paper we formulate the conditions under which a given subfamily of the family of functions with a connected graph is algebraically embedded in this family.

Recently, there have appeared a lot of interesting papers in which some classes of functions, related to the notion of connectedness, had been investigated. In particular, many interesting results were obtained when studying algebraic operations performed on functions belonging to these classes (cf. e.g. [1], [2], [3], [4], [6], [7], [8], [9], [10], [12]). In view of this fact, it seems essential to ask the question: what additional conditions must the class $\mathbb{K}$, consisting of functions with a connected graph, satisfy in order that the result of a finite number of algebraic operations performed on functions of this class should belong to the family of functions with a connected graph? A partial answer to the above question is included in the present paper.

Throughout the paper, we shall consider real functions defined on some subsets of the real line. By $\mathbb{R}$ we denote the set of real numbers (with the natural topology). Positive integers and rational numbers are denoted by $\mathbb{N}$ and $\mathbb{Q}$, respectively. The notation $[a,b]$ (resp. $[a,b)$, $(a,b]$ and the like) will be applied also in the case when $b < a$. If A is an arbitrary set, then by the symbol $cl_X(A)$ we shall denote the closure of the set A in X. Instead of $cl_{\mathbb{R}}(A)$ we shall write $\bar{A}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The derivative of the set A is denoted by A^d, whereas bc(A) stands for the set of all bilateral cluster points of A, and $\ell(A)$ is the family of all components of A.

If f is some function, then by the symbols G(f), C_f, $f_{|A}$ we denote, respectively, the graph of this function, the set of points of continuity of f and the restriction of f to the set A.

By $\mathbb{C}$ we denote the class of functions $f : \mathbb{R} \to \mathbb{R}$ possessing a connected graph. The family of all subsets of a fixed set A is denoted by 2^A.

Let H be any family of real functions of a real variable. Then we adopt the following notations:

$$H_{|A} = \{ h_{|A} : h \in H \},$$

$$C_H = \bigcap_{h \in H} C_h \quad \text{and} \quad \mathbb{D}_H = \mathbb{R} \setminus C_H.$$

Throughout the paper, we consider functions connected with the notion of the ideal. About the ideals under consideration we shall always assume that:

- if $I \subset 2^A$ is an ideal, then, for each $C \in \ell(A)$, $C \notin I$ and Int $A = \emptyset$ for any $A \in I$.

If $I \subset 2^A$ is an ideal, where $A \subset \mathbb{R}$, then $\overline{I} = \{ cl_A(B) : B \in I$ and $\forall_{C \in \ell(A)} \; C \not\subset cl_A(B) \}$.

Let $f : B \to \mathbb{R}$ (where $B \subset \mathbb{R}$) and let $A \subset B$. A number $\alpha \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is called a right-hand cluster point of the function f at the point x_0 with respect to the set A if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset A \cup \{x_0\}$ such that $x_n \geqslant x_0$, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = \alpha$. The set of all right-hand cluster points with respect to A is denoted by the symbol $L_A^+(f,x_0)$. In a similar way we define $L_A^-(f,x_0)$.

In the sequel, by an algebraic operation in the set of real functions of a real variable we shall understand addition, subtraction, multiplication, the maximum and the minimum of two functions.

So, let H stand for a family of functions $f : A \to \mathbb{R}$ (where $A \subset \mathbb{R}$).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Each result of an algebraic operation on functions of the family H is called a result of the first order on H. The set of all results of the first order on H will be denoted by the symbol $\Xi_1(H)$. Suppose that we have already defined by result of order n-1 ($n \geqslant 2$) and the family $\Xi_{n-1}(H)$. Each result of an algebraic operation on functions belonging to $\Xi_{n-1}(H)$ is called a result of the n-th order on H. The set of all results of the n-th order is denoted by $\Xi_n(H)$.

We say that a family of functions H is algebraically embedded in a family G if, for any $m \in \mathbb{N}$, $\Xi_m(H) \subset G$.

In monograph [5] R. Engelking defines a closed transformations as a continuous one transforming every closed set onto a closed set. The adoption of such a definition would not be convenient in our considerations. In this context, we shall adopt the following modification of this definition.

D e f i n i t i o n 1. Let $A_1 \subset A \subset \mathbb{R}$ and let $\mathcal{I} \subset 2^{A_1}$ be an ideal of sets. We say that a function $h : A \to \mathbb{R}$ is $\mathcal{I}$-closed with respect to the set A_1 if there exists a set $B \in \bar{\mathcal{I}}$ such that

1^0 for any $x \in A \setminus B$, the point $x \in bc(A_1 \setminus B)$ and $L^+_{A\setminus B}(h,x)(L^-_{A\setminus B}(h,x))$ is a dense set in each interval with end points belonging to this set (the situation when $+\infty$ or $-\infty$ is one of the end points of this interval is admitted here),

2^0 for any $D \in A \setminus B$ such that $D = cl_A(D)$, there exists $B_D \in \mathcal{I}$ such that $h(D \setminus B_D)$ is a closed set in $\mathbb{R}$.

D e f i n i t i o n 2. If $A_1 \subset A \subset \mathbb{R}$, then we say that the class H of functions defined on A is $\mathcal{I}$-closed with respect to the set A_1 if there exists an ideal $\mathcal{I} \subset 2^{A_1}$ such that, for some set $B \in \bar{\mathcal{I}}$, conditions 1^0 and 2^0 of definition 1 are satisfied for each function $h \in H$.

The adoption of the above definitions allows us to formulate the following theorem.

T h e o r e m 1. Let $\mathbb{K} \subset \mathbb{D}$ be a family of functions such that $C_{\mathbb{K}}$ is a dense set. Then, if the condition:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(*)\quad \begin{cases} \text{for any } x\in\mathbb{D}_{\mathbb{K}}, \text{ there exists } \delta>0 \text{ such that} \\ \mathbb{K}_{|[x-\delta,x+\delta]\cap C_{\mathbb{K}}\cup\{x\}} \text{ is an I-closed class with respect} \\ \text{to the set } [x-\delta,x+\delta]\cap C_{\mathbb{K}} \end{cases}$$

is satisfied, then $\mathbb{K}$ is algebraically embedded in $\mathbb{L}$.

P r o o f . Fix $x_0 \in \mathbb{D}_{\mathbb{K}}$ and let δ_{x_0} be a positive real number such that the class $\mathbb{K}_{|[x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}\cup\{x_0\}}$ is I-closed with respect to the set $[x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}$. This means that there exists an ideal I of subsets of the set $[x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}$ and a set $B_{x_0}\in \bar{I}$, such that, for any function $h\in\mathbb{K}_{|[x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}\cup\{x_0\}}$, conditions 1^0 and 2^0 of definition 1 are satisfied.

We shall first prove that:

$$(1)\quad \begin{cases} \text{there exists a set } A_{x_0} \text{ intersecting each component of} \\ \text{the set } [x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}, \text{ such that} \\ A_{x_0}\subset [x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}},\ x_0\in \mathrm{bc}(A_{x_0}) \text{ and } f_{|A_{x_0}\cup\{x_0\}} \\ \text{is continuous at } x_0 \text{ for any function } f\in\mathbb{K}. \end{cases}$$

Indeed, let us put

$$A_{x_0} = ([x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}})\setminus B_{x_0}.$$

Of course, $x_0\in \mathrm{bc}(A_{x_0})$ and, as can easily be seen, A_{x_0} intersects each component of the set $[x_0-\delta_{x_0},x_0+\delta_{x_0}]\cap C_{\mathbb{K}}$. So, let $f\in\mathbb{K}$. Denote $g = f_{|A_{x_0}\cup\{x_0\}}$. We shall show that g is continuous at x_0. For suppose to the contrary that there exists a sequence $\{y_n\}_{n=1}^{\infty}\subset A_{x_0}\cup\{x_0\}$ such that $y_n\longrightarrow x_0$ and $g(y_n)\neq g(x_0)$. Let $\alpha\neq g(x_0)$ be any cluster point of the sequence

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\{g(y_n)\}_{n=1}^{\infty}$ (we admit the possibility that $\alpha = +\infty$ or $\alpha = -\infty$). Without loss of generality let us assume that $\alpha > g(x_0)$. Then, in view of the assumptions we have made (cf. 1^0 of definition 1), it can be proved that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with the following properties: $\{x_n\}_{n=1}^{\infty} \subset A_{x_0}$, all terms of the sequence $\{x_n\}_{n=1}^{\infty}$ lie on the same side of the point x_0 (with no loss of generality we shall always assume in the sequel that $x_n \searrow x_0$), $g(x_n) \neq \alpha$ for $n = 1,2,\ldots$, $g(x_n) \to \alpha > g(x_0)$.

We shall now construct some sequence of sets $\{P_n\}_{n=1}^{\infty}$. For the purpose, let us consider two cases:

1. The singleton $\{x_n\}$ is a component of the set $[x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_{\mathbb{K}}$. Then $\{x_n\}$ is a component of the set A_{x_0} and we shall put $P_n = \{x_n\}$.

2. There exists $C_{x_n} \in \ell([x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_{\mathbb{K}})$ such that $x_n \in C_{x_n}$ and C_{x_n} is a nondegenerate interval. Then f is continuous on C_{x_n}. This means that the set $f^{-1}(\alpha) \cap C_{x_n}$ is closed in C_{x_n} and, in consequence, in view of the closedness of $B_{x_0} \cap C_{x_n}$ in C_{x_n}, we may infer that

$$G = C_{x_n} \setminus (B_{x_0} \cup f^{-1}(\alpha))$$

is an open set in C_{x_n} and $x_n \in G$. Then there exists an element $y \in G$ such that $[x_n,y] \subset G$ and $|y-x_n| < \frac{1}{2}|x_n-x_0|$. Let us then adopt $P_n = [x_n,y]$.

We shall now demonstrate that:

$$(2)\quad \begin{cases} \text{for any } \hat{I} \in I,\ x_0 \in \bigcup_{n=1}^{\infty} P_n \setminus \hat{I} \text{ and } \alpha \in f\Big(\overline{\bigcup_{n=1}^{\infty} P_n \setminus \hat{I}}\Big) \setminus \\ f\Big(\bigcup_{n=1}^{\infty} P_n \setminus \hat{I}\Big). \end{cases}$$

Indeed, let us fix $\hat{I} \in I$. Then $x_0 \notin \hat{I}$ and, of course, $x_0 \in \overline{\bigcup_{n=1}^{\infty} P_n}$.

Thus it remains to show that $\alpha \in f\Big(\overline{\bigcup_{n=1}^{\infty} P_n \setminus \hat{I}}\Big) \setminus f\Big(\bigcup_{n=1}^{\infty} P_n \setminus \hat{I}\Big)$.

For the purpose, note that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $$\overline{\bigcup_{n=1}^{\infty} P_n} = \bigcup_{n=1}^{\infty} P_n \cup \{x_0\}.$$

We shall now construct a sequence $\{x_n'\}_{n=1}^{\infty} \subset \bigcup_{n=1}^{\infty} P_n \setminus \hat{I}$ such that $x_n' \searrow x_0$ and $g(x_n') \to \alpha$.

So, let n be an arbitrary positive integer. If $x_n \in \hat{I}$, then $\{x_n\} \in \ell([x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_K)$ because, otherwise, $\{x_n\} \in I$ and, in view of the assumptions concerning the ideals considered, $\{x_n\}$ would not be a component of the set $[x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_K$. In this case, since $\operatorname{Int} \hat{I} = \emptyset$, there exists $x_n' \in P_n \setminus \hat{I}$, but (in view of the continuity of the function f on C_{x_n}) x_n' can be so chosen that

$$|f(x_n') - f(x_n)| < \frac{1}{n}.$$

Whereas if $x_n \notin \hat{I}$, then let us put $x_n' = x_n$. It can easily be verified that $\{x_n'\}_{n=1}^{\infty}$ satisfies the required conditions. The existence of such a sequence proves that $\alpha \in f\left(\overline{\bigcup_{n=1}^{\infty} P_n} \setminus \hat{I}\right)$. Since, in virtue of (3) and the definition of the intervals P_n, we may infer that $\alpha \notin f\left(\overline{\bigcup_{n=1}^{\infty} P_n} \setminus \hat{I}\right)$, therefore condition (2) has finally been proved.

Moreover, equality (3) proves that

$$\overline{\bigcup_{n=1}^{\infty} P_n} \subset ([x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_K \cup \{x_0\}) \setminus B_{x_0},$$

which allows us to deduce that

$$\overline{\bigcup_{n=1}^{\infty} P_n} = \operatorname{cl}_{[x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_K \cup \{x_0\}} \left(\bigcup_{n=1}^{\infty} P_n\right).$$

Relation (2) shows that condition (*) is not satisfied. (Of course, in (2) we may write $f_{|[x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap C_K \cup \{x_0\}}$ instead of f).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The contradiction obtained proves that the function g is continuous at the point x_0 and, thereby, the proof of (1) has been completed.

In virtue of (1), we may infer that, for any positive integer m, $\Xi_m(\mathbb{K}_{|A_{x_0}\cup\{x_0\}})$ is a family of functions continuous at x_0 (and thus, as can easily be seen, a family of continuous functions).

Making use of the above considerations, we shall now demonstrate that if h is a result of a finite order on $\mathbb{K}$, then $h\in\mathbb{L}$. To this effect, let us first notice that $C_h \supset C_{\mathbb{K}}$. Suppose that $h \notin \mathbb{L}$. Then there exist separated sets A and B such that $G(h) = A \cup B$. Denote $A' = \mathrm{proj}_{ox} A$, $B' = \mathrm{proj}_{ox} B$. Then $\mathbb{R} = A' \cup B'$, therefore A', B' are not separated. Assume, without loss of generality, that $A' \cap \overline{B'} \neq \emptyset$ and let $x_1 \in A' \cap \overline{B'}$. Then $x_1 \in D_{\mathbb{K}}$. For the point x_1, let us consider the number $\delta_{x_1} > 0$ and the set A_{x_1}, constructed in the first part of the proof. Then $x_1 \in \mathrm{bc}(A_{x_1})$, A_{x_1} intersects each component of the set $[x_1-\delta_{x_1}, x_1+\delta_{x_1}] \cap C_{\mathbb{K}}$ and $h_{|A_{x_1}\cup\{x_1\}}$ is continuous at x_1. Hence it appears that there exists a positive number $\delta_0 < \delta_{x_1}$ such that

(4)
$$A_{x_1} \cap (x_1 - \delta_0,\ x_1 + \delta_0) \subset A'.$$

Let $x_2 \in B'$. Then $x_1 \neq x_2$ and, with no loss of generality, let us adopt $x_2 > x_1$. In virtue of (4) and the evident fact that $A_{x_1} \cap [x_1, x_1+\delta_0] \neq \emptyset$, we may infer that there exists a point $x_1' \in C_{\mathbb{K}} \cap A' \cap [x_1, x_1+\delta_0]$. Let $x_1' \in C_{x_1'} \in \ell([x_1-\delta_{x_1}, x_1+\delta_{x_1}] \cap C_{\mathbb{K}})$. In view of the continuity of the function h on $C_{x_1'}$, we may deduce that $G(h_{|C_{x_1'}}) \subset A$ and, consequently, $C_{x_1'} \subset A'$.

So, let x be any element of the set $C_{\mathbb{K}} \cap [x_1, x_1') \setminus C_{x_1'}$ and let C_x stand for the component of the set $[x_1-\delta_{x_1}, x_1+\delta_{x_1}] \cap C_{\mathbb{K}}$ containing x. Then $A_{x_1} \cap C_x \neq \emptyset$ and, in consequence,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

reasoning analogously as above, we may conclude that $C_x \subset A'$. Consequently, there exists a number $\delta_o' \leqslant \delta_o$ such that

$$C_{\mathbb{K}} \cap [x_1,\ x_1 + \delta_o') \subset A'. \tag{5}$$

Let $\hat{x}_1 = \sup\{x \in (x_1, x_2] : C_{\mathbb{K}} \cap [x_1, x) \subset A'\}$ (by (5), such a bound exists and, of course, $\hat{x}_1 \leqslant x_2$). Then there may occur two cases:

I. $\hat{x}_1 \in B'$. In view of the definition of $\hat{x}_1$, it is not difficult to notice that $\hat{x}_1 \in \overline{A'} \cap B'$. Since $\hat{x}_1 > x_1$, therefore, reasoning analogously as in the case of the points x_1, x_2, one can show that

$$C_{\mathbb{K}} \cap (x_1'', \hat{x}_1) \subset B' \quad \text{for some} \quad x_1'' < \hat{x}_1.$$

The above inclusion, however, contradicts the definition of the point $\hat{x}_1$.

II. $\hat{x}_1 \in A'$. Then $\hat{x}_1 < x_2$. From the density of the set $C_{\mathbb{K}}$, in virtue of the definition of $\hat{x}_1$, one may infer that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ with the properties that: $\{z_n\}_{n=1}^{\infty} \subset B' \cap C_{\mathbb{K}}$ and $z_n \searrow \hat{x}_1$. This means that $\hat{x}_1 \in A' \cap \overline{B'}$, therefore $\hat{x}_1 \in \mathbb{D}_{\mathbb{K}}$ and, reasoning analogously as in the case of the point x_1, one can show that

$$C_{\mathbb{K}} \cap [\hat{x}_1, x^o) \subset A' \quad \text{for some} \quad x^o \in (\hat{x}_1, x_2].$$

However, this inclusion contradicts the fact that $\hat{x}_1 = \sup\{x \in (x_1, x_2] : C_{\mathbb{K}} \cap [x_1, x) \subset A'\}$.

Since the supposition that $h \notin \mathbb{K}$ has led us, in each of the cases possible to occur, to a contradiction, therefore $h \in \mathbb{K}$ and the proof of the theorem has thus been completed.

The theorem below will show that the assumption that the class of functions, considered in theorem 1, satisfies condition (*) is not accidental.

T h e o r e m 2. Let $\mathbb{K}$ be the family of functions, containing all constant functions and such that $\mathbb{D}^d = \emptyset$ and, for any $x \in \mathbb{R}$, only a finite number of functions of the fa-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

mily $\mathbb{K}$ are discontinuous at x. Then, if $\mathbb{K}$ is algebraically embedded in $\mathbb{L}$, it satisfies condition (*) from Theorem 1.

p r o o f . Let $x_0 \in \mathbb{D}_{\mathbb{K}}$ and let $\delta_{x_0} > 0$ be a number such that $[x_0-\delta_{x_0}, x_0+\delta_{x_0}] \setminus \{x_0\} \subset C_{\mathbb{K}}$. Let $\{f_1,\dots,f_l\}$ be the set of all functions of the family $\mathbb{K}$ which are not continuous at x_0. Consider the function

$$f = \sum_{i=1}^{l} (f_i - \mathrm{const}_{f_i(x_0)})^2. \tag{6}$$

Since $\mathbb{K}$ is algebraically embedded in $\mathbb{L}$, therefore f possesses a connected graph. Of course, f is a function of the first class of Baire, thus (cf. [1], [13]) there exist sequences $\{y_n^1\}_{n=1}^{\infty}$, $\{y_n^2\}_{n=1}^{\infty} \subset [x_0-\delta_{x_0}, x_0+\delta_{x_0}]$ such that $y_n^1 \nearrow x_0 \swarrow y_n^2$ and $\lim_{n\to\infty} f(y_n^1) = \lim_{n\to\infty} f(y_n^2) = f(x_0)$. In view of the continuity of the function f at each of the points y_n^i (i = 1,2; n = 1,2,...), we may infer that, for any i = 1, 2 and any $n \in \mathbb{N}$, there exist numbers ε_n^i with the properties that: $\lim_{n\to\infty} \varepsilon_n^i = 0$ (i=1,2); $(y_n^i-\varepsilon_n^i, y_n^i+\varepsilon_n^i) \subset C_{\mathbb{K}} \cap [x_0-\delta_{x_0}, x_0+\delta_{x_0}] \cap$ $\cap ((f(y_n^i) - \frac{1}{n}, f(y_n^i) + \frac{1}{n}))$, and the family $\{[y_n^i-\varepsilon_n^i, y_n^i+\varepsilon_n^i]\}_{n=1}^{\infty}$ is, for each i = 1, 2 the class of pairwise disjoint sets. Put

$$A_{x_0} = \bigcup_{n\in\mathbb{N}} \bigcup_{i=1}^{2} (y_n^i-\varepsilon_n^i, y_n^i+\varepsilon_n^i).$$

Then $x_0 \notin A_{x_0}$, $x_0 \in \mathrm{bc}(A_{x_0})$ and $A_{x_0} \subset C_{\mathbb{K}}$. It is easy to check that $f_{|A_{x_0} \cup \{x_0\}}$ is a continuous function, which, in view of (1), means that, for any $j \in \{1,\dots,l\}$, $f_{j|A_{x_0} \cup \{x_0\}}$ is a continuous function for any $k \in \mathbb{K}$.

We shall now demonstrate that $\mathbb{K}$ is I-closed with respect to the set $[x_0-\delta_{x_0}, x_0) \cup (x_0, x_0+\delta_{x_0}]$. Let I stand for the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ideal of countable subsets of the set $[x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}]$. Denote

$$B_1 = ([x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}]\setminus A_{x_0})\cap Q.$$

It is easily noticed that $([x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}])\setminus A_{x_0}$ is a closed set in $[x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}]$. Let us also notice that Q is dense in $[x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}]\setminus A_{x_0}$ (which follows from the disjointness of the intervals $[y_n^i-\varepsilon_n^i,y_n^i+\varepsilon_n^i]$). So, consider the set $B = cl_{[x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}]}(B_1)\in\bar{I}$.

It is easy to observe that

$$([x_0-\delta_{x_0},x_0)\cup(x_0,x_0+\delta_{x_0}])\setminus B = A_{x_0}\cup\{x_0\}.$$

We shall show that, with the set B thus defined, for each function $k\in K$, conditions 1° and 2° of definition 1 are satisfied. Indeed, if $x\in A_{x_0}\cup\{x_0\}$, then $x\in bc(A_{x_0})$. So, let $k\in K$ and let $h = k_{|[x_0-\delta_{x_0},x_0+\delta_{x_0}]}$. Then, in view of the continuity of $h_{|A_{x_0}\cup\{x_0\}}$, the sets $L^{\pm}_{A_{x_0}\cup\{x_0\}}(h,x)$ are singletons for any $x\in A_{x_0}\cup\{x_0\}$.

So, let $D\subset A_{x_0}\cup\{x_0\}$ be a closed set in $[x_0-\delta_{x_0},x_0+\delta_{x_0}]$. Then D is, of course, a compact set. Thus, in view of the continuity of $k_{|D}$, k(D) is a closed set, which ends the proof of the theorem.

REFERENCES

[1] A.M. Bruckner: Differentiation of real functions, Springer-Verlag (1978).

[2] A.M. Bruckner, J.G. Ceder: Darboux continuity, Jbr. Deutsch. Math. Verein 67 (1965) 93-117.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] A.M. Bruckner, J.G. Ceder: On the sum of Darboux functions, Proc. Amer. Math. Soc. 51 (1975) 97-102.

[4] A.M. Bruckner, J.G. Ceder, M. Weiss: Uniform limits of Darboux functions, Colloq. Math. 15 (1966) 65-67.

[5] R. Engelking: General Topology, Warszawa 1977.

[6] I. Farkova: Uber Darbouxsche Funktionen, Mat. Čas. 20 (1970) 185-194.

[7] R. Fleissner: A note on Baire 1 Darboux function, Real Anal. Exchange, 3 (1977-78) 104-106.

[8] R.G. Gibson: The uniform limit of connectivity functions, Real Anal. Exchange 11 (1985-86) 254-258.

[9] J.M. Jędrzejewski: Własności funkcji związane z pojęciem spójności, (habilitation thesis) - Acta Univ. Lodz. (1984) 1-84.

[10] J.S. Lipiński: On a problem of Bruckner and Ceder concerning the sum of Darboux functions, Proc. Amer. Math. Soc. 62 (1977) 57-61.

[11] H. Pawlak, R. Pawlak: Fundamental rings for the classes of Darboux functions (in press).

[12] R. Pawlak: On rings of Darboux functions, Colloq. Math. (in press).

[13] J. Young: A theorem in the theory of functions of a real variable, Rend. Circ. Mat. Palermo 24 (1907) 187-192.

INSTITUTE OF MATHEMATICS, ŁÓDŹ UNIVERSITY, 90-238 ŁÓDŹ, POLAND

Received May 16, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA

Vol. XXIII No 1 1990

Dan Tudor Vuza

UNIFORM CONTINUITY AND UNIFORM CONVERGENCE

In [2], p. 308, the relation between the uniform continuity of a function $f:\mathbb{R} \to \mathbb{R}$ and the uniform convergence of the sequence of functions $(f_n)_{n\geq 1}$ attached to f, where $f_n(x) = f(x + n^{-1})$, is considered. It is remarked that the uniform continuity of f implies the uniform convergence of (f_n), while the converse is not true. The counterexample given there uses Dirichlet's function ($f(x) = 1$ for x rational, $f(x) = 0$ for x irrational) for which the sequence (f_n) is constant at every $x \in \mathbb{R}$. Dirichlet's function is discontinuous at every point; it is therefore naturally to ask whether the converse implication is true when f belongs to the class of continuous functions. The purpose of this note is to show that the answer is negative even for the class of infinitely differentiable functions. In fact, we shall consider the problem in a more general setting, namely that of functions defined on a topological group.

T h e o r e m . Let G be a commutative metrizable nondiscrete and noncompact topological group and let $(x_n)_{n\geq 1} \subset G$ be a sequence such that $\lim_{n\to\infty} x_n = 0$. There is a function $f:G \to \mathbb{R}$ with the following properties:

i) f is continuous and bounded.

ii) $\lim_{n\to\infty} \sup_{x\in G} |f(x + x_n) - f(x)| = 0$.

iii) f is not uniformly continuous.

iv) If G is a Lie group, then f is infinitely differentiable.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . The proof is based on a series of lemmas. First of all, some preliminaries. By replacing, if necessary, G by its completion, we may assume that G is topologically complete. It is known [1] that there is a symmetric subadditive function $d:G \to \mathbb{R}_+$ which vanishes at 0 so that $(x,y) \mapsto d(x-y)$ is a distance which defines the topology of G.

For every $r>0$ and every $M \subset G$ we shall denote by $B_r(M)$ the set of those $x \in G$ such that there is $y \in M$ with $d(x-y)<r$. For every integer $k \geqslant 1$ we define $M^{(k)}$ as the set of all sums $x_1 + \dots + x_k$ with $x_1,\dots,x_k \in M$.

By a <u>filtration</u> on G we mean a sequence $\hat{V} = (V_n)_{n\geqslant 1}$ of subsets of G with the following properties:

a) $0 \in V_n$,

b) $V_n = -V_n$,

c) $V_{n+1}^{(3)} \subset V_n$.

Properties a) - c) are supposed to hold for every $n \geqslant 1$.

We shall say that the filtration $\hat{V}$ has a certain property provided that every V_n has that property; for instance, "$\hat{V}$ is nowhere dense" means that every V_n is nowhere dense.

To every filtration $\hat{V} = (V_n)_{n\geqslant 1}$ we associate the function $p_{\hat{V}}:G \to \mathbb{R}$ defined by

$$p_{\hat{V}}(x) = \inf\left\{\sum_{i=1}^{n} g_{\hat{V}}(x_i) \,\middle|\, n \geqslant 1,\ x_i \in G,\ \sum_{i=1}^{n} x_i = x\right\}$$

where

$$g_{\hat{V}}(x) = 1 \quad \text{if} \quad x \in G \setminus V_1,$$

$$g_{\hat{V}}(x) = 2^{-n} \quad \text{if} \quad x \in V_n \setminus V_{n+1},$$

$$g_{\hat{V}}(x) = 0 \quad \text{if} \quad x \in \bigcap_{n=1}^{\infty} V_n.$$

The properties of $p_{\hat{V}}$ we shall make use are gathered in the following lemma.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

L e m m a 1.

i) $0 \leqslant p_{\hat{V}}(x) \leqslant 1$ for every $x \in G$ and $p_{\hat{V}}(0) = 0$.

ii) $|p_{\hat{V}}(x) - p_{\hat{V}}(y)| \leqslant p_{\hat{V}}(x - y)$ for every $x, y \in G$,

iii) $V_n \subset p_{\hat{V}}^{-1}([0, 2^{-n}])$ for every $n \geqslant 1$.

iv) $p_{\hat{V}}^{-1}([0, 2^{-2}]) \subset V_1$.

v) If $\hat{V}$ is open, then $p_{\hat{V}}$ is continuous.

vi) If $\hat{V}_1 = (V_{1n})_{n \geqslant 1}$, $\hat{V}_2 = (V_{2n})_{n \geqslant 1}$ and $V_{1n} \subset V_{2n}$ for every $n \geqslant 1$, then $p_{\hat{V}_2}(x) \leqslant p_{\hat{V}_1}(x)$ for every $x \in G$.

P r o o f . i), iii) and vi) are immediate.

ii) follows from the fact that $p_{\hat{V}}$ is symmetric and subadditive.

For v), use first iii) to deduce the continuity of $p_{\hat{V}}$ at 0 and then use ii) to deduce its continuity at every point.

The proof of iv) is essentially based on the argument of Proposition 2 in [1], §1, n° 4 and proceeds as follows. We shall prove the inequality

$$(1) \qquad 2^{-1} g_{\hat{V}}\left(\sum_{i=1}^{n} x_i\right) \leqslant \sum_{i=1}^{n} g_{\hat{V}}(x_i).$$

Once this is proved, we shall derive from it the relation $2^{-1} g_{\hat{V}}(x) \leqslant p_{\hat{V}}(x)$ for every $x \in G$ and the latter will imply the assertion in iv).

The verification of (1) is done by induction on n. To avoid complications, we shall write g instead of $g_{\hat{V}}$. Clearly, (1) is true for $n = 1$. We assume it to be true for $n - 1$ and we prove it for n. Let

$$s = \sum_{i=1}^{n} g(x_i),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$K = \left\{ j \mid 1 \leqslant j \leqslant n, \sum_{i=1}^{j} g(x_i) < 2^{-1}s \right\},$$

$$k = \max K \text{ if } K \neq \emptyset,$$
$$= 0 \quad \text{if } K = \emptyset.$$

We have

$$\sum_{i=1}^{n} g(x_i) = \sum_{1 \leqslant i \leqslant k} g(x_i) + g(x_{k+1}) + \sum_{k+2 \leqslant i \leqslant n} g(x_i)$$

(convention: $\sum_{i \in \emptyset} g(x_i) = 0$). Applying the induction hypothesis to the sums in the right side of the above relation we obtain

$$2^{-1} g\left(\sum_{1 \leqslant i \leqslant k} x_i \right) \leqslant \sum_{1 \leqslant i \leqslant k} g(x_i),$$

$$2^{-1} g\left(\sum_{k+2 \leqslant i \leqslant n} x_i \right) \leqslant \sum_{k+2 \leqslant i \leqslant n} g(x_i).$$

But $\sum_{1 \leqslant i \leqslant k} g(x_i) < 2^{-1}s$ and $\sum_{1 \leqslant i \leqslant k+1} g(x_i) \geqslant 2^{-1}s$, by the definition of k; hence, $\sum_{k+2 \leqslant i \leqslant n} g(x_i) \leqslant 2^{-1}s$. It follows from the above relations that

$$(2) \qquad \begin{cases} g\left(\sum_{1 \leqslant i \leqslant k} x_i \right) < s, \\ g(x_{k+1}) \leqslant s, \\ g\left(\sum_{k+2 \leqslant i \leqslant n} x_i \right) \leqslant s. \end{cases}$$

The definition of a filtration implies that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $$2^{-1}g(y_1 + y_2 + y_3) \leqslant \max(g(y_1), g(y_2), g(y_3))$$

for every $y_1, y_2, y_3 \in G$. Combining (2) and (3) we obtain that (1) is true for n.

L e m m a 2. Let $(x_n)_{n\geqslant 1} \subset G$ be such that $\lim_{n\to\infty} x_n = 0$. There is a closed nowhere dense filtration $\hat{V}$ such that $\lim_{n\to\infty} p_{\hat{V}}(x_n) = 0$.

P r o o f . We shall denote by X_n the compact set $\{x_n | n \geqslant m\} \cup \{-x_n | n \geqslant m\} \cup \{0\}$. We distinguish between two cases.

Case A : G is not locally compact. We determine a sequence $(m_k)_{k\geqslant 1}$ of integers such that

(4) $$\sum_{k=1}^{\infty} 3^{k-1} \sup d(X_{m_k}) < \infty$$

and we define $\hat{V} = (V_n)_{n\geqslant 1}$ by letting V_n to be the set of all sums $\sum_{k=n}^{\infty} y_k$ with $y_k \in X_{m_k}^{(3^{k-n})}$ for $k \geqslant n$ (the sums converge by virtue of (4) and the completeness of G). It follows from (4) that the map

$$(y_k)_{k\geqslant n} \longmapsto \sum_{k=n}^{\infty} y_k$$

from the compact space $\prod_{k=n}^{\infty} X_{m_k}^{(3^{k-n})}$ to G is continuous; hence, V_n is compact. As G is not locally compact, V_n is nowhere dense.

Case B : G is locally compact. Let μ be a Haar measure on G. We shall construct inductively a sequence $(m_k)_{k\geqslant 1}$ of integers and a sequence $(r_k)_{k\geqslant 1}$ of strictly positive real numbers so that the following hold:

a) $$\mu\left(B_{r_k}\left(\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}\right)\right) \leqslant k^{-1} \quad \text{for every } k \geqslant 1.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

b) $\sum_{i=j+1}^{k} 3^{i-1} \sup d(X_{m_i}) \leqslant 2^{-1} r_j$ for every $k \geqslant 2$ and $1 \leqslant j \leqslant k-1$.

This is done as follows. Put $m_1 = 1$. Assuming $m_1,\dots,m_k$ and $r_1,\dots,r_{k-1}$ to be defined, let us define m_{k+1} and r_k. As $\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}$ is an at most countable compact set and G is non-discrete, we have

$$0 = \mu\left(\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}\right) = \lim_{\substack{r \to 0 \\ r>0}} \mu\left(B_r\left(\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}\right)\right).$$

There is therefore $r_k > 0$ such that

$$\mu\left(B_{r_k}\left(\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}\right)\right) \leqslant k^{-1}.$$

As we have by the induction hypothesis

$$\sum_{i=j+1}^{k} 3^{i-1} \sup d(X_{m_i}) < 2^{-1} r_j, \quad 1 \leqslant j \leqslant k-1,$$

we may choose m_{k+1} large enough so that

$$\sum_{i=j+1}^{k+1} 3^{i-1} \sup d(X_{m_i}) < 2^{-1} r_j, \quad 1 \leqslant j \leqslant k.$$

We define the filtration $\hat{V} = (V_n)_{n \geqslant 1}$ by letting V_n to be the set of all sums $\sum_{k=n}^{\infty} y_k$ with $y_k \in X_{m_k}^{(3^{k-n})}$ for $k \geqslant n$. As in Case 1, one sees that V is compact. To see that V is nowhere dense, remark that property b) implies

$$V_1 \subset B_{r_k}\left(\sum_{i=1}^{k} X_{m_i}^{(3^{i-1})}\right)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for every $k \geqslant 1$. Consequently, $\mu(V_1) = 0$ by property a), which implies that V_n is nowhere dense for every $n \geqslant 1$.

To see that $\lim_{n\to\infty} p_{\hat{V}}(x_n) = 0$ in both Cases A and B, observe that the relation $X_{m_n} \subset V_n$ implies, by virtue of iii) in Lemma 1, that $p_{\hat{V}}(x_k) \leqslant 2^{-n}$ whenever $k \geqslant m_n$.

L e m m a 3. Let $(x_n)_{n\geqslant 1} \subset G$ be such that $\lim_{n\to\infty} x_n = 0$ and let $r > 0$. There is a sequence $(f_m)_{m\geqslant 1}$ of functions $f_m : G \to \mathbb{R}$ with the following properties:

i) The support of each f_m is contained in $\{x \mid x \in G, d(x) < r\}$.

ii) $0 \leqslant f_m(x) \leqslant 1$ for $x \in G$.

iii) Each f_m is continuous.

iv) $\lim_{n\to\infty} \sup_{\substack{m\geqslant 1 \\ x\in G}} |f_m(x + x_n) - f_m(x)| = 0$.

v) The sequence (f_m) is not equicontinuous at 0.

vi) If G is a Lie group, then each f_m is infinitely differentiable.

P r o o f . By Lemma 2, there is a closed nowhere dense filtration $\hat{V} = (V_n)_{n\geqslant 1}$ such that $\lim_{n\to\infty} p_{\hat{V}}(x_n) = 0$. Define the sequence $\hat{V}_m = (V_{mn})_{n\geqslant 1}$ of open filtrations by

$$V_{mn} = B_{(3^{n_m})^{-1}}(V_n).$$

Let $\varphi : G \to \mathbb{R}$ be a uniformly continuous function such that $0 \leqslant \varphi(x) \leqslant 1$ for $x \in G$, $\varphi(x) = 1$ for $d(x) \leqslant 3^{-1}r$ and $\varphi(x) = 0$ for $d(x) \geqslant 2^{-1}r$. We take $\varphi p_{\hat{V}_m}$ as f_m. By v) in Lemma 1, f_m is continuous. We have

$$|f_m(x+x_n) - f_m(x)| \leqslant |\varphi(x+x_n) - \varphi(x)| + |p_{\hat{V}_m}(x+x_n) - p_{\hat{V}_m}(x)| \leqslant$$

$$\leqslant |\varphi(x+x_n) - \varphi(x)| + p_{\hat{V}_m}(x_n) \leqslant$$

$$\leqslant |\varphi(x+x_n) - \varphi(x)| + p_{\hat{V}}(x_n)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

by ii) and vi) in Lemma 1. The property iv) is then a consequence of the above inequalities.

To prove v), it suffices to show that the sequence $(p_{\hat{V}_m})$ is not equicontinuous at 0. Otherwise, the set

$$W = \bigcap_{m=1}^{\infty} p_{\hat{V}_m}^{-1}([0,2^{-2}])$$

would be a neighbourhood of 0. But, according to iv) in Lemma 1, we have $p_{\hat{V}_m}^{-1}([0,2^{-2}]) \subset V_{m1}$ which implies

$$W \subset \bigcap_{m=1}^{\infty} V_{m1} = V_1.$$

As V_1 is nowhere dense, we have arrived at a contradiction.

In the case when G is a Lie group, we may choose φ to be infinitely differentiable and we may apply a regularization procedure to $p_{\hat{V}_m}$. Namely, there is a sequence $(\varphi_k)_{k\geq 1}$ of infinitely differentiable functions $\varphi_k : G \to \mathbb{R}_+$ with the following properties:

a) The support of φ_k is contained in $\{x \mid x \in G,\ d(x) < k^{-1}\}$.

b) $\int \varphi_k d\mu = 1$ (μ being a Haar measure on G).

It is well known that for every continuous function $f : G \to \mathbb{R}$, each function $f * \varphi_k$ is infinitely differentiable and the sequence $(f * \varphi_k)_{k\geq 1}$ converges to f uniformly on compact subsets of G (* denotes convolution). We then define f_m by

$$f_m = \varphi(p_{\hat{V}_m} * \varphi_{k_m})$$

where the sequence of integers (k_m) is determined so that the sequence $(p_{\hat{V}_m} * \varphi_{k_m})$ would not be equicontinuous at 0. The property iv) is a consequence of the inequalities

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sup_{x\in G} |(f*\varphi_k)(x+y) - (f*\varphi_k)(x)| \leqslant$$

$$\leqslant \sup_{x\in G} \int |f(x+y-z) - f(x-z)|\ \varphi_k(z)\ d\mu(z) \leqslant$$

$$\leqslant \sup_{x,z\in G} |f(x+y-z) - f(x-z)| \leqslant \sup_{u\in G} |f(u+y) - f(u)|.$$

We may now conclude the proof of our theorem. As G is not compact, there are $\delta > 0$ and $(y_m)_{m\geqslant 1} \subset G$ such that $d(y_m - y_{m'}) \geqslant \delta$ for $m \neq m'$. Let $(f_m)_{m\geqslant 1}$ be the sequence provided by Lemma 3, with $r = 3^{-1}\delta$. We define $f:G \longrightarrow \mathbb{R}$ by

$$f(x) = f_m(x - y_m) \quad \text{for} \quad x \in B_{3^{-1}\delta}(\{y_m\}),$$

$$f(x) = 0 \quad \text{for} \quad x \in G \setminus \bigcup_{m=1}^{\infty} B_{3^{-1}\delta}(\{y_m\}).$$

The function f has all the required properties.

REFERENCES

[1] N. B o u r b a k i : Topologie générale, ch. 9. Hermann, Paris, 1958.

[2] S. M a r c u s : Şocul matematicii [The mathematical shock. In Romanian, with an abstract in English]. Ed. Albatros, Bucharest, 1987.

DEPARTMENT OF MATHEMATICS, THE NATIONAL INSTITUTE FOR SCIENTIFIC AND TECHNICAL CREATION, 79622 BUCHAREST, ROMANIA

Received May 16, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Joanna Szymańska-Zagrodny

SOME SUFFICIENT CONDITIONS IN CONSTRAINED NONSMOOTH OPTIMIZATION

1. Introduction

It is well-known that, in the inequality constrained optimization, the Kuhn-Tucker conditions are sufficient for optimality, if the functions involved are convex. Various classes of functions have been investigated for the purpose of weakening this assumption. Recently, Hanson [2] has considered a differentiable function f from $\mathbb{R}^n$ into $\mathbb{R}$ for which there exists a vector function $\eta(x,u) \in \mathbb{R}^n$ such that

$$f(x) - f(u) \geqslant \eta^T(x,u)\, \nabla f(u), \tag{1.1}$$

where ∇f denotes the gradient of f. He has shown that, if the objective function and the constraints of a nonlinear programming problem satisfy (1.1) with the same $\eta(x,u)$, then the Kuhn-Tucker conditions are sufficient for optimality. Correspondingly to the definitions of quasi-convexity and pseudo-convexity, he has also defined two more general classes of functions, namely the class for which

$$f(x) \leqslant f(u) \Longrightarrow \eta^T(x,u)\, \nabla f(u) \leqslant 0 \tag{1.2}$$

and that for which

$$\eta^T(x,u)\, \nabla f(u) \geqslant 0 \Longrightarrow f(x) \geqslant f(u). \tag{1.3}$$

It turns out that these definitions are surprisingly general (see [3], [4]).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Jeyakumar [4] has proved the Kuhn-Tucker conditions for the nondifferentiable case, assuming that the functions involved are invex.

In this paper we extend the concept of invexity, giving some more comprehensive class of functions which ensure that the Kuhn-Tucker conditions are sufficient for optimality. We express them in terms of Dini generalized subdifferentials and employ for nonsmooth problems. We use the variable Lagrange multipliers method considered by Studniarski [5]. An example is included to show that the ordinary Lagrange multipliers method fails in the situation considered here and the more subtle tools must be used.

2. Notation and preliminaries

Let X be a topological vector space and f a function from X into $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ stands for the set $\mathbb{R} \cup \{-\infty, +\infty\}$.

D e f i n i t i o n 2.1. The lower and upper Dini directional derivatives of the function f at $a \in \operatorname{dom} f = \{x : f(x) \neq \pm\infty\}$ in the direction $x \in X$ are defined by

$$D'_-f(a;x) := \liminf_{t \downarrow 0} [f(a+tx) - f(a)]\, t^{-1}$$

and

$$D'_+f(a;x) := \limsup_{t \downarrow 0} [f(a+tx) - f(a)]\, t^{-1},$$

respectively. If the function f has the Fréchet derivative $f'(a)$ at the point a, then

$$D'_-f(a;x) = f'(a)x = D'_+f(a;x) \quad \text{for all} \quad x \in X. \tag{2.1}$$

D e f i n i t i o n 2.2. A function $f: \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be approximately quasidifferentiable at $a \in X$, if f is continuous in a neighbourhood of a and if there exists a nonempty convex compact subset $\underset{\sim}{\partial} f(a) \subset \mathbb{R}^n$ such that

$$D'_+f(a;x) \leqslant \max_{v \in \underset{\sim}{\partial} f(a)} v^T x \quad \text{for all} \quad x \in \mathbb{R}^n.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We will call the set $\widetilde{\partial}f(a)$ an approximate quasidifferential of f at a.

The notion of approximately quasidifferentiable function is used in [4] to obtain necessary and sufficient optimality conditions. This is homologous to the concept of function possessing an upper convex approximation used by Studniarski in [6].

D e f i n i t i o n 2.3. A function $f: \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be invex on a subset $C \subset \mathbb{R}^n$ at $a \in C$ with respect to $\widetilde{\partial}f(a)$, if for some function $\eta : X \times X \to \mathbb{R}^n$ the following relations hold

$$f(x) \geqslant f(a) + \eta^T(x,a)v \quad \text{for all} \quad x \in C,$$

$$\eta(x,a) \in \operatorname{Cone}(C-a) \quad \text{for all} \quad x \in C,$$

$$v \in \widetilde{\partial}f(a),$$

where $\operatorname{Cone}(C-a) := \bigcup_{\lambda>0} \lambda(C-a)$.

D e f i n i t i o n 2.4. The normal cone to a subset $C \subset \mathbb{R}^n$ at a point a is defined by

$$N(C,a) := \left\{ v \in \mathbb{R}^n;\ v^T(x-a) \leqslant 0 \quad \text{for all } x \in C \right\}.$$

Let $f_0, f_1, \ldots, f_m$ be extended real-valued functions, i.e. $f_i : X \to \overline{\mathbb{R}}$, and $a \in \operatorname{dom} f \cap C$, where C is a nonempty subset of X. By $f(x)$ we denote the vector $(f_1(x), f_2(x), \ldots, f_m(x))^T$.

Let us consider the problem

$$P(f) : \begin{cases} \text{minimize } f_0(x), \\ \quad x \in C \\ \text{subject to } f(x) \leqslant 0. \end{cases}$$

Let $L(x,\lambda): X \times \mathbb{R}^n \to \overline{\mathbb{R}}$ denote, as usual, the Lagrangian function, i.e.

$$L(x,\lambda) := f_0(x) + \lambda^T f(x).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Sufficiency for global optimality

We shall present an extension of Hanson's result [2] (Theorem 2.1) to a nondifferentiable problem. The result involves Studniarski's method of variable multipliers (see [5]). As an application we shall show that it yields Theorem 5.1 from [4].

T h e o r e m 3.1. If for $x_0 \in C \cap \operatorname{dom} f$ the following two assumptions hold

(i) there exists a function $\eta: C \to X$ such that, for all $x \in C$,

$$D'_- f_0(x_0;\eta(x)) \geqslant 0 \Longrightarrow f_0(x) \geqslant f_0(x_0), \tag{3.1}$$

$$f_i(x) \leqslant f_i(x_0) \Longrightarrow D'_+ f_i(x_0;\eta(x)) \leqslant 0, \quad i=1,\dots,m, \tag{3.2}$$

(ii) for each real sequence $t_k \downarrow 0$ there exists vector $\lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m$ satisfying the conditions

$$\lambda \geqslant 0, \tag{3.3}$$

$$\lambda^T f(x_0) = 0, \tag{3.4}$$

$$\limsup_{k\to\infty} \left[L(x_0+k_k\eta(x),\lambda) - L(x_0,\lambda)\right] t_k^{-1} \geqslant 0 \text{ for all } x \in C, \tag{3.5}$$

then x_0 is a minimizer of $P(f)$.

P r o o f . Suppose that x_0 is not minimal. Then there exists a point $x \in C$ such that $f_0(x) < f_0(x_0)$ and $f(x) \leqslant 0$ and, since (3.1) hold, we get $D'_- f_0(x_0;\eta(x)) < 0$. We can choose a sequence $t_k \downarrow 0$ such that

$$\lim_{k\to\infty} \left[f_0(x_0+t_k\eta(x)) - f_0(x_0)\right] t_k^{-1} = D'_- f_0(x;\eta(x)).$$

For this sequence we can choose $\lambda \in \mathbb{R}^m$ satisfying (3.3)-(3.5).

As a consequence of (3.3), (3.4) for all $i \in I := \{i;\lambda_I \neq 0\}$ we obtain that $f_i(x_0) = 0$ and this gives the inequality $f_i(x) \leqslant f_i(x_0)$, since $f_i(x) \leqslant 0$. Thus, by (3.2), we get

$$D'_+ f_i(x_0;\eta(x)) \leqslant 0 \quad \text{for all } i \in I.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence, we have

$$\limsup_{k\to\infty} \left[L(x_o+t_k\eta(x),\lambda) - L(x_o,\lambda)\right] t_k^{-1} =$$

$$= \limsup_{k\to\infty} \left\{ f_o(x_o+t_k\eta(x)) - f_o(x_o) + \sum_{i\in I} \lambda_i \left[f_i(x_o+t_k\eta(x)) - f_i(x_o)\right] \right\} t_k^{-1} \leqslant$$

$$\leqslant D'_- f_o(x_o;\eta(x)) + \sum_{i\in I} \lambda_i D'_+ f_i(x_o;\eta(x)) < 0,$$

but this gives a contradiction with (3.5).

E x a m p l e . Let $f_i : \mathbb{R}^2 \longrightarrow \bar{\mathbb{R}}$, $i = 0,1,2$. We shall consider the following problem

$$\begin{cases} \underset{C}{\text{minimize}}\ f_o(x,y), \\ \text{subject to } f_1(x,y) \leqslant 0, \\ f_2(x,y) \leqslant 0, \end{cases}$$

where $C := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0\}$,

$$f_o(x,y) = \begin{cases} y^2 - y & \text{for} \quad x = 0, \\ \left(\frac{1}{2}\right)^i + y^2 - y & \text{for} \quad |x| \in \left[\left(\frac{1}{2}\right)^{i+1}, \left(\frac{1}{2}\right)^i\right],\ i=1,2,\ldots, \\ 0 & \text{for} \quad |x| > \frac{1}{2}, \end{cases}$$

$$f_1(x,y) = \begin{cases} y - \frac{3}{4}x & \text{for} \quad x \geqslant 0, \\ +\infty & \text{for} \quad x < 0, \end{cases}$$

$$f_2(x,y) = \begin{cases} y - x & \text{for} \quad x \geqslant 0, \\ +\infty & \text{for} \quad x < 0. \end{cases}$$

One can check that the function f_o is not differentiable at $(0,0)$ in the direction $(1,0)$ and $f_o(0,0) = 0$. We shall prove that the origin is a minimizer of the problem. Let $\eta(x,y) := (x,y)$ and fix $(x,y) \in C$. Taking $t_i = \left(\frac{1}{2}\right)^{i-1} x$ for $x > 0$, we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$D'_- f_0((0,0);(x,y)) = \lim_{i\to\infty} \left[f_0((0,0) + t_i(x,y))\right] t_i^{-1} = x-y,$$

since for $x = 0$ we have $D'_- f_0((0,0);(0,y)) = -y$; therefore f_0 satisfies (3.1) on C. Because we have $D'_+ f_1((0,0);\eta(x,y)) = y - \frac{3}{4}x$ and $D'_+ f_2((0,0);\eta(x,y)) = y-x$, so f_1, f_2 satisfy (3.2) on C. We have

$$(3.6)\qquad \limsup_{k\to\infty} \left[L((0,0)+t_k(x,y),\lambda) - L((0,0),\lambda)\right] t_k^{-1} = \alpha x - y + \lambda_1\left(y - \frac{3}{4}x\right) + \lambda_2(y-x),$$

where $\alpha \in [1,2]$ and α depends on the chosen sequence t_k. Notice that for all $(x,y) \in C$ we can choose vectors $\lambda_1, \lambda_2 \geq 0$ such that (3.6) is nonnegative. However we have

$$\text{(i)}\quad \limsup_{k\to\infty} \left[L((0,0)+t_k(\tfrac{5}{3},1),\lambda) - L((0,0),\lambda)\right] t_k^{-1} = \alpha\frac{5}{3} - 1 + \lambda_1(1-\frac{5}{4}) + \lambda_2(1-\frac{5}{3}) \text{ for } x = \frac{5}{3},\ y = 1,$$

so for this direction $\lambda_1 = \lambda_2 = 0$;

$$\text{(ii)}\quad \limsup_{k\to\infty} \left[L((0,0)+t_k(0,1),\lambda) - L((0,0),\lambda)\right] t_k^{-1} = -1 + \lambda_1 + \lambda_2 \quad \text{for } x = 0,\ y = 1;$$

so λ_1, λ_2 must satisfy the inequality $\lambda_1 + \lambda_2 \geq 1$.

The above example shows that, in this case, the method of using given a priori multipliers does not have a desirable effect. The more subtle tools given by Theorem 3.1 has to be used.

4. Consequences

4.1. Differentiable problems

From Theorem 3.1 we can conclude the sufficient conditions obtained in the differentiable case in [2], [3]. Below we shall present them.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $f_0, f_1, \ldots, f_m$ be the Fréchet differentiable functions defined on a set $C \subset \mathbb{R}^n$. The problem P(f) subject to above additional assumptions will be denoted by $P^1(f)$. The following result is an extension of Theorem 2.1 in [2].

C o r o l l a r y 4.1. Let a given vector-function $\eta(x_1,x_2)\colon C\times C \to \mathbb{R}$ satisfy for all $x_1, x_2 \in C$ the following conditions

$$\eta^T(x_1,x_2)\nabla f_0(x_2) \geqslant 0 \Longrightarrow f_0(x_1) \geqslant f_0(x_2), \tag{4.1}$$

$$f_i(x_1) \leqslant f_i(x_2) \Longrightarrow \eta^T(x_1,x_2)\nabla f_i(x_2) \leqslant 0, \quad i=1,\ldots,m. \tag{4.2}$$

If there exist $x_0 \in C$ and a vector $v_0 \in \mathbb{R}^m$ such that

$$v_0 \geqslant 0, \tag{4.3}$$

$$v_0^T f(x_0) = 0, \tag{4.4}$$

$$\nabla f_0(x_0) + v_0^T f(x_0) = 0, \tag{4.5}$$

then x_0 is minimal in $P^1(f)$.

P r o o f . According to (2.1), we have $D'_- f_i(x_0;v) = v^T\nabla f_i(x_0) = D'_+ f_i(x_0;v)$, $i=0,1,\ldots,m$, for all $v \in \mathbb{R}^m$. Putting $x_2 = x_0$ in (4.1) and (4.2), we obtain (3.1) and (3.2), respectively. Conditions (3.3) and (3.4) are satisfied because of (4.3) and (4.4), respectively. To check (3.5) let us take $\lambda = v_0$. Then for an arbitrary sequence $t_k \downarrow 0$, by (4.5), we have

$$\limsup_{k\to\infty} \left[L(x_0+t_k\eta(x),v_0) - L(x_0,v_0)\right] t_k^{-1} =$$

$$= \limsup_{k\to\infty} \left\{ f_0(x_0+t_k\eta(x)) - f_0(x_0) + \sum_{i\in J} v_i \left[f_i(x_0+t_k\eta(x)) - f_i(x_0)\right] \right\} t_k^{-1} =$$

$$= \eta^T(x)\left[\nabla f_0(x_0) + v_0^T\nabla f(x_0)\right] = 0.$$

By using the same method as in the proof of Corollary 4.1, we obtain .

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

C o r o l l a r y 4.2 (see [3], Proposition 2.1). A sufficient condition for x_0 to be optimal in $P^1(f)$ is that $f_0(x)$ and $f(x)$ satisfy, with respect to a common function $\eta(x)\colon C \to \mathbb{R}$, the following conditions

$$f_0(x)-f_0(x_0) \geqslant \eta^T(x)\nabla f_0(x_0) \quad \text{for all } x \in \{x \in C; f(x) \leqslant 0\}, \tag{4.6}$$

$$-f_i(x_0) \geqslant \eta^T(x)\nabla f_i(x_0) \quad \text{for all } i \in \{i;\ f_i(x_0)=0\}, \tag{4.7}$$

and the Kuhn-Tucker conditions (4.3)-(4.5) hold for both $x_0 \in C$ and $v_0 \in \mathbb{R}^m$.

4.2. Invex problem

Now, as an application of the Theorem 3.1 we are able to obtain some sufficient optimality conditions in nondifferentiable case, more precisely, when functions are invex.

Consider the following problem (cf. [4], Theorem 5.1)

$$P^2(f)\colon \begin{cases} \text{minimize } f_0(x), \\ x \in X_0 \subset \mathbb{R}^n \\ \text{subject to } f_i(x) \leqslant 0, \quad i = 1,2,\ldots,m, \ x \in C, \end{cases}$$

where $C \subset \mathbb{R}^n$ is a nonempty convex set and f_0, f_i, $i=1,2,\ldots,m$, are real functions defined on an open set $X_0 \subset \mathbb{R}^n$.

Assume that all functions f_0, f_i are approximately quasidifferentiable at $a \in C$ with approximate quasidifferentials $\widetilde{\partial} f_i(a)$, $i = 0,1,\ldots,m$.

C o r o l l a r y 4.4. Let a be a feasible point of $P^2(f)$. Assume that there exists a vector $\lambda \in \mathbb{R}^m$ such that the Kuhn-Tucker conditions

$$\lambda_i \geqslant 0, \tag{4.8}$$

$$\lambda_i f_i(a) = 0, \quad i = 1,2,\ldots,m, \tag{4.9}$$

$$0 \in \widetilde{\partial} f_0(a) + \sum_{i=1}^{m} \lambda_i \widetilde{\partial} f_i(a) + N(C,a) \tag{4.10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

hold at a. If the functions f_0, f_i, $i = 1,2,\ldots,m$, are invex at a on C with respect to $\widetilde{\partial} f_i(a)$ for some function $\eta : X_0 \times X_0 \to \mathbb{R}^n$, $X_0 \subset \mathbb{R}^n$, then a is a minimizer of $\bar{P}^2(f)$.

P r o o f . Let $\eta(x) := \eta(x,a)$ for all $x \in C$ and let $D'_+ f_0(a;\eta(x)) \geqslant 0$ at a fixed point $x \in C$. If $f(x) < f(x_0)$, then, by the assumptions, we obtain

$$0 > f(x) - f(a) \geqslant \max_{v \in \widetilde{\partial} f(a)} \eta^T(x) v \geqslant D'_+ f_0(a;(x)).$$

Hence $D'_+ f_0(a;\eta(x)) < 0$; so we have a contradiction to the supposition. Therefore, (3.1) holds.

Let $f_i(x) \leqslant f_i(a)$ for some i and $x \in C$. Then we obtain

$$0 \geqslant f_i(x) - f_i(a) \geqslant \max_{v \in \widetilde{\partial} f_i(a)} \eta^T(x) v \geqslant D'_+ f_i(a;\eta(x))$$

what implies (3.2). Formulas (4.8) and (4.9) force (3.3) and (3.4), respectively. To prove (3.5) let us fix $x \in C$ and a sequence $t_k \downarrow 0$. Taking $\lambda = (\lambda_1,\ldots,\lambda_m)$ satisfying (4.8)÷(4.10), we get

$$\left\{ f_0(a+t_k\eta(x)) - f_0(a) + \sum_{i=1}^{m} \lambda_i \left[f_i(a+t_k\eta(x)) - f_i(a) \right] \right\} t_i^{-1} \geqslant$$

$$\max_{v \in \widetilde{\partial} f_0(a)} \eta^T(x) v + \sum_{i=1}^{m} \lambda_i \max_{v \in \widetilde{\partial} f_i} \eta^T(x) v.$$

Since $0 \in \widetilde{\partial} f_0(a) + \sum_{i=1}^{m} \lambda_i \widetilde{\partial} f_i(a) + N(C,a)$, we get

$$\limsup_{k \to \infty} \left[L(a+t_k\eta(x),\lambda) - L(a,\lambda) \right] t_k^{-1} \geqslant 0.$$

This ends the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] F.H. Clarke: Optimization and nonsmooth analysis, Wiley Introscience, New York, 1983.

[2] M.A. Hanson: On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. 80 (1981) 545-550.

[3] M.A. Hanson, B. Mond: Necessary and sufficient conditions in constrained optimization, Math. Program. 37 (1987) 51-58.

[4] V. Jeyakumar: On optimality conditions in nonsmooth inequality constrained minimization, Numer. Funct. Anal. Optim. 9 (1987) 535-546.

[5] M. Studniarski: Warunki optymalności wyższych rzędów dla niegładkich zadań programowania matematycznego, Acta Univ. Lodz. (1987)

[6] M. Studniarski: Mean value theorems and sufficient optimality conditions for nonsmooth functions, J. Math. Anal. Appl. 2 (1985) 313-326.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF ŁÓDŹ,
93-590 ŁÓDŹ, POLAND

Received May 23, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Rüdiger Günttner

ON THE UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY THE EULER MEANS OF FOURIER SERIES

1. The theorems

Let C be the space of all 2π-periodic real- or complex-valued functions continuous in $[0,2\pi]$ and supplied with the usual supremum norm $\|\cdot\|$. Given any function $f \in C$, let $S_n[f]$ be the n-th partial sum of its Fourier series. Denote by $E_{n,q}[f]$ the Euler means of the series of f, defined by

$$(1) \qquad E_{n,q}[f](t) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k[f](t), \quad (q > 0).$$

Holland and Sahney [5] proved that

$$(2) \qquad \| E_{n,q}[f] - f \| = O\left(\sum_{r=1}^{n} \frac{\omega(f,1/r)}{r} \right), \quad (f \in C),$$

where $\omega(f,\cdot)$ denotes the modulus of continuity of f. Pych-Taberska [8] among other results gives a refinement of (2) in the form

$$(3) \qquad \| E_{n,q}[f] - f \| \leq c(q) \cdot \log n \cdot \omega\left(f, \frac{\pi}{n}\right).$$

A somewhat different estimate is given by Efimov [1], q = 1. The main purpose of our paper is to derive an improvement of (3) if q = 1, especially we get the exact value of the constant c(1). Denoting by $\| E_{n,1} \|$ the "Lebesgue-constant" or norm of the operator $E_{n,1}$, we prove

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

T h e o r e m 1. For every fixed t we have

$$(4)\qquad \sup_{\substack{f\in C\\ f\neq\text{const.}}} \frac{|E_{n,1}[f](t) - f(t)|}{\omega(f,2\pi/(n+1))} = \frac{1}{2}\,(1 + \|E_{n,1}\|).$$

It is known that (see [7] or [6])

$$(5)\qquad \|E_{n,1}\| = \frac{2}{\pi^2}\log n + \alpha + o(1), \qquad (\alpha = 0{,}73200\ldots).$$

Now let us rewrite (4) as

$$|E_{n,1}[f](t)-f(t)| \leqslant \frac{1}{2}\,\omega\Big(f, \frac{2\pi}{n+1}\Big)\cdot(1 + \|E_{n,1}\|), \qquad (f\in C),$$

which, for every fixed n, cannot be improved. We show this even being asymptotically true which is expressed by

T h e o r e m 2. For every fixed t and for every $\varepsilon > 0$ there exist $f\in C$ and an infinite subsequence $n_1, n_2, n_3, \ldots$ such that

$$(6)\qquad |E_{n,1}[f](t)-f(t)| > \frac{1-\varepsilon}{2}\,\omega\Big(f, \frac{2\pi}{n+1}\Big)\cdot(1 + \|E_{n,1}\|),$$

$$(n=n_1, n_2, n_3, \ldots).$$

2. The proofs

In the following we omit the subscript $q = 1$. It is known that (e.g. [4], [9], [1])

$$(7)\qquad \begin{cases} E_n[f](t) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t+x)\cdot K_n(x)\,dx, \\[2ex] K_n(x) = \cos^n\dfrac{x}{2}\cdot\dfrac{\sin((n+1)x/2)}{\sin(x/2)}. \end{cases}$$

Of course we have

$$(8)\qquad \|E_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(x)|\,dx, \qquad 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx.$$

Let $n = 2r$ be even and $h = 2\pi/(n+1)$. We start the proof of Theorem 1 noting that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(9)\quad E_n[f](t)-f(t) = \frac{1}{2\pi}\left(\sum_{k=-r}^{0} + \sum_{k=1}^{r}\right) \int_{(k-1)h}^{kh} [f(t+x)-f(t)]\cdot K_n(x)\,dx$$

(if $n = 2r-1$ we only have to replace $-r$ by $-r+1$ under the sign of summation). Let t be fixed. Setting

$$x_k = t + x + (k-1)\cdot h$$

we rewrite (9) in the form

$$(10)\quad E_n[f](t)-f(t) = \frac{1}{2\pi}\int_0^h \left(\sum_{k=-r}^{0} + \sum_{k=1}^{r}\right)[f(x_k)-f(t)]\cdot K_n(t-x_k)\,dx.$$

Abbreviating $(0 \leqslant x \leqslant h)$

$$(11)\quad a_i = a_i(x) := K_n(t,x_i)$$

and

$$(12)\quad b_k = b_k(x) := \begin{cases} \sum_{i=1}^{r} a_i(x), & k = 1,2,\ldots,r, \\ \sum_{i=-r}^{k} a_i(x), & k = -r,-r+1,\ldots,0, \end{cases}$$

we obtain from (10) by Abel's transformation

$$E_n[f](t)-f(t) = \frac{1}{2\pi}\int_0^h \left\{\sum_{k=-r}^{-1} [f(x_k)-f(x_{k+1})]\cdot b_k + [f(x_0)-f(t)]\cdot b_0 + \right.$$

$$\left. + \sum_{k=2}^{r} [f(x_k)-f(x_{k-1})]\cdot b_k + [f(x_1)-f(t)]\cdot b_1 \right\} dx,$$

thus

$$(13)\quad |E_n[f](t)-f(t)| \leqslant \omega(f,h)\cdot\frac{1}{2\pi}\int_0^h \left(\sum_{k=-r}^{0} + \sum_{k=1}^{r}\right)|b_k|\,dx.$$

From (7), (11) and (12) we conclude that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(14)\quad \operatorname{sgn} b_k = \operatorname{sgn} a_k = \begin{cases} (-1)^{k+1} & \text{for } k=1,2,\dots,r, \\ (-1)^{k} & \text{for } k=-r,-r+1,\dots,0. \end{cases}$$

Therefore we have (lower index not exceeding r in absolute value)

$$\left(\sum_{k=-r}^{0} + \sum_{k=1}^{r}\right)|b_k| = (a_0+a_{-2}+a_{-4}+\dots) + (a_1+a_3+a_5+\dots) =$$

$$= \frac{1}{2}\left[(a_0-a_{-1}+a_{-2}-a_{-3}+a_{-4}-+\dots)+(a_1-a_2+a_3-a_4+a_5-+\dots)\right] +$$

$$+ \frac{1}{2}\left[(a_0+a_{-1}+a_{-2}+a_{-3}+a_{-4}+\dots) +(a_1+a_2+a_3+a_4+a_5+\dots)\right] =$$

$$= \frac{1}{2}\sum_{k=-r}^{r}|a_k| + \frac{1}{2}\sum_{k=-r}^{r} a_k.$$

But

$$\frac{1}{2\pi}\int_0^h \sum_{k=-r}^{r}|a_k|\,dx = \|E_n\|, \qquad \frac{1}{2\pi}\int_0^h \sum_{k=-r}^{r} a_k\,dx = 1,$$

so it follows from (13) and (15) that

$$|E_n[f](t)-f(t)| \leqslant \omega(f,h)\cdot\frac{1}{2}\,(1+\|E_n\|).$$

To show this estimate being optimal in the class C we consider (7) and for fixed t

$$g(t+x) := \begin{cases} \operatorname{sgn} K_n(x), & x \neq 0 \quad -\pi \leqslant x \leqslant \pi, \\ -1, & x = 0. \end{cases}$$

Obviously $1 = \omega(g,h)/2$ and

$$(16)\quad E_n[g](t)-g(t) = \|E_n\| + 1 = \frac{1}{2}\,\omega(g,h)\cdot(\|E_n\|+1).$$

Though g is not continuous it is possible to replace g by continuous functions reaching the right hand-side of (16)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

in the limit (for a more general proof of (16) see e.g. Steckin [10], (13)-(15)). This completes the proof of Theorem 1.

To prove Theorem 2 we consider the functionals (t fixed)

$$r_n(f) := |E_n[f](t) - f(t)| / (\|E_n\| + 1), \qquad (f \in C)$$

$$q_n(f) := \omega(f,h)/2, \quad h = 2\pi/(n+1),$$

which both fullfill the properties of a seminorm p with norm $\|p\| = 1$ in the Banach space C, which in particular means for p that

$$p(f) \geqslant 0, \; p(\alpha \cdot f) = |\alpha| \cdot p(f), \; p(f+g) \leqslant p(f) + p(g),$$

$$\|p\| = \inf\{M \mid p(f) \leqslant M \cdot \|f\|, \; f \in C\} = 1.$$

Now Theorem 2 is an easy consequence of the following theorem proved in [2].

T h e o r e m . Let r_n, q_n be seminorms, $\|r_n\| = \|q_n\| = 1$, defined on a Banach space X, satisfying $r_n(f) \leqslant q_n(f)$, $n = 1,2,3,\ldots$, and $q_n(f) \longrightarrow 0$, $(n \longrightarrow \infty)$, for every fixed $f \in X$. Then let $\varepsilon > 0$ be given there exist an infinite sequence $n_1, n_2, n_3, \ldots$ and an element $f \in X$ such that $q_{n_k}(f) > 0$ and

$$r_n(f) > (1-\varepsilon) \cdot q_n(f), \qquad (n = n_1, n_2, n_3, \ldots).$$

3. Remark

Theorem 1 has first been treated in [3] with emphasis on discrete operators such as interpolation operators.

REFERENCES

[1] A.A. E f i m o v : Approximation of continuous functions by Euler means, Anal. Math. 12 (1986), 97-113, (Russian).

[2] R. G ü n t t n e r : Asymptotisch optimale Abschätzungen zwischen Halbnormen, Studia Sci. Math. Hungar. 8 (1973), 479-483.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] R. Güntner: Operatoren mit Abelscher Alternanteneigenschaft, Habilitationsschrift, Uni. Osnabrück 1987, FB Mathematik/Informatik.

[4] G.H. Hardy: Divergent Series, Oxford, 1973 (1949), 364.

[5] A.S.B. Holland, B.N. Sahney: On degree of approximation by Euler (E,q)-means, Studia Sci. Math. Hungar. 11 (1976), 431-435.

[6] A.E. Livingston: The Lebesgue constants for Euler (E,p) summation of Fourier series, Duke Math. J. 21 (1954), 309-313.

[7] L. Lorch: The Lebesgue constants for (E,1) summation of Fourier series, Duke Math. J. 19 (1952), 45-50.

[8] P. Pych-Taberska: Approximation of periodic functions by the Euler and Borel means of Fourier series, Demonstratio Math. 18 (1985), 115-129.

[9] C.K. Chui, A.S.B. Holland: On the order of aporoximation by Euler and Taylor means, J. Approximation Theory 39 (1983), 24-38.

[10] S.B. Stečkin: Approximation of continuous periodic functions by Favard sums, Trudy Mat. Inst. AN SSSR 109 (1971), 26-34, (Russian).

FB MATHEMATIK/INFORMATIK, UNIVERSITÄT OSNABRÜCK, POSTFACH 4469
D-4500 OSNABRÜCK, FRG

Received June 10, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

J. Baranowicz, S. Walczak

NECESSARY CONDITIONS FOR THE EXISTENCE OF A SOLUTION OF SOME EXTREMAL PROBLEMS IN THE FAMILIES OF HOLOMORPHIC FUNCTIONS

Introduction

Let F denote a fixed family of holomorphic functions. By $\Phi_0,\Phi_1,\ldots,\Phi_m$ we mean a sequence of functionals defined in the family F. The paper considers a general extremal problem of finding a minimum of the functional $\Phi_0(f)$ under the constraints $\Phi_j(f) \leqslant 0$, $j = 1,2,\ldots,m$; $f \in F$, (the detailed assumptions imposed on the functionals Φ_j, $j = 0,1,\ldots,m$, are given in Section 1). The case of constraints of equality type was considered, among others, in papers [10], [11], [14], [16]. To investigate the problem formulated above, we use, after some modifications, the method presented in [2]-[4], [9]. The basic definitions and notions connected with this methods are given in Section 2.

In Section 3 we shall prove a general necessary condition for the existence of extremum called the Euler-Lagrange equation (Th. 3.4). In proving this condition we adapt the above-mentioned method, its application to the space of holomorphic functions was possible due to Lemmas 3.1, 3.2, 3.3.

More specific necessary condition for a conditional extremum is proved in Theorem 4.2. This condition refers to those families of complex functions which have an integral representation given by L. Brickman, T.H. MacGregor and D.R. Wilken (cf. [1]).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In paper [1] a necessary condition for the existence of extremum, in the linear case, was given. The case of convex functionals was investigated in paper [7]. Theorem 4.2 of the present paper is a generalization of these results to the case of arbitrary smooth functionals. From this theorem one can obtain the classical results concerning extremal functions for the families of Caratheodory, typically-real, starlike and other functions (see [5] and its references).

In Section 5, we give an example which illustrates the application of the necessary condition provided by Theorem 4.2.

1. Formulation of extremal problem

Let us denote by C a complex plane, and let K stand for the disc $\{z \in C : |z| < 1\}$. We use H to denote the complex linear space of all functions which are holomorphic in K, with the topology of almost uniform convergence. It is known (cf. [13, §1.45]) that the space H endowed with the above topology is a locally convex Hausdorff space.

By F we mean any family of holomorphic functions which forms a closed and convex subset of the space H.

Let $m, n_1, n_2, \ldots, n_k$ be fixed natural numbers. Consider a finite sequence of real functions $F_0, F_1, \ldots, F_m$ defined on the space R^{2n} where $n = n_1+n_2+\ldots+n_k+k$. We shall always assume that these functions are of the class $C^{(1)}$, i.e. they are Fréchet differentiable and $F_j'(x)$, $j=0,1,\ldots,m$, is a continuous function in the space R^{2n}. Let $\xi_1, \xi_2, \ldots, \xi_k$ be fixed distinct points of the disc K. Denote by η a mapping defined on H by the formula

$$\eta(f) = \Big[\operatorname{re} f^{(0)}(\xi_1), \operatorname{im} f^{(0)}(\xi_1), \ldots, \operatorname{re} f^{(n_1)}(\xi_1), \operatorname{im} f^{(n_1)}(\xi_1); \ldots; \operatorname{re} f^{(0)}(\xi_k), \operatorname{im} f^{(0)}(\xi_k), \ldots, \operatorname{re} f^{(n_k)}(\xi_k), \operatorname{im} f^{(n_k)}(\xi_k)\Big] \tag{1.1}$$

where $f^{(0)} = f$. Let us note that η, $\eta : H \longrightarrow R^{2n}$, is a linear continuous mapping which maps H onto the space R^{2n}. In what

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

follows we write ηf instead of $\eta(f)$. We introduce the notations

$$\operatorname{re} f^{(i)}(\zeta_j) = u_j^i, \quad \operatorname{im} f^{(i)}(\zeta_j) = v_j^i$$

for $i = 0,1,\dots,n_n$ and $j = 1,2,\dots,k$.

Now, the vector ηf has the form

$$\eta f = \left[u_1^0, v_1^0, \dots, u_1^{n_1}, v_1^{n_1}; \dots; u_k^0, v_k^0, \dots, u_k^{n_k}, v_k^{n_k}\right].$$

We construct a finite sequence of real functionals

$$\Phi_j(f) = F_j(\eta f), \quad j = 0,1,\dots,m. \tag{1.2}$$

defined on the space H.

We shall consider the following extremal problem

P r o b l e m 1. Determine a minimum of the functional $\Phi_0(f)$ subject to the constraints

$$\Phi_1(f) \leqslant 0, \dots, \Phi_m(f) \leqslant 0, \tag{1.3}$$

$$f \in F. \tag{1.4}$$

From now on, we shall assume Problem 1 to have a solution f_0, $f_0 \in F$. This is really the case if, for instance F is compact and the set $\bigcap_{j=1}^{m+1} Q_j$ is non-empty, where

$$Q_j = \left\{ f \in H : \Phi_j(f) \leqslant 0 \right\}, \quad j=1,2,\dots,m, \tag{1.5}$$

$$Q_{m+1} = F. \tag{1.6}$$

2. Concepts and theorems of the method

Now, we shall briefly give the most important notions and facts we are going to use (cf. [4], §§5-10).

D e f i n i t i o n 2.1. We say that $f_1 \in H$ is a direction of decrease of the functional Φ, $\Phi : H \longrightarrow R$, at the point f_0 if there exist a neighbourhood U of f_1 and numbers $\alpha < 0$ and $\varepsilon_0 > 0$, such that

$\Phi(f + \varepsilon f) \leqslant \Phi(f_0) + \varepsilon\alpha$ for every $f \in U$ and $0 < \varepsilon < \varepsilon_0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D e f i n i t i o n 2.2. We say that $f_1 \in H$ is a feasible direction for a set $Q \subset H$ at the point f_0 if there exist a neighbourhood U of f_1 and a number $\varepsilon_0 > 0$, such that $f_0 + \varepsilon U \subset Q$ for every $0 < \varepsilon < \varepsilon_0$.

D e f i n i t i o n 2.3. We say that $f_1 \in H$ is a direction tangent to a set $Q \subset H$ at the point f_0 if there exists a number $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there is a point $f(\varepsilon) \in Q$ satisfying the condition $f(\varepsilon) = f_0 + \varepsilon f_1 + r(\varepsilon)$ where $\frac{r(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0^+$ (i.e. if U is an arbitrary neighbourhood of 0, then $\frac{r(\varepsilon)}{\varepsilon} \in U$ for sufficiently small $\varepsilon > 0$).

We introduce the following notations:

$K_d(\Phi, f_0)$ - the set of all directions of decrease of the functional Φ at the point f_0;

$K_f(Q, f_0)$ - the set of all feasible directions for the set Q at the point f_0;

$K_t(Q, f_0)$ - the set of all directions tangent to the set Q at the point f_0.

It was proved ([4] §6) that $K_d(\Phi, f_0)$ and $K_f(Q, f_0)$ are open cones with vertices at 0, whereas $K_t(Q, f_0)$ is a cone with vertex at 0.

D e f i n i t i o n 2.4. Let $K \subset H$ be a cone with vertex at 0. The set

$$K^* = \{ \varphi \in H^* : \langle \varphi, f \rangle \geq 0 \quad \text{for every} \quad f \in K \}$$

is called a cone dual to K. H^* stands here for the space dual to H.

D e f i n i t i o n 2.5. Let $Q \subset H$ be a convex set and $f_0 \in Q$. The set

$$N(f_0|Q) = \{ \varphi \in H^* : \langle \varphi, f - f_0 \rangle \geq 0 \quad \text{for every} \quad f \in Q \}$$

is called a cone of functionals supporting the set Q at the point f_0.

T h e o r e m 2.6. ([4], §6). Let the functional $\Phi, \Phi : H \to R$, attain at the point f_0 is local minimum on the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

set $\bigcap_{j=1}^{m+1} Q_j$. Assume that the cones $K_d(\Phi, f_o)$, $K_f(Q_j, f_o)$ $(j=1,2,\ldots,m)$ and $K_t(Q_{m+1}, f_o)$ are convex and $K_d(\Phi, f_o) \neq \emptyset$. Then there exist $\varphi_o \in K_d^*(\Phi, f_o)$, $\varphi_j \in K_f^*(Q_j, f_o)$ $(j=1,2,\ldots,m)$, $\varphi_{m+1} \in K_t^*(Q_{m+1}, f_o)$ not all zero, such that $\sum_{j=0}^{m+1} \varphi_j = 0$.

Moreover, if $\bigcap_{j=1}^{m} K_f(Q_j, f_o) \cap K_t(Q_{m+1}, f_o) \neq \emptyset$, then $\varphi_o \neq 0$.

T h e o r e m 2.7 ([4], Th. 10.2). Let $\varphi \in H^*$, $\varphi \neq 0$. Consider a cone K defined as follows

$$K = \{ g \in H : \varphi(g) < 0 \}.$$

Then the cone K^*, dual to K, has the form

$$K^* = \{ \lambda\varphi : \lambda \leqslant 0 \}.$$

T h e o r e m 2.8. ([4], Th. 10.2). Let Q be a closed convex subset of H. Then the cone dual to $K_t(Q, f_o)$ coincides with the cone of functionals supporting the set Q at the point f_o, i.e. the following equality holds

$$K_t^*(Q, f_o) = N(f_o | Q).$$

3. Necessary condition for extremum in the family

To investigate Problem 1, we apply method presented in [4], §2.

Let f_o be a solution of Problem 1. In order to make use of Theorem 2.6, we are to find:

(i) the cone of directions of decrease of the functional $\Phi_0(f)$ at the point f_o and the cone dual to it;

(ii) the cones of feasible directions for constraints (1.3) and the cones dual to them;

(iii) the cone of directions tangent to constraint (1.4) and the cone dual to it.

L e m m a 3.1. Assume that $F_o'(x_o) \neq 0$ where $x_o = \eta f_o$. Then the cone $K_d(\Phi_o, f_o)$ is given by the formula

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$K_d(\Phi_0,f_0) = \{ g \in H : \langle F_0'(x_0),y\rangle < 0;\ y = \eta g\}.$$

P r o o f . We shall start by letting

$$G = \{ g \in H : \langle F_0'(x_0),\ y\rangle < 0;\ y = \eta g\}.$$

To prove that $K_d(\Phi_0,f_0) \subset G$, take $g \in K_d(\Phi_0,f_0)$. According to Definition 2.1, there are a neighbourhood U_g of g and numbers $\alpha<0$, $\varepsilon_0>0$, such that

$$\Phi_0(f_0 + \varepsilon\bar{g}) \leqslant \Phi_0(f_0) + \varepsilon\alpha \tag{3.1}$$

for any $\bar{g} \in U_g$ and any $\varepsilon \in (0,\varepsilon_0)$. Setting $\bar{g} = g$ in (3.1) and following the definition of the functional $\Phi_0(f)$ (see (1.1), (1.2)), we can write

$$\frac{F_0(x_0 + \varepsilon y) - F_0(x_0)}{\varepsilon} \leqslant \alpha < 0. \tag{3.2}$$

As $\varepsilon \to 0^+$ in (3.2), we obtain that the directional derivative of the function F_0 at the point x_0 in the y direction (i.e. $F_0'(x_0,y)$) is negative. Since F_0 is Fréchet differentiable at x_0 we have $F_0'(x_0,y) = \langle F_0'(x_0),y\rangle$. Hence (3.2) implies

$$\langle F_0'(x_0),y\rangle < 0,$$

but the last inequality means that $g \in G$.

Now, we come to the proof of the inclusion $G \subset K_d(\Phi_0,f_0)$. Let $g_0 \in G$. Denote $y_0 = \eta g_0$ and consider the set

$$C = \{ y \in R^{2n} : \langle F_0'(x_0),y\rangle < 0\}.$$

Obviously, $y_0 \in C$. Since F_0 is differentiable and $F_0'(x_0) \neq 0$, one gets that C is a cone of directions of decrease of the function F_0 at the point x_0 (see [4], Th. 7.5). It follows that y_0 is the direction of decrease of F_0 at x_0. From this one deduces the existence of a neighbourhood V_{y_0} of y_0 and numbers $\alpha<0$, $\varepsilon_0>0$, such that

$$F_0(x_0 + \varepsilon\bar{y}) \leqslant F_0(x_0) + \varepsilon\alpha$$

for any $\bar{y} \in V_{y_0}$ and any $\varepsilon \in (0,\varepsilon_0)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Denote by U_{g_o} the inverse image of V_{y_o} under the mapping η. Applying the continuity of η, we obtain that U_{g_o} is an open subset of H and that $g_o \in U_{g_o}$. We take U_{g_o} as a neighbourhood of the point g_o.

We have thus shown the existence of a neighbourhood U_{g_o} of g_o and numbers $\alpha < 0$, $\varepsilon_o > 0$, such that, for any $\bar{g} \in U_{g_o}$ ($\eta\bar{g} = \bar{y}$ for $\bar{y} \in V_{y_o}$) and any $\varepsilon \in (0, \varepsilon_o)$, we have

$$F_o(\eta f_o + \varepsilon\lambda\bar{g}) \leqslant F_o(\eta f_o) + \varepsilon\alpha.$$

The above inequality is equivalent to the following one:

$$\Phi_o(f_o + \varepsilon\bar{g}) \leqslant \Phi_o(f_o) + \varepsilon\alpha$$

which means that $g_o \in K_d(\Phi_o, f_o)$. This completes the proof of Lemma 3.1.

Now, we consider any constraint of (1.3). Let us put

$$\tilde{Q}_j = \{f \in H : \Phi_j(f) \leqslant \Phi_j(f_o)\}; \quad j=1,2,\ldots,m.$$

L e m m a 3.2. Assume that $F_j'(x_o) \neq 0$ where $x_o = \eta f_o$. Then

$$K_f(\tilde{Q}_j, f_o) = K_d(\Phi_j, f_o) = \{g \in H: \langle F_j'(x_o), y\rangle < 0,\ y = \eta g\}.$$

P r o o f . We first note that $K_d(\Phi_j, f_o) \subset K_f(\tilde{Q}_j, f_o)$ ([4], Lemma 8.1). To prove the opposite inclusion, we apply [4], Th. 8.1. So, we need to establish that the functional $\Phi_j(f)$, $f \in H$, is differentiable at the point f_o in any direction g and that the function $g \longrightarrow \Phi_j'(f_o, g)$ is convex. We have

$$\frac{\Phi_j(f_o + \varepsilon g) - \Phi_j(f_o)}{\varepsilon} = \frac{F_j(x_o + \varepsilon y) - F_j(x_o)}{\varepsilon}. \tag{3.3}$$

As $\varepsilon \to 0^+$, the right-hand side of (3.3) tends to the directional derivative $F_j'(x_o, y)$. Since $F_j'(x_o, y) = \langle F_j'(x_o), y\rangle$, therefore by (3.3) we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\Phi_j(f_0,g) = \langle F_j'(x_0), y\rangle. \tag{3.4}$$

From the assumption $F_j'(x_0) \neq 0$ follows the existence of a vector $\tilde{y}$, $\tilde{y} \in R^{2n}$, such that $\langle F_j'(x_0), y\rangle < 0$. Since η is "onto", it follows that there is some $\tilde{g}$, $\tilde{g} \in H$, such that $\tilde{y} = \eta\tilde{g}$. Thus, making use of (3.4), we have proved the existence of such a direction $\tilde{g}$ that

$$\Phi_j'(f_0,\tilde{g}) < 0.$$

We can now conclude that, for the set $\tilde{Q}_j$, all the assumptions of Th. 8.1 [4] are fulfilled. From this theorem we get $K_f(\tilde{Q}_j,f_0) \subset K_d(\Phi_j,f_0)$, which completes the proof.

We now turn to the cones dual to $K_d(\Phi_0,f_0)$ and $K_f(\tilde{Q}_j,f_0)$. From Theorem 2.7 and Lemma 3.2 we get

L e m m a 3.3. If $F_j'(x_0) \neq 0$, $j = 0,1,\ldots,m$; $x_0 = \eta f_0$, then the following equalities

$$K_d^*(\Phi_0,f_0) = \{\lambda_0 F_0'(x_0) : \lambda_0 \leq 0\},$$

$$K_f^*(Q_j,f_0) = \{\lambda_j F_j'(x_0) : \lambda_j \leq 0\}, \quad j = 1,2,\ldots,m$$

hold.

In connection with constraint (1.4), we consider the cone $K_t(F,f_0)$. It turns out, however, that there is no need to describe this cone since we are able to find the dual cone without the previous knowledge of $K_t(F,f_0)$. Namely, we observe, using Theorem 2.8, that

$$K_t^*(F,f_0) = N(f_0|F). \tag{3.5}$$

We are now ready to prove a general necessary condition.

T h e o r e m 3.4. If the function f_0 is a solution of Problem 1, then there exist Lagrange multipliers $\lambda_i \leq 0$, $i = 0,1,\ldots,m$, and $\varphi_{m+1} \in N(f_0|F)$, not vanishing simultaneously, such that

$$\varphi_{m+1} + \sum_{j=0}^{m} \lambda_j F_j'(x_0) = 0; \quad x_0 = \eta f_0 \tag{3.6}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$\lambda_j F_j(x_0) = 0, \quad j = 1,2,\ldots,m. \tag{3.7}$$

Moreover, if there exist an $\tilde{f}$ such that $\Phi_j(\tilde{f}) < 0$, $j=1,\ldots,m$ then $\lambda_0 \neq 0$.

P r o o f . First of all, we note that if there is an index j, $0 \leqslant j \leqslant m$, such that $F_j'(x_0) = 0$, then, by letting $\lambda_j = -1$, $\lambda_i = 0$ for $i = 0,1,\ldots,j-1,j+1,\ldots,m$, we obtain that conditions (3.6), (3.7) are satisfied (since always $0 \in N(f_0|F)$). Therefore, we may assume that $F_j'(x_0) \neq 0$ for $j = 0,1,\ldots,m$. From this assumption, by Lemma 3.1, it then follows that $K_d(\Phi_j,f_0) \neq \emptyset$ for $j = 0,1,\ldots,m$.

Changing, if necessary, the numeration of the functionals, we may assume that $\Phi_1(f_0) = 0,\ldots,\Phi_r(f_0) = 0$, $\Phi_{r+1}(f_0) < 0,\ldots,$ $\Phi_m(f_0) < 0$. By the continuity of the functionals $\Phi_{r+1},\ldots,\Phi_m$ at the point f_0, we find the existence of a neighbourhood U of f_0 such that $\Phi_{r+1}(f) < 0,\ldots,\Phi_m(f) < 0$ for every $f \in U$. Consequently, since f_0 is a local minimum point in Problem 1, so it is also a local minimum point in the problem

$$\Phi_0(f) \longrightarrow \min, \tag{3.8}$$

$$\Phi_1(f) \leqslant 0,\ldots,\Phi_r(f) \leqslant 0, \tag{3.9}$$

$$f \in F. \tag{3.10}$$

It is readily seen that, for problem (3.8)-(3.10), all the assumptions of Th.2.6 are satisfied. By this theorem, there exist functionals $\varphi_0,\varphi_1,\ldots,\varphi_r$ and φ_{m+1}, not vanishing simultaneously, such that $\varphi_0 \in K_d^*(\Phi_0,f_0)$; $\varphi_j \in K_f^*(\Phi_j,f_0)$, $j = 1,2,\ldots,r$; $\varphi_{m+1} \in N(f_0|F)$ and

$$\varphi_{m+1} + \sum_{j=0}^{r} \varphi_j = 0. \tag{3.11}$$

Let us note that among the functionals $\varphi_0,\varphi_1,\ldots,\varphi_r$ there is at least one which is nonzero.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From Lemma 3.3 we find the form of the functional φ_0, namely,

$$\varphi_0 = \lambda_0 F_0'(x_0) \quad \text{for some number} \quad \lambda_0,\ \lambda_0 \leqslant 0.$$

Observe that constraint (3.9) has the form

$$f \in \bigcap_{j=1}^{r} \tilde{Q}_j;\ \tilde{Q}_j = \left\{ f \in H:\ \Phi_j(f) \leqslant \Phi_j(f_0) \right\}, \quad j=1,2,\ldots,r,$$

and hence, making use of Lemmas 3.2 and 3.3, we obtain that

$$\varphi_j = \lambda_j F_j'(x_0) \quad \text{for} \quad j=1,2,\ldots,r.$$

Consequently, (3.11) takes the following form

$$\varphi_{m+1} + \sum_{j=0}^{r} \lambda_j F_j(x_0) = 0,$$

and equalities (3.7) for $j = 1,2,\ldots,r$ are clearly satisfied. Since the functionals $\varphi_0,\varphi_1,\ldots,\varphi_r$ are not vanishing simultaneously, among the numbers $\lambda_0,\lambda_1,\ldots,\lambda_r$ there is at least one which is nonzero. Now, it obviously suffices to put $\lambda_{r+1} = \ldots = \lambda_m = 0$ in order to obtain (3.6) and (3.7). The proof is completed.

4. Solution of extremal problem in the family $\hat{F}$

We shall now consider a special case of the family F having an integral representation (see [1]).

Let us denote by X any compact Hausdorff space. We assume that the mapping $q : K \times X \longrightarrow C$ possesses the following properties:

(i) for each $t \in X$, the mapping $z \longrightarrow q(z,t)$ is analytic in K;

(ii) for each $z \in K$, the mapping $t \longrightarrow q(z,t)$ is continuous on X;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iii) for each r, $0 < r < 1$, there exists a number $M_r > 0$ such that $|q(t,z)| \leqslant M_r$ for $|z| \leqslant r$ and for $t \in X$.

Denote by ρ a set of probabilistic measures defined on Borel subsets of the space X. For $\mu \in \rho$, let

$$f_\mu(z) = \int_X q(z,t)\,d\mu(t), \quad z \in K, \tag{4.1}$$

and let $\hat{F} = \{ f_\mu : \mu \in \rho \}$.

For properties of $\hat{F}$ see [1], [7], [11]. Extremal problems in the family $\hat{F}$ or in some of its subfamilies were investigated in [7], [8], [14]-[17].

Consider the following extremal problem in the family $\hat{F}$:

P r o b l e m 2. Determine a minimum of the functional $\Phi_0(f)$ subject to the constraints

$$\Phi_1(f) \leqslant 0, \ldots, \Phi_m(f) \leqslant 0$$

$$f \in \hat{F}.$$

Let f_0, $f_0 = f_{\mu_0}(z) = \int_X q(z,t)\,d\mu_0(t)$, be a solution of Problem 2. We observe that such a solution exists since the family $\hat{F}$ is compact (see [1]) and the functional Φ_j, $j = 0,1,\ldots,m$, are continuous.

Let $v = v(t)$ be a continuous real function defined on the space X. Denote by E the set $E = \{ \tau \in X : v(\tau) = M \}$ where $M = \max_{t \in X} v(t)$.

T h e o r e m 4.1 ([7], Th. 2, p. 150). If

$$\int_X v(t)\,d\mu_0(t) = \max_{\mu \in \rho} \int_X v(t)\,d\mu(t),$$

then the measure μ_0 is cumulated on the set E.

We introduce the notations

$$L(f,\lambda_0,\lambda_1,\ldots,\lambda_m) = \sum_{j=0}^{m} \lambda_j F_j(x), \quad x = \eta f;$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4.2)\qquad w(t) = \sum_{s=1}^{k} \sum_{i=0}^{n_s} \left(A_s^i \operatorname{re}\, q^{(i)}(\zeta_s,t) + B_s^i \operatorname{im}\, q^{(i)}(\zeta_s,t) \right), \qquad t \in X,$$

where

$$A_s^i = \left. \frac{\partial L(f,\lambda_0,\lambda_1,\ldots,\lambda_m)}{\partial u_s^i} \right|_{x=x_0}, \qquad B_s^i = \left. \frac{\partial L(f,\lambda_0,\lambda_1,\ldots,\lambda_m)}{\partial v_s^i} \right|_{x=x_0};$$

$$x_0 = \eta f_0$$

and

$$q^{(i)}(\zeta_s,t) = \left. \frac{\partial^i}{\partial \zeta^i}\, q(\zeta,t) \right|_{\zeta=\zeta_s}.$$

Denote by E_w a set of the form $E_w = \left\{ t \in X : w(t) = \max_{\tau \in X} w(\tau) \right\}$. The set is non-empty since the function w is continuous and the space X is compact.

T h e o r e m 4.2. If the function f_0, $f_0 = f_{\mu_0}(z) = \int_X q(z,t)\, d\mu_0(t)$, is a solution of Problem 2, then the measure μ_0 is cumulated on the set E_w.

P r o o f . Since the family $\hat{F}$ is compact and convex, Theorem 3.4 may be applied to Problem 2. From this theorem it follows that the functional φ_{m+1}, where

$$\varphi_{m+1} = -\sum_{j=0}^{m} \lambda_j F_j'(x_0), \qquad x_0 = \eta f_0,$$

is a functional supporting the set $\hat{F}$ at the point f_0. Hence we get

$$\langle \varphi_{m+1}, f \rangle \geqslant \langle \varphi_{m+1}, f_0 \rangle \quad \text{for any} \quad f \in \hat{F},$$

i.e.

$$(4.3)\qquad \left\langle \sum_{j=0}^{m} \lambda_j F_j'(x_0), x \right\rangle \leqslant \left\langle \sum_{j=0}^{m} \lambda_j F_j'(x_0), x_0 \right\rangle$$

for any $x \in R^{2n}$ of the form $x = \eta f$ where $f \in \hat{F}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

After simple calculations we obtain

$$\left\langle \sum_{j=0}^{m} \lambda_j F_j'(x_0), x \right\rangle = \int_X w(t)\, d\mu(t) \tag{4.4}$$

where $w(t)$ is the function given by (4.2).

Hence, taking account of (4.3) and (4.4), we obtain

$$\int_X w(t)\, d\mu(t) \leqslant \int_X w(t)\, d\mu_0(t)$$

where $\mu, \mu_0 \in P$.

Making use of Theorem 4.1, we conclude the proof.

R e m a r k . Theorem 4.2 is a generalization of Theorem 3, [7], which was proved (with the use of the extremum principle) under the additional condition of convexity of the function $\Phi = [\Phi_0, \Phi_1, \ldots, \Phi_m]$.

5. An example

Let us put in formula (4.1)

$$q(z,t) = \frac{e^{it} + z}{e^{it} - z}, \quad X = [-\pi, \pi]. \tag{5.1}$$

In this case the set $\hat{F}$ coincides with the well known family of Caratheodory functions, usually denoted by C. In the family C we consider the following extremal problem:

P r o b l e m P. Find the minimum of the functional

$$\Phi_0(f) = \operatorname{re} \frac{1}{3!} f'''(0) = \operatorname{re} a_3, \tag{5.2}$$

under the constraint

$$\operatorname{re} f'(0) \cdot \operatorname{re} \frac{1}{2} f''(0) = \operatorname{re} a_1 \cdot \operatorname{re} a_2 \geqslant k. \tag{5.3}$$

We note that if $k > 4$ the condition (5.3) is contradictory to the widely known estimations: $-2 \leqslant \operatorname{re} a_1 \leqslant 2$, $-2 \leqslant \operatorname{re} a_2 \leqslant 2$. For $k = 4$ the only function, of class C, that satisfies (5.3) is $f(z) = \frac{1+z}{1-z}$. Therefore, we can assume that $k < 4$. For sufficiently small k, for instance $k \leqslant -4$, the condition (5.3) is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

fullfield by all Caratheodory functions and in this case the solution of Problem P is $f_0(z) = \frac{1-z}{1+z}$.

We observe that, in general case, the Problem P seems to be a nontrivial one. Because of the constraint (5.3) one can not apply to it any of the variational methods (see [5], [10], [12]).

According to previously established notations we now put

$$\Phi_1(f) = k - \operatorname{re} f'(0) \operatorname{re} \frac{1}{2} f''(0)$$

$$\xi_1 = 0, \quad u_1^1 = \operatorname{re} f'(0) = \operatorname{re} a_1, \quad u_1^2 = \operatorname{re} \frac{1}{2} f''(0) = \operatorname{re} a_2,$$

$$u_1^3 = \operatorname{re} \frac{1}{3!} f'''(0) = \operatorname{re} a_3.$$

Problem P we may now reformulate as follows:

P r o b l e m P'. Find the minimum of the functional

$$\Phi_0(f) = u_1^3$$

under the constraint

$$\Phi_1(f) = k - u_1^1 u_1^2 \leqslant 0.$$

It is easy to see that the function $w(t)$ (comp. (4.2)), in the case under consideration, has the form

$$w(t) = 2\lambda_0 \cos 3t + C_2 \cos 2t + C_1 \cos t,$$

where C_1, C_2 are some constants and $\lambda_0 \leqslant 0$ is a multiplier that corresponds to the functional Φ_0.

It is not difficult to observe that if $k < 4$ then there exists a function $\tilde{f} \in C$ such that $\Phi_1(\tilde{f}) < 0$. Hence $\lambda_0 \neq 0$ (see Th. 3.4) and therefore the set E_w (comp. Th. 4.2) contains at most six elements. Function f_0 which is a solution of P, has the form

$$f_0(z) = \sum_{k=1}^{6} \alpha_k \frac{e^{it_k} + z}{e^{it_k} - z}$$

where $\alpha_k \geqslant 0$, $\sum_{k=1}^{6} \alpha_k = 1$, $t_k \in [-\pi, \pi]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We have thus shown that Problem P′, after Theorem 4.2 has been applied, reduces to a mathematical programming problem.

REFERENCES

[1] L. Brickman, T.H. MacGregor, D.R. Wiken: Convex hulls of some classical families of univalent functions, Trans. Amer. Math. Soc. 156 (1971), 91-107.

[2] A.Ya. Dubovitskii, A.A. Milyutin: Extremal problems with constraints, Dokl. AN SSSR 149, (1963), 759-762 (in Russian).

[3] A.Ya. Dubovitskii, A.A. Milyutin: Extremal problems with constraints, Zh. Vychisl. Mat. i Mat. Fiz. 5, No 3 (1965), 395-453 (in Russian).

[4] I.V. Girsanov: Lectures on mathematical theory of extremum problems, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[5] G.M. Golusin: Geometric theory of functions of a complex variable, Moscow, 1966, English transl., Trans. Math. Monographs, vol. 26, Amer. Math. Soc., Providence, R.I., 1969.

[6] L. Mikołajczyk: A theorem on distortion for univalent p-symmetrical functions bounded in the circle $|z|<1$, Comment. Math. Prace Mat. 12 (1968), 35-51.

[7] L. Mikołajczyk, S. Walczak: Application of the extremum principle to investigating certain extremal problems, Trans. Amer. Math. Soc. 259 No 1 (1980), 147-155.

[8] L. Mikołajczyk, S. Walczak: On application of the Dubovitskii-Milyutin method to investigating certain extremal problems, Demonstratio Math. 13 (1980), 509-530.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[9] L.W. Neustadt: An abstract variational theory with applications to a broad class optimization problems I, II, SIAM J. Control 4 (1966), 505-527, 5 (1967), 90-137.

[10] J. Pfaltzgraff, B. Pinczuk: A variational method for classes of meromorphic functions, J. Anal. Math. 24 (1971), 101-150.

[11] Ch. Pommerenke: Univalent functions, Göttingen, 1975.

[12] M.S. Robertson: Variational method for function with positive real part, Trans. Amer. Math. Soc. 102 (1962), No 1, 82-93.

[13] W. Rudin: Functional Analysis, MacGraw-Hill Book Company, 1973.

[14] S. Walczak: Method of examining conditional extrema in some families of complex functions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 24 (1976).

[15] S. Walczak: Euler-Lagrange conditions for controls with bounded variation, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., Vol. 26, No 2 (1978), 125-128.

[16] S. Walczak: Investigations of conditional extrema in the families of functions defined by Lebesgue and Stieltjes integrals, Bull. Soc. Sci. Lett. Łódź, 28, 8 (1976), 1-15.

[17] S. Walczak: On some problem of calculus of variations, Bull. Soc. Sci. Lett. Łódź, 21, 4 (1974), 1-10.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ, 90-238 ŁÓDŹ, POLAND

Received June 17, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Zygfryd Kominek

ON APPROXIMATELY SUBADDITIVE FUNCTIONS

Let $(X,+)$ be a commutative, 2-divisible group. Then every approximately subadditive functional (i.e. $f: X \to \mathbb{R}$, $\mathbb{R}$ denotes the real line, such that $f(x+y) \leqslant f(x)+f(y)+\varepsilon$ for some $\varepsilon \geqslant 0$ and all $x,y \in X$), is minorized by a Jensen functional. Moreover, if $f: X \to \mathbb{R}$ is approximately convex (in the sense of Jensen) function, then there exists convex (in the sense of Jensen) function $h: X \to \mathbb{R}$ and a constant K such that $|f-h| \leqslant K$.

In 1952 D.H. Hyers and S.M. Ulam ([2]) proved the stability of the inequality

$$f(tx+(1-t)y) \leqslant tf(x) + (1-t)f(y), \quad t \in [0,1], \quad x,y \in D,$$

where D is a convex subset of $\mathbb{R}^n$. In 1984 P.W. Cholewa ([1]) constructed the example showing that the stability property fails to hold for ε-J-convex functions. More precisely, Cholewa constructed a function $f: D \to \mathbb{R}$ satisfying the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x) + f(y)}{2} + 1 \quad \text{for all} \quad x,y \in D,$$

and such that there is no J-convex function uniformly close to f, i.e. for every J-convex function $g: D \to \mathbb{R}$ the inequality

$$|f(x) - g(x)| \leqslant M, \quad x \in D,$$

fails to hold for any constant $M > 0$. In this example the set $D \subset \mathbb{R}$ was, in particular, a commutative semigroup divisible by 2 and f was, additionally, subadditive. Our Theorem 2 shows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

that if we assume some stronger condition, such construction will not be possible. Moreover, we shall prove that every ε- subadditive functional defined on a commutative, 2-divisible group is minorized by a Jensen function (Theorem 1).

Let $(S,+)$ be a semigroup and let ε be a fixed non-negative real number. A function $f: S \to \mathbb{R}$ is said to be ε-subadditive iff

$$f(x+y) \leqslant f(x) + f(y) + \varepsilon \tag{1}$$

for all $x,y \in S$. 0-subadditive functions are called, shortly, subadditive. Similarly, if $(S,+)$ is 2-divisible and $\eta \geqslant 0$ we shall say that $f: S \to \mathbb{R}$ is η-J-convex iff it satisfies the following inequality:

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x) + f(y)}{2} + \eta \tag{2}$$

for all $x,y \in S$. If $\eta = 0$ in (2), we shall say that f is a J-convex function. If $\eta = 0$ and "$\leqslant$" is replaced by "$\geqslant$", then f is said to be J-concave function. A function being both J-convex and J-concave will be called Jensen function.

In the proof of Theorem 1 we shall use the following theorem of R. Kaufman (cf. [3])

T h e o r e m. Let $(S,+)$ be a commutative semigroup. If $M: S \to \mathbb{R}$ is a subadditive function, $m: S \to \mathbb{R}$ is a superadditive function (i.e. $m(x+y) \geqslant m(x) + m(y)$, $x,y \in S$) and $m \leqslant M$, then there exists an additive function (i.e. both subadditive and superadditive) $a: S \to \mathbb{R}$ such that $m \leqslant a \leqslant M$.

T h e o r e m 1. Let $(X,+)$ be a commutative, 2-divisible group. If $f: X \to \mathbb{R}$ is an ε-subadditive function then there exists a Jensen function $h: X \to \mathbb{R}$ such that

$$h(x) \leqslant f(x) \quad \text{for every} \quad x \in X. \tag{3}$$

P r o o f. Let us put

$$M(x) := f(x) + \varepsilon, \quad x \in X$$

and

$$m(x) := -M(-x), \quad x \in X.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

According to (1) we get

$$M(x+y) \leqslant M(x) + M(y), \quad x,y \in X$$

and hence

$$m(x+y) \geqslant m(x) + m(y), \quad x,y \in X.$$

Moreover, for each $x \in X$ we have

$$m(x) = -M(-x) \leqslant -M(0) + M(x) \leqslant M(x).$$

By Theorem (of Kaufman) there exists an additive function $a: X \to \mathbb{R}$ such that $m \leqslant a \leqslant M$. The function $h := a - \varepsilon$ is clearly a Jensen function and satisfies condition (3).

As an immediate consequence of the proof and the theorem of Kaufman we obtain

C o r o l l a r y 1. Let $(X,+)$ be a commutative group and let $f: X \to \mathbb{R}$ be a subadditive function. Then there exists an additive function $a: X \to \mathbb{R}$ such that $a \leqslant f$.

C o r o l l a r y 2. Let $(X,+)$ be a commutative, 2-divisible group. If $f: X \to \mathbb{R}$ is ε-subadditive and J-concave function then f is a Jensen function.

P r o o f . It follows from Theorem 1 that there exists a Jensen function h satisfying condition (3). Thus the function

(4)
$$\psi(x) := f(x) - h(x), \quad x \in X,$$

is a J-concave function bounded below by zero. It is rather well known that such ψ has to be constant. However, we give an elementary proof of this fact.

It follows from non-negativity and concavity of ψ that

$$\psi(x) \leqslant 2\psi\left(\frac{x+y}{2}\right), \quad x,y \in X.$$

Hence

$$\psi(x) \leqslant 2\psi(z), \quad x,z \in X$$

and, therefore

(5)
$$\frac{1}{2}\psi(0) \leqslant \psi(x) \leqslant 2\psi(0), \quad x \in X.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It means that the inequality

$$\frac{2^n - 1}{2^n}\psi(0) \leqslant \psi(x) \leqslant \frac{2^{n-1} + 1}{2^{n-1}}\psi(0), \quad x \in X \tag{6}$$

is true for $n = 1$. Assume (6) for some positive integer $n \geqslant 1$ and put

$$\psi_1(x) := \psi(x) - \frac{2^n - 1}{2^n}\psi(0), \quad x \in X.$$

Of course, ψ_1 is a J-concave function bounded below by zero, hence ψ_1 satisfies (5):

$$\frac{1}{2}\psi_1(0) \leqslant \psi_1(x) \leqslant 2\psi_1(0), \quad x \in X,$$

which implies that

$$\frac{2^{n+1} - 1}{2^{n+1}}\psi_1(0) \leqslant \psi_1(x) \leqslant \frac{2^n + 1}{2^n}\psi_1(0)$$

and ends the proof of (6) for any positive integer n. As n tends to infinity in (6) we get

$$\psi(x) = \psi(0), \quad x \in X,$$

which, together with definition of ψ gives

$$f(x) = h(x) + f(0) - h(0), \quad x \in X.$$

That completes the proof.

T h e o r e m 2. Let $(S,+)$ be a commutative, 2-divisible semigroup and assume that $0 \in S$. If a function $f: S \to \mathbb{R}$ is ε-subadditive and η-J-convex, then there exists a J-convex function $h: S \to \mathbb{R}$ and a constant K such that

$$|h(x) - f(x)| \leqslant K, \quad \text{for all} \quad x \in X. \tag{7}$$

P r o o f . Similarly as in the proof of Theorem 1 we put

$$M(x) := f(x) + \varepsilon, \quad x \in S$$

and

$$h_n(x) := \frac{1}{2^n} M(2^n x), \quad x \in S$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for any positive integer n. By subadditivity of M, for each $x \in S$ and every positive integer n we get

$$h_{n+1}(x) \leqslant h_n(x) \leqslant M(x). \tag{8}$$

According to (2)

$$M(x) \leqslant \frac{1}{2} M(2x) + \frac{1}{2} M(0) + \eta, \quad x \in S$$

and hence

$$h_1(x) \geqslant M(x) - \frac{1}{2} M(0) - \eta, \quad x \in S.$$

By induction we obtain

$$h_n(x) \geqslant M(x) - \left(\frac{1}{2} + \ldots + \frac{1}{2^n}\right) M(0) - \left(1 + \ldots + \frac{1}{2^{n-1}}\right)\eta \tag{9}$$

for any $x \in S$ and each positive integer n, which, together with (8), proves that the sequence $(h_n(x))$ is convergent for every $x \in S$. Let us put

$$h(x) := \lim_{n \to \infty} h_n(x).$$

Fix arbitrary $x, y \in S$ and a positive integer n. Then

$$h_n\left(\frac{x+y}{2}\right) = \frac{1}{2^n} f\left(\frac{2^n x + 2^n y}{2}\right) \leqslant \frac{1}{2^n}\left[\frac{f(2^n x) + f(2^n y)}{2} + \eta\right] =$$

$$= \frac{h_n(x) + h_n(y)}{2} + \frac{1}{2^n}\eta.$$

Thus h is J-convex. Moreover, by (8) and (9) we have

$$M(x) - M(0) - 2\eta \leqslant h(x) \leqslant M(x), \quad x \in S.$$

Setting $K := \max\{f(0) + 2\eta, \varepsilon\}$ we observe that (7) is also true and the proof is finished.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] P.W. C h o l e w a : Remarks on the stability of functional equations, Aequationes Math. 27 (1984) 76-86.

[2] D.H. H y e r s , S.M. U l a m : Approximately convex functions, Proc. Amer. Math. Soc. 3 (1952) 821-828.

[3] R. K a u f m a n : Interpolation of additive functionals, Studia Math. 27 (1966) 269-272.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY IN KATOWICE, 40-007 KATOWICE, POLAND

Received June 17, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Jan Kurek

ON THE NATURAL OPERATORS OF VERTICAL PROLONGATION TYPE ON 1 – CONNECTION BUNDLE

Recently R. Matthes introduced a concept of an r-connection on fibred manifold as generalization of the classical connections [5].

In this paper we consider the vertical prolongation of 1-connections on fibred manifold $Y \to M$ onto the vertical bundle $VY \to M$ by flows. We prove that the only natural affine operator transforming 1-connections on fibred manifold Y onto 1-connections on the vertical bundle VY is the vertical prolongation by flows.

In the paper, we use a general method for finding all natural operators of certain types elaborated by I. Kolář in [2], [3].

The author is grateful to Professor I. Kolář for suggesting the problem, valuable remarks and useful discussions and to Dr J. Slovak for private communication.

1. Let $\pi : Y \to M$ be a fibred manifold, $\dim Y = n + m$, $\dim M = n$. Denote by $\beta_M : J^1TM \to TM$ the target jet projection of the first jet prolongation and by $p_Y : TY \to Y$ the tangent bundle projection.

Definition 1, [5]. 1-connection on fibred manifold $\pi : Y \to M$ is a map

(1) $$\Gamma : J^1TM \oplus Y \to TY$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

satisfying

$$(2) \qquad (T\pi,\ p_Y) \circ \Gamma = \beta_M \oplus id_Y : J^1TM \oplus Y \longrightarrow TM \oplus Y$$

and linear in J^1TM.

Any 1-connection Γ on Y determines a linear fibred map $\Gamma : J^1TM \longrightarrow TY$ over identity of Y and satisfying $T\pi \circ \Gamma = \beta_M$.

D e f i n i t i o n 2. A fibred manifold $q_Y : Con_1 Y \to Y$ over Y of all linear maps $\Gamma : J^1TM \longrightarrow TY$ satisfying $\pi \circ \Gamma = \beta_M$ is called the bundle of 1-connections on Y.

Hence, 1-connections Γ on Y coincide with the sections $\Gamma : Y \longrightarrow Con_1 Y$. In local induced coordinates (x^i, y^p, dx^i, dy^p) on TY and (x^i, X^i, X^i_j) on J^1TM, the equations of 1-connection Γ on Y are of the form:

$$(3) \qquad \Gamma : dx^i = X^i,\ dy^p = \Gamma^p_i(x,y)X^i + \Gamma^{pj}_i(x,y)X^i_j$$

with arbitrary smooth functions $\Gamma^p_i(x,y)$, $\Gamma^{pj}_i(x,y)$ on Y. Denote the components of 1-connection Γ by

$$(4) \qquad y^p_i = \Gamma^p_i(x,y), \qquad y^{pj}_i = \Gamma^{pj}_i(x,y).$$

Considering a change of fibre coordinates on Y : $\bar{x}^i = \bar{x}^i(x)$, $\bar{y}^p = \bar{y}^p(x,y)$ and a change of induced coordinates on J^1TM : $\bar{x}^i = \bar{x}^i(x)$, $\bar{X}^i = a^i_j X^j$, $\bar{X}^i_j = a^i_{lk}\tilde{a}^k_j X^l + a^i_l X^l_k \tilde{a}^k_j$, we obtain the following transformation rules for the components of 1-connection:

$$(5) \qquad \bar{\Gamma}^p_i = a^p_q \Gamma^q_l \tilde{a}^l_i + a^p_l \tilde{a}^l_i + a^p_q \Gamma^{ql}_m \tilde{a}^m_{ij} a^j_l,$$

$$\bar{\Gamma}^{pj}_i = a^p_q \Gamma^{qk}_l \tilde{a}^l_i a^j_k.$$

The fibre bundle $q_Y : Con_1 Y \longrightarrow Y$ of 1-connections on Y is an affine bundle with the associated vector bundle $Hom(J^1TM, VY)$.

If ξ is a vector field on M, its horizontal lift $\Gamma\xi$ with respect to 1-connection Γ on Y is defined by

$$(6) \qquad \Gamma\xi = \Gamma \circ (j^1\xi \oplus id_Y) : Y \longrightarrow TY.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In local coordinates, if $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$, then its horizontal lift is of the form:

$$(7)\quad \Gamma\xi = \xi^i(x) \frac{\partial}{\partial x^i} + \left(\Gamma^p{}_i(x,y)\xi^i + \Gamma^{pj}{}_i(x,y) \frac{\partial \xi^i}{\partial x^j} \right) \frac{\partial}{\partial y^p} .$$

Consider the vertical prolongation by flows $V\Gamma\xi$ on VY of the horizontal lift $\Gamma\xi$:

$$(8)\quad V\Gamma\xi = \xi^i \frac{\partial}{\partial x^i} + \left(\Gamma^p{}_i \xi^i + \Gamma^{pj}{}_i \frac{\partial \xi^i}{\partial x^j} \right) \frac{\partial}{\partial y^p} +$$

$$+ \left(\frac{\partial \Gamma^p{}_i}{\partial y^q} Y^q \xi^i + \frac{\partial \Gamma^{pj}{}_i}{\partial y^q} Y^q \frac{\partial \xi^i}{\partial x^j} \right) \frac{\partial}{\partial y^p} ,$$

where $(x^i, y^p, Y^p = dy^p)$ are the induced coordinates on VY. The vertical prolongation $V\Gamma\xi$ defines a horizontal lift of ξ with respect to a unique 1-connection $V\Gamma$ on VY.

D e f i n i t i o n 3. The unique 1-connection $V\Gamma$ on VY such that the horizontal lift $V\Gamma\xi$ of a vector field ξ on M is of the form (8) is called the vertical prolongation of 1-connection Γ on Y.

In local induced coordinates, the vertical prolongation $V\Gamma$ on VY has equations of the form

$$(9)\quad V\Gamma : dx^i = X^i, \quad dy^p = \Gamma^p{}_i X^i + \Gamma^{pj}{}_i X^i{}_j$$

$$dY^p = \left(\frac{\partial \Gamma^p{}_i}{\partial y^q} Y^q \right) X^i + \left(\frac{\partial \Gamma^{pj}{}_i}{\partial y^q} Y^q \right) X^i{}_j .$$

Denote by

$$(10)\quad \begin{cases} w^p{}_i = \Gamma^p{}_i(x,y), & w^{pj}{}_i = \Gamma^{pj}{}_i(x,y), \\ W^p{}_i = \dfrac{\partial \Gamma^p{}_i}{\partial y^q} Y^q, & W^{pj}{}_i = \dfrac{\partial \Gamma^{pj}{}_i}{\partial y^q} Y^q, \end{cases}$$

the components of the vertical prolongation $V\Gamma$.

Consider a fibred manifold $Con_1 VY$ of all 1-connections on VY. The fibre bundle $q_{VY} : Con_1 VY \longrightarrow VY$ of 1-connections on VY is an affine bundle.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D e f i n i t i o n 4. A natural affine operator $A : Con_1 \longrightarrow Con_1V$ is a system of maps $A_Y : C^\infty(Con_1Y \longrightarrow Y) \longrightarrow C^\infty(Con_1VY \longrightarrow VY)$ for all fibred manifolds Y such that:

i) for every section Γ of Con_1Y and every isomorphism $f : Y \longrightarrow \bar{Y}$ it holds:

$$A_{\bar{Y}}(Con_1 f \circ \Gamma \circ f^{-1}) = (Con_1 Vf) \circ A_Y \Gamma \circ (Vf)^{-1},$$

where $Vf : VY \longrightarrow V\bar{Y}$ and $Con_1 Vf : Con_1 VY \longrightarrow Con_1 V\bar{Y}$ and $Con_1 f : Con_1 Y \longrightarrow Con_1 Y$ are induced maps by f,

ii) $A_U(\Gamma|_U) = (A_Y \Gamma)_{p_Y^{-1}(U)}$ for every open submanifolds $U \subset Y$ and every $\Gamma \in C^\infty(Con_1 Y)$, where $p_Y : VY \longrightarrow Y$,

iii) every A_Y is regular i.e. A_Y transforms every smoothly parametrized family of sections into smoothly parametrized family,

iv) every A_Y is affine i.e.

$$A_Y(t\Gamma + (1 - t)\Lambda) = t\, A_Y(\Gamma) + (1 - t)\, A_Y(\Lambda)$$

for any $t \in R$ and $\Gamma, \Lambda \in C^\infty(Con_1 Y)$.

Note, that the natural affine operator $A : Con_1 \longrightarrow Con_1 V$ is a system of base - extending operators, [3]:

$$A_Y : C^\infty(Con_1 Y \longrightarrow Y) \longrightarrow C^\infty(Con_1 VY \longrightarrow VY).$$

If $A : Con_1 \longrightarrow Con_1 V$ is an r-th order operator, its associated map $\underline{A}_Y : J^r(Con_1 Y) \oplus VY \longrightarrow Con_1 VY$ is a fibred map over identity of VY and there is a bijection between natural operators A and natural transformations $\underline{A}$ formed by maps $\underline{A}_Y$, [3].

We will need the following:

L e m m a , [2]. There is a bijection between the r-th order natural operators $A : Con_1 \longrightarrow Con_1 V$ and $G^{r+2}_{n,m}$-equivariant maps $f : (J^r(Con_1 R^{n+m}))_0 \times (VR^{n+m})_0 \longrightarrow (Con_1 VR^{n+m})_0$.

T h e o r e m . The only natural affine operator $Con_1 \longrightarrow Con_1 V$ is the vertical prolongation.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . We shall prove the theorem in 3 steps. First we describe all first order operators, then we prove that all finite order operators are of order 1. Finally, we prove that all considered operators have a finite order.

I. The first order natural affine operators:

$A_Y : C^\infty(Con_1 Y \to Y) \to C^\infty(Con_1 VY \to VY)$ are in bijection with $G^3_{n,m}$-equivariant maps:

$$f : (J^1(Con_1 R^{n+m}))_0 \times (VR^{n+m})_0 \to (Con_1 VR^{n+m})_0$$

and affine in $(J^1(Con_1 R^{n+m}))_0$.

Because $Con_1 Y \to Y$ is an affine bundle, its first jet prolongation $J^1(Con_1 Y) \to Y$ is an affine bundle, too.

$G^3_{n,m}$ is the group of all 3-jets at origin of the diffeomorphisms $R^{n+m} \to R^{n+m}$ preserving origin and the fibration $p : R^{n+m} \to R^n$. The coordinates on $G^3_{n,m}$, which correspond to the values of the partial derivatives of functions $\bar{x}^i(x)$, $\bar{y}^p(x,y)$ at the origin are a^i_j, a^i_{jk}, a^i_{jkl}, a^p_i, a^p_{ij}, a^p_{ijk}, a^p_q, a^p_{qi}, a^p_{qr}, a^p_{qri}, a^p_{qij}, a^p_{qrs}. The symbol $\tilde{a} = a^{-1}$ means the inverse element in $G^3_{n,m}$.

The canonical coordinates on the standard fibre $(J^1(Con_1 R^{n+m}))_0$ of the first jet prolongation are: y^p_i, y^{pj}_i, y^p_{ik}, y^p_{iq}, y^{pj}_{ik}, y^{pj}_{iq}. The coordinates on $(VR^{n+m})_0$ are Y^p. Denote by: w^p_i, w^{pj}_i, W^p_i, W^{pj}_i the coordinates on standard fibre $(Con_1 VR^{n+m})_0$.

Using standard evaluations, we find the actions of $G^3_{n,m}$ on the standard fibres $(J^1(Con_1 R^{n+m}))_0$, $(VR^{n+m})_0$, $(Con_1 VR^{n+m})_0$ by the following formulae (11), (12), (13), respectively:

$$\bar{y}^p_i = a^p_q\, y^q_l\, \tilde{a}^l_i + a^p_l\, \tilde{a}^l_i + a^p_q\, y^{ql}_m\, \tilde{a}^m_{ij}\, a^j_l, \tag{11}$$

$$\bar{y}^{pj}_i = a^p_q\, y^{qk}_l\, \tilde{a}^l_i\, a^j_k,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\bar{y}^p{}_{ij} = a^p{}_q\, y^q{}_{lm}\, \tilde{a}^l{}_i\, \tilde{a}^m{}_j + a^p{}_{qm}\, \tilde{a}^m{}_j\, y^q{}_l\, \tilde{a}^l{}_i + a^p{}_q\, y^q{}_l\, \tilde{a}^l{}_{ij} +$$
$$+ a^p{}_{lm}\, \tilde{a}^l{}_i\, \tilde{a}^m{}_j + a^p{}_l\, \tilde{a}^l{}_{ij} + a^p{}_{qm}\, \tilde{a}^m{}_j\, y^{ql}{}_n\, \tilde{a}^n{}_{ik}\, a^k{}_l +$$
$$+ a^p{}_q\, y^{ql}{}_{mn}\, \tilde{a}^m{}_{ik}\, a^k{}_l\, \tilde{a}^n{}_j + a^p{}_q\, y^{ql}{}_m\, \tilde{a}^m{}_{inj}\, a^n{}_l +$$
$$+ a^p{}_q\, y^{ql}{}_m\, \tilde{a}^m{}_{ik}\, a^k{}_{ln}\, \tilde{a}^n{}_j - a^p{}_{qs}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_j\, y^q{}_l\, \tilde{a}^l{}_i -$$
$$- a^p{}_q\, y^q{}_{ls}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_j\, \tilde{a}^l{}_i - a^p{}_{ls}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_j\, \tilde{a}^l{}_i -$$
$$- a^p{}_{qs}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_j\, y^{ql}{}_m\, \tilde{a}^m{}_{ik}\, a^k{}_l -$$
$$- a^p{}_q\, y^{ql}{}_{ms}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_j\, \tilde{a}^m{}_{ik}\, a^k{}_l,$$

$$\bar{y}^p{}_{ir} = a^p{}_q\, y^q{}_{ls}\, \tilde{a}^s{}_r\, \tilde{a}^l{}_i + a^p{}_{qs}\, \tilde{a}^s{}_r\, y^q{}_l\, \tilde{a}^l{}_i + a^p{}_{ls}\, \tilde{a}^s{}_r\, \tilde{a}^l{}_i +$$
$$+ a^p{}_{qs}\, \tilde{a}^s{}_r\, y^{ql}{}_m\, \tilde{a}^m{}_{ij}\, a^j{}_l + a^p{}_q\, y^{ql}{}_{ms}\, \tilde{a}^s{}_n\, \tilde{a}^m{}_{ij}\, a^j{}_l,$$

$$\bar{y}^{pj}{}_{ik} = a^p{}_q\, y^{ql}{}_{mn}\, \tilde{a}^m{}_i\, \tilde{a}^n{}_k\, a^j{}_l + a^p{}_{qn}\, \tilde{a}^n{}_k\, y^{ql}{}_m\, \tilde{a}^m{}_i\, a^j{}_l +$$
$$+ a^p{}_q\, y^{ql}{}_m\, \tilde{a}^m{}_{ik}\, a^j{}_l + a^p{}_q\, y^{ql}{}_m\, \tilde{a}^m{}_i\, a^j{}_{ln}\, \tilde{a}^n{}_k -$$
$$- a^p{}_{qs}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_k\, y^{ql}{}_m\, \tilde{a}^m{}_i\, a^j{}_l -$$
$$- a^p{}_q\, y^{ql}{}_{ms}\, \tilde{a}^s{}_r\, a^r{}_n\, \tilde{a}^n{}_k\, \tilde{a}^m{}_i\, a^j{}_l,$$

$$\bar{y}^{pj}{}_{ir} = a^p{}_q\, y^{ql}{}_{ms}\, \tilde{a}^s{}_r\, \tilde{a}^m{}_i\, a^j{}_l + a^p{}_{qs}\, \tilde{a}^s{}_r\, y^{ql}{}_m\, \tilde{a}^m{}_i\, a^j{}_l,$$

(12) $$\bar{Y}^p = a^p{}_q\, Y^q,$$

(13) $$\bar{w}^p{}_i = a^p{}_q\, w^q{}_l\, \tilde{a}^l{}_i + a^p{}_l\, \tilde{a}^l{}_i + a^p{}_q\, w^{ql}{}_m\, \tilde{a}^m{}_{ij}\, a^j{}_l,$$
$$\bar{w}^{pj}{}_i = a^p{}_q\, w^{qk}{}_l\, \tilde{a}^l{}_i\, a^j{}_k,$$
$$\bar{W}^p{}_i = a^p{}_q\, W^q{}_l\, \tilde{a}^l{}_i + a^p{}_q\, W^{qk}{}_l\, \tilde{a}^l{}_{im}\, a^m{}_k + a^p{}_{qr}\, w^q{}_l\, \tilde{a}^l{}_i\, Y^r +$$
$$+ a^p{}_{lr}\, \tilde{a}^l{}_i\, Y^r + a^p{}_{qr}\, y^{qk}{}_l\, \tilde{a}^l{}_{im}\, a^m{}_k\, Y^r,$$
$$\bar{W}^{pj}{}_i = a^p{}_q\, W^{qk}{}_l\, \tilde{a}^l{}_i\, a^j{}_k + a^p{}_{qr}\, y^{qk}{}_l\, \tilde{a}^l{}_i\, a^j{}_k\, Y^r.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Consider a $G^3_{n,m}$-equivariant map

$$f : (J^1(Con_1R^{n+m}))_0 \times R^m \to (Con_1VR^{n+m})_0$$

with a coordinate expression

$$(14)\quad \begin{aligned} w^p{}_i &= f^p{}_i(Y^p, y^p{}_i, y^{pj}{}_i, y^p{}_{ik}, y^p{}_{ir}, y^{pj}{}_{ik}, y^{pj}{}_{ir}),\\ w^{pj}{}_i &= g^{pj}{}_i(Y^p, y^p{}_i, y^{pj}{}_i, y^p{}_{ik}, y^p{}_{ir}, y^{pj}{}_{ik}, y^{pj}{}_{ir}),\\ W^p{}_i &= F^p{}_i(Y^p, y^p{}_i, y^{pj}{}_i, y^p{}_{ik}, y^p{}_{ir}, y^{pj}{}_{ik}, y^{pj}{}_{ir}),\\ W^{pj}{}_i &= G^{pj}{}_i(Y^p, y^p{}_i, y^{pj}{}_i, y^p{}_{ik}, y^p{}_{ir}, y^{pj}{}_{ik}, y^{pj}{}_{ir}). \end{aligned}$$

The affine property of the map f in $(J^1(Con_1R^{n+m}))_0$ means that all coordinate functions $f^p{}_i$, $g^{pj}{}_i$, $F^p{}_i$, $G^{pj}{}_i$ are affine in $y^p{}_i, y^{pj}{}_i, y^p{}_{ik}, y^p{}_{ir}, y^{pj}{}_{ik}, y^{pj}{}_{ir}$ for every fixed Y^p.

First consider equivariancy of functions

$$(15)\quad g^{pj}{}_i = \alpha(Y^p)y^p{}_i + \beta(Y^p)y^{pj}{}_i + \gamma(Y^p)y^p{}_{ik} + \delta(Y^p)y^p{}_{ir} + \\ + \varepsilon(Y^p)y^{pj}{}_{ik} + \eta(Y^p)y^{pj}{}_{iq} + \zeta(Y^p)$$

with respect to the base homotheties: $\tilde{a}^i{}_j = k\delta^i{}_j$, $a^p{}_q = \delta^p{}_q$, and all other a's vanishing. The homogeneity condition

$$(16)\quad g^{pj}{}_i = g^{pj}{}_i(Y^p, ky^p{}_i, y^{pj}{}_i, k^2y^p{}_{ij}, ky^p{}_{ir}, ky^{pj}{}_{ik}, y^{pj}{}_{ir})$$

gives

$$\alpha(Y^p) = 0,\ \gamma(Y^p) = 0,\ \delta(Y^p) = 0,\ \varepsilon(Y^p) = 0,$$

and

$$g^{pj}{}_i = \beta(Y^p)y^{pj}{}_i + \eta(Y^p)y^{pj}{}_{ir} + \zeta(Y^p).$$

Equivariancy condition of $g^{pj}{}_i$ with respect to the fibre homotheties $\tilde{a}^i{}_j = \delta^i{}_j$, $a^p{}_q = k\delta^p{}_q$, and all other a's vanishing

$$(17)\quad k\,g^{pj}{}_i = g^{pj}{}_i(kY^p, ky^p{}_i, ky^{pj}{}_i, ky^p{}_{ij}, y^p{}_{ir}, ky^{pj}{}_{ik}, y^{pj}{}_{ir})$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

gives $\beta(Y^p) = \text{const}$ and $\eta(kY^p) = k\eta(Y^p)$, $\xi(kY^p) = k\xi(Y^p)$ are linear in Y^p. Hence, functions $g^{pj}{}_i$ are of the form

$$(18)\qquad g^{pj}{}_i = \beta^{pjl}_{qki}\, y^{qk}{}_l + \eta^{psjl}_{qrki}\, Y^q\, y^{rk}{}_{ls} + \xi^{pj}_{qi}\, Y^q.$$

Using the theory of invariant tensors, [3], we deduce that

$$(19)\qquad g^{pj}{}_i = \lambda_1\, y^{pj}{}_i + \lambda_2\, \delta^j{}_i\, y^{pk}{}_k + \lambda_3\, y^{rj}{}_{ir}\, Y^p + \lambda_4\, y^{pj}{}_{ir}\, Y^r + \lambda_5\, \delta^j{}_i\, y^{rk}{}_{kr}\, Y^p + \lambda_6\, \delta^j{}_i\, y^{pk}{}_{kr}\, Y^r + \lambda_7\, \delta^j{}_i\, Y^p.$$

Considering equivariancy of $g^{pj}{}_i$ with respect to the subgroup K: $\tilde{a}^i{}_j = \delta^i{}_j$, $a^p{}_q = \delta^p{}_q$ and all other a's arbitrary, we get $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ and

$$(20)\qquad g^{pj}{}_i = \lambda_1\, y^{pj}{}_i + \lambda_2\, \delta^j{}_i\, y^{pk}{}_k + \lambda_7\, \delta^j{}_i\, Y^p.$$

We determine functions $G^{pj}{}_i$ using the same homogeneity conditions with respect to the base homoteties and fibre homoteties and we get

$$(21)\qquad G^{pj}{}_i = \mu_1\, y^{pj}{}_i + \mu_2\, \delta^j{}_i\, y^{pk}{}_k + \mu_3\, y^{rj}{}_{ir}\, Y^p + \mu_4\, y^{pj}{}_{ir}\, Y^r + \mu_5\, \delta^j{}_i\, y^{rk}{}_{kr}\, Y^p + \mu_6\, \delta^j{}_i\, y^{pk}{}_{kr}\, Y^r + \mu_7\, \delta^j{}_i\, Y^p.$$

Considering equivariancy of $G^{pj}{}_i$ with respect to the above subgroup K, we obtain the relations: $\mu_4 = \lambda_1$, $\mu_6 = \lambda_2$, $\mu_3 = \mu_5 = \lambda_7 = 0$, and

$$(22)\qquad G^{pj}{}_i = \mu_1\, y^{pj}{}_i + \mu_2\, \delta^j{}_i\, y^{pk}{}_k + \lambda_1\, y^{pj}{}_{ir}\, Y^r + \lambda_2\, \delta^j{}_i\, y^{pk}{}_{kr}\, Y^r + \mu_7\, \delta^j{}_i\, Y^p,$$

$$g^{pj}{}_i = \lambda_1\, y^{pj}{}_i + \lambda_2\, \delta^j{}_i\, y^{pk}{}_k.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Analogously, we determine functions

$$(23)\quad f^p{}_i = \alpha(Y^p)y^p{}_i + \beta(Y^p)y^{pj}{}_i + \gamma(Y^p)y^p{}_{ij} + \delta(Y^p)y^p{}_{ir} + \\ + \varepsilon(Y^p)y^{pj}{}_{ik} + \eta(Y^p)y^{pj}{}_{ir} + \zeta(Y^p)$$

using the homogeneity condition with respect to the base homoteties

$$(24)\quad k\, f^p{}_i = f^p{}_i(Y^p,\ ky^p{}_i,\ y^{pj}{}_i,\ k^2y^p{}_{ij},\ ky^p{}_{ir},\ ky^{pj}{}_{ik},\ y^{pj}{}_{ir}).$$

Hence, we get: $\beta(Y^p) = 0$, $\gamma(Y^p) = 0$, $\eta(Y^p) = 0$, $\zeta(Y^p) = 0$, and

$$(25)\quad f^p{}_i = \alpha(Y^p)y^p{}_i + \delta(Y^p)y^p{}_{ir} + \varepsilon(Y^p)y^{pj}{}_{ik}.$$

The homogeneity condition with respect to fibre homoteties

$$(26)\quad k\, f^p{}_i = f^p{}_i(kY^p, ky^p{}_i, ky^{pj}{}_i, ky^p{}_{ij}, y^p{}_{ir}, ky^{pj}{}_{ik}, y^{pj}{}_{ir})$$

gives $\alpha(Y^p) = \text{const}$, $\varepsilon(Y^p) = \text{const}$, $\delta(kY^p) = k\delta(Y^p)$. Hence, we get

$$(27)\quad f^p{}_i = \alpha^{pl}_{qi}\, y^q{}_l + \delta^{psl}_{qri}\, Y^q\, y^r{}_{ls} + \varepsilon^{pkl}_{qij}\, y^{qj}_{kl}.$$

Using the theory of invariant tensors, [3], we deduce that

$$(28)\quad f^p{}_i = \tau_1\, y^p{}_i + \tau_2\, y^p{}_{iq}\, Y^q + \tau_3\, y^q{}_{iq}\, Y^p + \tau_4\, y^{pm}{}_{mi} + \\ + \tau_5\, y^{pm}{}_{im}.$$

Equivariancy of $f^p{}_i$ with respect to the subgroup K gives $\tau_1 = 1$, $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$, $\lambda_1 = 1$, $\lambda_2 = 0$, and

$$(29)\quad f^p{}_i = y^p{}_i, \qquad g^{pj}{}_i = y^{pj}{}_i.$$

Finally, we determine functions $F^p{}_i$ using the same homogeneity conditions with respect to base homoteties and fibre homoteties. We get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(30)\quad F^p{}_i = \varrho_1 y^p{}_i + \varrho_2 y^p{}_{iq} Y^q + \varrho_3 y^q{}_{iq} Y^p + \varrho_4 y^{pm}{}_{mi} + \varrho_5 y^{pm}{}_{im}.$$

Equivariancy of $F^p{}_i$ with respect to the subgroup K gives $\varrho_2 = 1$, $\varrho_1 = \varrho_3 = \varrho_4 = \varrho_5 = 0$, $\mu_1 = \varrho_1 = 0$, $\mu_2 = 0$, $\mu_7 = 0$, $\lambda_1 = 1$, $\lambda_2 = 0$.

In this way, we have obtained

$$(31)\quad f^p{}_i = y^p{}_i,\ g^{pj}{}_i = y^{pj}{}_i,\ F^p{}_i = y^p{}_{iq}Y^q,\ G^{pj}{}_i = y^{pj}{}_{iq}Y^q.$$

So that, we have deduced that the vertical prolongation (9) is the only natural affine operator of the order 1 transforming any 1-connection Γ on Y into a 1-connection on VY.

II. An r-th order affine operator $A : \mathrm{Con}_1 \longrightarrow \mathrm{Con}_1 V$ corresponds to a $G^{r+2}{}_{n,m}$-equivariant map

$$f : (J^r(\mathrm{Con}_1 R^{n+m}))_0 \times R^m \longrightarrow (\mathrm{Con}_1 VR^{n+m})_0$$

and affine in $(J^r(\mathrm{Con}_1 R^{n+m}))_0$.

Denote by $y^p{}_{i\alpha\beta}$, $y^{pj}{}_{i\alpha\beta}$ the partial derivatives of $y^p{}_i$, $y^{pj}{}_i$ with respect to a multiindex α in x^i and a multiindex β in y^p.

Any map $f : (J^r(\mathrm{Con}_1 R^{n+m}))_0 \times R^m \longrightarrow (\mathrm{Con}_1 VR^{n+m})_0$ is of the form

$$(32)\quad \begin{aligned} w^p{}_i &= f^p{}_i(Y^p, y^p{}_{i\alpha\beta}, y^{pj}{}_{i\alpha\beta}),\\ w^{pj}{}_i &= g^{pj}{}_i(Y^p, y^p{}_{i\alpha\beta}, y^{pj}{}_{i\alpha\beta}),\\ W^p{}_i &= F^p{}_i(Y^p, y^p{}_{i\alpha\beta}, y^{pj}{}_{i\alpha\beta}),\\ W^{pj}{}_i &= G^{pj}{}_i(Y^p, y^p{}_{i\alpha\beta}, y^{pj}{}_{i\alpha\beta}). \end{aligned}$$

The affine property of f in $(J^r(\mathrm{Con}_1 R^{n+m}))_0$ means that all coordinate functions $f^p{}_i$, $g^{pj}{}_i$, $F^p{}_i$, $G^{pj}{}_i$ are affine in $y^p{}_{i\alpha\beta}$, $y^{pj}{}_{i\alpha\beta}$ for every fixed Y^p.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We determine affine functions

$$g^{pj}{}_i = \varepsilon(Y^p)y^p{}_{i\alpha\beta} + \eta(Y^p)y^{pj}{}_{i\alpha\beta} + \zeta(Y^p) \tag{33}$$

using a homogeneity condition with respect to the base homoteties

$$g^{pj}{}_i = g^{pj}{}_i(Y^p,\ k^{1+|\alpha|}y^p{}_{i\alpha\beta},\ k^{|\alpha|}y^{pj}{}_{i\alpha\beta}). \tag{34}$$

Hence, we obtain that: $\varepsilon(Y^p) = 0$, $\eta(Y^p)$ are arbitrary for $|\alpha| = 0$ and

$$g^{pj}{}_i = \eta(Y^p)y^{pj}{}_{i\beta} + \zeta(Y^p). \tag{35}$$

A homogeneity condition with respect to fibre homoteties

$$k\,g^{pj}{}_i = g^{pj}{}_i(k\,Y^p,\ k^{1-|\beta|}y^p{}_{i\alpha\beta},\ k^{1-|\beta|}y^{pj}{}_{i\alpha\beta}) \tag{36}$$

gives the relations: $\eta(Y^p) = \text{const}$ for $|\beta| = 0$, $\eta(k\,Y^p) = k\eta(Y^p)$ for $|\beta| = 1$ and $\zeta(k\,Y^p) = k\zeta(Y^p)$ and $g^{pj}{}_i$ do not depend on $y^{pj}{}_i$ for $|\beta| > 1$. Hence, the functions $g^{pj}{}_i$ depend on the 1-jets only and they are of the same form as in case I. In the same way we deduce that $G^{pj}{}_i$ only depend on the 1-jets and they are of the same form as in case I.

We determine affine functions

$$f^p{}_i = \varepsilon(Y^p)y^p{}_{i\alpha\beta} + \eta(Y^p)y^{pj}{}_{i\alpha\beta} + \zeta(Y^p) \tag{37}$$

using a homogeneity condition with respect to base homoteties:

$$k\,f^p{}_i = f^p{}_i(Y^p,\ k^{1+|\alpha|}y^p{}_{i\alpha\beta},\ k^{|\alpha|}y^{pj}{}_{i\alpha\beta}). \tag{38}$$

Hence, we obtain

$\zeta(Y^p) = 0$, $\varepsilon(Y^p)$ are arbitrary, $\eta(Y^p) = 0$ for $|\alpha| = 0$ and $\zeta(Y^p) = 0$, $\varepsilon(Y^p) = 0$, $\eta(Y^p)$ are arbitrary for $|\alpha| = 1$ and $f^p{}_i$ is of the form

$$f^p{}_i = \varepsilon(Y^p)y^p{}_{i\beta} + \eta(Y^p)y^{pj}{}_{ik\beta}. \tag{39}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

A homogeneity condition with respect to fibre homoteties

$$(40) \qquad k\, f^p_i = f^p_i(k\, Y^p,\ k^{1-|\beta|} y^p_{i\alpha\beta},\ k^{1-|\beta|} y^{pj}_{i\alpha\beta})$$

gives $\varepsilon(Y^p) = \text{const}$, $\eta(Y^p) = \text{const}$ for $|\beta| = 0$, and $\varepsilon(k\, Y^p) = k\varepsilon(Y^p)$ for $|\beta| = 1$ and f^p_i do not depend on $y^p_{i\beta}$, $y^{pj}_{ik\beta}$ for $|\beta| > 1$. Hence, functions f^p_i are not dependend on higher order jets and they are of the same form as in the case I. In the same way, we deduce that also F^p_i are not dependend on higher order jets and they are of the same form as in the case I. So, we have reduced the whole problem to the case I.

III. Any natural affine operator $A : \mathrm{Con}_1 \longrightarrow \mathrm{Con}_1 V$ has a finite order. This follows from a generalization of the well known Peetre theorem due to [6], [4]. The assertion in our affine case is a direct consequence of the results contained in [6], [4] and was privately communicated with J. Slovak.

REFERENCES

[1] I. Kolář : Functorial prolongations of r-connections. Naturwissenschaftlich-Theoretisches Zentrum, Karl-Marx-Universität DDR Leipzig, 1988.

[2] I. Kolář : Some natural operations with connections, to appear.

[3] I. Kolář : Some natural operators in differential geometry, Proc. Conf. Diff. Geom. and its Appl. Brno, 1986, Dordrecht (1987), 91-110.

[4] I. Kolář : Lie derivatives and higher order lagrangians, Proc. Conf. Diff. Geom. and its Appl. Nove Mesto 1980, Praha (1982), 117-123.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] R. M a t t h e s : Generalized connections, Algebra and Geometry (to appear in Reports on Math. Phys.).

[6] J. S l o v a k : On the finite order of some operators, Proc. Conf. Diff. Geom. and its Appl., Brno 1986, Dordrecht (1987), 283-294.

INSTITUTE OF MATHEMATICS, MARIA CURIE SKŁODOWSKA UNIVERSITY, 20-031 LUBLIN, POLAND

Received June 17, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Andrzej Kasperski

THE MODULAR SPACES OF MULTIFUNCTIONS, II

1. Introduction

In this paper we introduce the spaces of multifunctions X_φ and X_φ^n, where n is a natural number and φ is a φ-function. We also introduce the functions $m(F)$, $\underline{f}(F)$ and $\overline{f}(F)$. We introduce a convergence of $\{F_n\}$ in X_φ, X_φ^1 and X_φ^2 by the convergence of $\{m(F_n)\}$, $\{\underline{f}(F_n)\}$, $\{\overline{f}(F_n)\}$ in L^φ. We introduce a convergence in X_φ^k, $k \geq 3$, by the convergence in X_φ^{k-1} or X_φ^2. We also study the completeness of these spaces.

Let (Ω, Σ, μ) be a measure space with nonnegative, nontrivial σ-finite and complete measure μ. Every function from Ω to 2^R will be called the multifunction. Let $F: \Omega \to 2^R$ and let m_1 be the Lebesgue measure on 2^R. We introduce the functions $m(F)$, $\underline{f}(F)$ and $\overline{f}(F)$ by the formulas:

$$m(F)(t) = m_1(F(t)), \quad \underline{f}(F)(t) = \inf_{x \in F(t)} x, \quad \overline{f}(F)(t) = \sup_{x \in F(t)} x$$

for every $t \in \Omega$.

Let X be the space of all multifunctions F from Ω to 2^R such that: $F(t)$ are nonempty and compact for every $t \in \Omega$. Two multifunctions F, G such that $F(t) = G(t)$ μ-a.e. will be treated as the same element of X.

Let N be the set of all natural numbers.

Let φ be φ-function and let $\varrho(x) = \int_\Omega \varphi(|x(t)|)d\mu$ for every $x \in L^\varphi$. Convergence $x_n \to x$ in $L^\varphi(\Omega, \Sigma, \mu)$ is equivalent to the condition $\varrho(a(x_n - x)) \to 0$ as $n \to \infty$ for every $a > 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. On the space X^1_φ

Let $X^1_\varphi = \{F \in X : F(t)$ is a convex set for every $t \in \Omega$,

$$\underline{f}(F), \bar{f}(F) \in L^\varphi(\Omega, \Sigma, \mu)\}.$$

R e m a r k . If $F \in X^1_\varphi$, then $m(F) \in L^\varphi(\Omega, \Sigma, \mu)$.

D e f i n i t i o n 1. Let $F_n \in X^1_\varphi$ for every $n \in N$ and let $F \in X^1_\varphi$. The sequence $\{F_n\}$ is φ-convergent to F in X^1_φ ($F_n \xrightarrow{\frac{1}{\varphi}} F$), if $\underline{f}(F_n) \to \underline{f}(F)$ and $\bar{f}(F_n) \to \bar{f}(F)$ in $L^\varphi(\Omega, \Sigma, \mu)$ as $n \to \infty$.

D e f i n i t i o n 2. Let $F_n \in X^1_\varphi$ for every $n \in N$. The sequence $\{F_n\}$ fulfils the (C,φ)-condition in X^1_φ, if $\{\underline{f}(F_n)\}$ and $\{\bar{f}(F_n)\}$ are Cauchy sequences in $L^\varphi(\Omega, \Sigma, \mu)$.

T h e o r e m 1. X^1_φ is $(C,\varphi)^1$-complete, i.e. every sequence which fulfils the (C,φ)-condition in X^1_φ is φ-convergent in X^1_φ.

P r o o f . Let $\{F_n\}$ fulfil the (C,φ)-condition in X^1_φ. Hence $L^\varphi(\Omega, \Sigma, \mu)$ is a complete space, then there exist $f, g \in L^\varphi(\Omega, \Sigma, \mu)$ such that $\underline{f}(F_n) \to f$ and $\bar{f}(F_n) \to g$ in $L^\varphi(\Omega, \Sigma, \mu)$ as $n \to \infty$ and $f(t) \leqslant g(t)$ for every $t \in \Omega$. Let $F(t) = [f(t), g(t)]$ for every $t \in \Omega$. It is easy to see that $F_n \xrightarrow{\frac{1}{\varphi}} F$.

R e m a r k 2. $F_n \xrightarrow{\frac{1}{\varphi}} F$ iff $F_n \xrightarrow{\frac{1}{\varrho_\varphi}} F$ (see [1]).

3. On the space X^2_φ

Let $X^2_\varphi = \{F \in X : F(t) = F^1(t) \cup F^2(t)$, $\bar{f}(F^1)(t) \leqslant \underline{f}(F^2)(t)$ for every $t \in \Omega$, $F^1, F^2 \in X^1_\varphi\}$.

R e m a r k 3. If $F \in X^2_\varphi$, then $m(F) \in L^\varphi(\Omega, \Sigma, \mu)$.

Let now $F \in X^2_\varphi$. We introduce the multifunctions $O(F)$, $W(F)$ and $\bar{W}(F)$ by the formulas:

$$O(F)(t) = [\underline{f}(F^1)(t), \bar{f}(F^2)(t)],$$

$$W(F)(t) = [\bar{f}(F^1)(t), \underline{f}(F^2)(t)],$$

$$\bar{W}(F)(t) = [0, \underline{f}(F^2)(t) - \bar{f}(F^1)(t)] \text{ for every } t \in \Omega.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D e f i n i t i o n 3. Let $F_n \in X_\varphi^2$ for every $n \in N$ and let $F \in X_\varphi^2$. The sequence $\{F_n\}$ is φ-convergent to F in X_φ^2 ($F_n \xrightarrow{2}_{\varphi} F$), if $\underline{f}(F_n) \to \underline{f}(F)$, $\bar{f}(F_n) \to \bar{f}(F)$ and $m(F_n \dot{\vee} F) \to 0$ in $L^\varphi(\Omega, \Sigma, \mu)$ as $n \to \infty$, where $m(F_n \dot{\vee} F)(t) = m_1((F_n(t) \setminus F(t)) \cup (F(t) \setminus F_n(t)))$ for every $t \in \Omega$.

R e m a r k 4. If $F_n \xrightarrow{1}_{\varphi} F$, then $F_n \xrightarrow{2}_{\varphi} F$.

P r o o f . It is easy to see that, if $F_n, F \in X_\varphi^1$, then

$$m(F_n \dot{\vee} F)(t) \leq |\underline{f}(F_n)(t) - \underline{f}(F)(t)| + |\bar{f}(F_n)(t) - \bar{f}(F)(t)|$$

for every $t \in \Omega$.

D e f i n i t i o n 4. Let $F_n \in X_\varphi^2$ for every $n \in N$. The sequence $\{F_n\}$ fulfils the (C,φ)-condition in X_φ^2, if $\{\underline{f}(F_n)\}$ and $\{\bar{f}(F_n)\}$ are Cauchy sequences in $L^\varphi(\Omega, \Sigma, \mu)$ and $m(F_n \dot{\vee} F_m) \to 0$ in $L^\varphi(\Omega, \Sigma, \mu)$ as $n, m \to \infty$ ($m(F_n \dot{\vee} F_m) \to 0$ in $L^\varphi(\Omega, \Sigma, \mu)$ as $n, m \to \infty$, if for every $\varepsilon > 0$ and every $a > 0$ there exists $M > 0$, such that $\varrho(am(F_n \dot{\vee} F_m)) < \varepsilon$ as $n, m > M$).

L e m m a 1. If $\{F_n\}$ fulfils the (C,φ)-condition in X_φ^2 and there exist the nonfinite subsets $I_1, I_2 \subset N$, such that $m_1(W(F_n)(t) \cap W(F_m)(t)) = 0$ for all $n \in I_1$, $m \in I_2$ and every $t \in \Omega$, then:

a) $\bar{W}(F_n) \xrightarrow{1}_{\varphi} 0$, for $n \in I_1$, as $n \to \infty$,

b) $\bar{W}(F_m) \xrightarrow{1}_{\varphi} 0$, for $m \in I_2$, as $m \to \infty$,

where $0(t) = 0$ for every $t \in \Omega$.

P r o o f . If $m_1(W(F_n)(t) \cap W(F_m)(t)) = 0$, then

$$|\underline{f}(F_n^2)(t) - \bar{f}(F_n^1)(t)| \leq m(F_n \dot{\vee} F_m)(t) + |\bar{f}(F_n^2)(t) - \bar{f}(F_m^2)(t)| + |\underline{f}(F_m^1)(t) - \underline{f}(F_n^1)(t)|,$$

$$|\underline{f}(F_m^2)(t) - \bar{f}(F_m^1)(t)| \leq m(F_n \dot{\vee} F_m)(t) + |\bar{f}(F_m^2)(t) - \bar{f}(F_n^2)(t)| + |\underline{f}(F_m^1)(t) - \underline{f}(F_n^1)(t)|.$$

So $\bar{W}(F_n) \xrightarrow{1}_{\varphi} 0$, for $n \in I_1$, as $n \to \infty$ and $W(F_m) \xrightarrow{1}_{\varphi} 0$, for $m \in I_2$, as $m \to \infty$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

R e m a r k 5. If $\{F_n\}$ fulfils the (C,φ)-condition in X^2_φ and there exists a nonfinite subset $I \subset N$ such that $\overline{W}(F_n) \xrightarrow{1}_{\varphi} 0$, for $n \in I$, as $n \to \infty$, then $\overline{W}(F_m) \xrightarrow{1}_{\varphi} 0$ as $m \to \infty$, $m \in N$.

P r o o f . It is easy to see that, if $\{F_n\}$ fulfils the (C,φ)-condition in X^2_φ, then $|m(F_n) - m(F_m)| \longrightarrow 0$ and $|m(O(F_n)) - m(O(F_m))| \longrightarrow 0$ in $L^\varphi(\Omega,\Sigma,\mu)$ as $n,m \to \infty$. So, if $\overline{W}(F_n) \xrightarrow{1}_{\varphi} 0$, for $n \in I$, as $n \to \infty$, then the same holds for all $m \in N$.

L e m m a 2. If $\{F_n\}$ fulfils the (C,φ)-condition in X^2_φ and $\overline{W}(F_n) \xrightarrow{1}_{\varphi} 0$, then there exists $F \in X^1_\varphi$ such that $F_n \xrightarrow{2}_{\varphi} F$.

P r o o f . If $\{F_n\}$ fulfils the (C,φ)-condition in X^2_φ, then $\{O(F_n)\}$ fulfils the (C,φ)-condition in X^1_φ. From Theorem 1 there exists $F \in X^1_\varphi$ such that $O(F_n) \xrightarrow{1}_{\varphi} F$. It is easy to see that now

$$m(F_n \asymp F)(t) \leqslant |\bar{f}(F_n)(t) - \bar{f}(F)(t)| + |\underline{f}(F_n)(t) - \underline{f}(F)(t)| + m(\overline{W}(F_n))(t).$$

So $F_n \xrightarrow{2}_{\varphi} F$ as $n \to \infty$.

L e m m a 3. If $\{F_n\}$ fulfils the (C,φ)-condition in X^2_φ and there exists $M > 0$ such that $m_1(W(F_n)(t) \cap W(F_m)(t)) > 0$ for all $n,m \geqslant M$, $n \neq m$, and every $t \in \Omega$, then $\{\underline{f}(F^2_n)\}$ and $\{\bar{f}(F^1_n)\}$ are Cauchy sequences in $L^\varphi(\Omega,\Sigma,\mu)$.

P r o o f . It is easy to see that, if $m_1(W(F_n)(t) \cap W(F_m)(t)) > 0$, then

$$|\bar{f}(F^1_n)(t) - \bar{f}(F^1_m)(t)| \leqslant m(F_n \asymp F_m)(t) + |\underline{f}(F_n)(t) - \underline{f}(F_m)(t)| + |\bar{f}(F_n)(t) - \bar{f}(F_m)(t)|,$$

$$|\underline{f}(F^2_n)(t) - \underline{f}(F^2_m)(t)| \leqslant m(F_n \asymp F_m)(t) + |\underline{f}(F_n)(t) - \underline{f}(F_m)(t)| + |\bar{f}(F_n)(t) - \bar{f}(F_m)(t)|.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

C o r o l l a r y 1. If the assumptions of the Lemma 3 hold, then there exists $F \in X_\varphi^2$ such that $F_n \xrightarrow[\varphi]{2} F$.

P r o o f . We obtain, from the Lemma 3, that there exist $g_1, g_2 \in L^\varphi(\Omega, \Sigma, \mu)$ such that $\underline{f}(F_n^2) \longrightarrow g_2$, $\bar{f}(F_n^1) \longrightarrow g_1$ in $L^\varphi(\Omega, \Sigma, \mu)$ and $g_1(t) \leqslant g_2(t)$ for every $t \in \Omega$. If $\{F_n\}$ fulfils the (C,φ)-condition in X_φ^2, then there exist $f_1, f_2 \in L^\varphi(\Omega, \Sigma, \mu)$ such that $\underline{f}(F_n) \longrightarrow f_1$, $\bar{f}(F_n) \longrightarrow f_2$ in $L^\varphi(\Omega, \Sigma, \mu)$ and $f_1(t) \leqslant g_1(t) \leqslant g_2(t) \leqslant f_2(t)$ for every $t \in \Omega$. Let $F(t) = ([f_1(t), g_1(t)] \cup [g_2(t), f_2(t)])$ for every $t \in \Omega$. It is easy to see that $F \in X_\varphi^2$ and $F_n \xrightarrow[\varphi]{2} F$.

From Lemmas 1, 2, 3, Remark 5 and Corollary 1 we obtain

T h e o r e m 2. X_φ^2 is the $(C,\varphi)^2$-complete i.e. every sequence which fulfiled the (C,φ)-condition in X_φ^2 is φ-convergent in X_φ^2.

R e m a r k 6.

a) If $F_n \xrightarrow[\varrho_\varphi]{2,1} F$, then $F_n \xrightarrow[\varphi]{2} F$ (see [1]).

b) If $F_n \xrightarrow[(s,\varrho_\varphi)]{} F$, then $F_n \xrightarrow[\varphi]{2} F$ (see [1]).

c) If $F_n \xrightarrow[\varrho_\varphi]{2} F$, then $F_n \xrightarrow[\varphi]{2} F$ (see [1]).

d) If $F_n \xrightarrow[(w,\varrho_\varphi)]{} F$, then $F_n \xrightarrow[\varphi]{2} F$ (see [1]).

3. On the space X_φ^3

Let $X_\varphi^3 = \{F \in X : F(t) = F^1(t) \cup F^2(t) \cup F^3(t), \bar{f}(F^1)(t) \leqslant \underline{f}(F^2)(t), \bar{f}(F^2)(t) \leqslant \underline{f}(F^3)(t)$ for every $t \in \Omega$, $F^1, F^2, F^3 \in X_\varphi^1\}$.

Let $F \in X_\varphi^3$. We introduce the multifunctions $F^{(1)}$ and $F^{(2)}$ by the formulas:

$$F^{(1)}(t) = \operatorname{conv}(F^1(t) \cup F^2(t)) \cup F^{(3)}(t),$$

$$F^{(2)}(t) = F^1(t) \cup \operatorname{conv}(F^2(t) \cup F^3(t)) \quad \text{for every } t \in \Omega.$$

D e f i n i t i o n 5. Let $F_n \in X_\varphi^3$ for every $n \in N$ and let $F \in X_\varphi^3$. The sequence $\{F_n\}$ is φ-convergent to F in X_φ^3 $\left(F_n \xrightarrow[\varphi]{3} F\right)$, if there exist $G_1, G_2 \in X_\varphi^2$ such that $F_n^{(1)} \xrightarrow[\varphi]{2} G_1$, $F_n^{(2)} \xrightarrow[\varphi]{2} G_2$ and $F(t) = G_1(t) \cap G_2(t)$ for every $t \in \Omega$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

R e m a r k 3. If $F_n \xrightarrow{3}_{\varphi} F$, then $m(F_n \nsim F) \to 0$ in $L^{\varphi}(\Omega, \Sigma, \mu)$ as $n \to \infty$.

R e m a r k 8. If $F_n \xrightarrow{2}_{\varphi} F$, then $F_n \xrightarrow{3}_{\varphi} F$.

D e f i n i t i o n 6. Let $F_n \in X^3_{\varphi}$ for every $n \in \mathbb{N}$. The sequence $\{F_n\}$ fulfils the (C,φ)-condition in X^3_{φ}, if $\{F_n^{(1)}\}$ and $\{F_n^{(2)}\}$ fulfil the (C,φ)-condition in X^2_{φ}.

T h e o r e m 3. X^3_{φ} is the $(C,\varphi)^3$-complete i.e. every sequence which fulfiled the (C,φ)-condition in X^3_{φ} is φ-convergent in X^3_{φ}.

P r o o f. If $\{F_n\}$ fulfils the (C,φ)-condition in X^3_{φ}, then there exist $G_1, G_2 \in X^2_{\varphi}$ such that $F_n^{(1)} \xrightarrow{2}_{\varphi} G_1$ and $F_n^{(2)} \xrightarrow{2}_{\varphi} G_2$. Let $F(t) = G_1(t) \cap G_2(t)$. It is easy to see that $F_n \xrightarrow{3}_{\varphi} F$.

4. <u>On the space X^k_{φ}</u>

Let $k \in \mathbb{N}$, $k \geqslant 4$ and let

$$X^k_{\varphi} = \Big\{ F \in X : F(t) = \bigcup_{i=1}^{k} F^i(t),\ \bar{f}(F^i)(t) \leqslant \underline{f}(F^{i+1})(t) \text{ for } i = 1,\dots,k-1, \text{ for every } t \in \Omega,\ F^1,\dots,F^k \in X^1_{\varphi} \Big\}.$$

Let $F \in X^k_{\varphi}$. We introduce the multifunctions $F^{(1)},\dots,F^{(k-1)}$ by the formulas:

$$F^{(1)}(t) = \operatorname{conv}(F^1(t) \cup F^2(t)) \cup \bigcup_{i=3}^{k} F^i(t),$$

$$F^{(2)}(t) = F^1(t) \cup \operatorname{conv}(F^2(t) \cup F^3(t)) \cup \bigcup_{i=4}^{k} F^i(t),$$

$$F^{(3)}(t) = F^1(t) \cup F^2(t) \cup \operatorname{conv}(F^3(t) \cup F^4(t)) \cup \bigcup_{i=5}^{k} F^i(t),$$

...

$$F^{(k-1)}(t) = \bigcup_{i=1}^{k-2} F^i(t) \cup \operatorname{conv}(F^{k-1}(t) \cup F^k(t)),$$

for every $t \in \Omega$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D e f i n i t i o n 7. Let $F_n \in X_\varphi^k$ for every $n \in N$ and let $F \in X_\varphi^k$. The sequence $\{F_n\}$ is φ-convergent to F in X_φ^k ($F_n \xrightarrow[\varphi]{k} F$), if there exist $G_1,\dots,G_{k-1} \in X_\varphi^{k-1}$ such that $F_n^{(1)} \xrightarrow[\varphi]{k-1} G_1$, $F_n^{(2)} \xrightarrow[\varphi]{k-1} G_2,\dots,F_n^{(k-1)} \xrightarrow[\varphi]{k-1} G_{k-1}$, $F(t) = \bigcap_{i=1}^{k-1} G_i(t)$ for every $t \in \Omega$.

Let $F \in X_\varphi^k$. We introduce the multifunctions $\underline{F}^{(1)},\dots,\underline{F}^{(k-1)}$ by the formulas:

$$\underline{F}^{(1)}(t) = F^1(t) \cup \operatorname{conv}\left(\bigcup_{i=2}^{k} F^i(t)\right),$$

$$\underline{F}^{(2)}(t) = \operatorname{conv}((F^1(t) \cup F^2(t)) \cup \operatorname{conv}\left(\bigcup_{i=3}^{k} F^i(t)\right),$$

$$\underline{F}^{(3)}(t) = \operatorname{conv}(F^1(t) \cup F^2(t) \cup F^3(t)) \cup \operatorname{conv}\left(\bigcup_{i=4}^{k} F^i(t)\right),$$

...

$$\underline{F}^{(k-2)}(t) = \operatorname{conv}\left(\bigcup_{i=1}^{k-2} F^i(t)\right) \cup \operatorname{conv}(F^{k-1}(t) \cup F^k(t)),$$

$$\underline{F}^{(k-1)}(t) = \operatorname{conv}\left(\bigcup_{i=1}^{k-1} F^i(t)\right) \cup F^k(t) \quad \text{for every } t \in \Omega.$$

D e f i n i t i o n 7. Let $F_n \in X_\varphi^k$ for every $n \in N$ and let $F \in X_\varphi^k$. The sequence $\{F_n\}$ is $(\varphi,2)$-convergent to F in X_φ^k ($F_n \xrightarrow[\varphi,2]{k} F$), if there exist $G_1,G_2,\dots,G_{k-1} \in X_\varphi^2$ such that $\underline{F}_n^{(i)} \xrightarrow[\varphi]{2} G_i$ for $i = 1,\dots,k-1$ and $F(t) = \bigcap_{i=1}^{k-1} G_i(t)$ for every $t \in \Omega$.

R e m a r k 9. If $F_n \xrightarrow[\varphi]{i} F$, then $F_n \xrightarrow[\varphi]{i+1} F$ for every $i \in N$.

R e m a r k 10. If $F_n \xrightarrow[\varphi,2]{i} F$, then $F_n \xrightarrow[\varphi,2]{i+1} F$ for $i \geqslant 3$ and $i \in N$.

R e m a r k 11. $F_n \xrightarrow[\varphi]{i} F$ iff $F_n \xrightarrow[\varphi,2]{i} F$ for $i \geqslant 3$ and $i \in N$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D e f i n i t i o n 8. Let $F_n \in X_\varphi^k$ for every $n \in N$. The sequence $\{F_n\}$ fulfils the $(C,\varphi,2)$-condition in X_φ^k, if $\{\underline{F}_n^{(i)}\}$ fulfils the (C,φ)-condition in X_φ^2 for $i = 1,2,\ldots,k-1$.

T h e o r e m 4. X_φ^k is $(C,\varphi,2)^k$-complete i.e. every sequence which fulfiled the $(C,\varphi,2)$-condition in X_φ^k is $(\varphi,2)$-convergent in X_φ^k.

The proof is quite analogous to that of Theorem 3 and we omit it.

5. On the space X_φ

Let $X_\varphi = \{F \in X: \underline{f}(F), \bar{f}(F), m(F) \in L^\varphi(\Omega,\Sigma,\mu)\}$.

D e f i n i t i o n 9. Let $F_n \in X_\varphi$ for every $n \in N$ and let $F \in X_\varphi$. The sequence $\{F_n\}$ is φ-convergent to F in X_φ ($F_n \xrightarrow{\varphi} F$), if:

a) $\underline{f}(F_n) \longrightarrow \underline{f}(F)$, $\bar{f}(F_n) \longrightarrow \bar{f}(F)$, $m(F_n \nsim F) \longrightarrow 0$ in $L(\Omega,\Sigma,\mu)$ as $n \to \infty$,

b) if $f_n(t) \in F_n(t)$ for all $n \in N$ and μ-a.e. $t \in \Omega$ and $f_n \longrightarrow f$ in $L^\varphi(\Omega,\Sigma,\mu)$ as $n \to \infty$, then $f(t) \in F(t)$ for μ-a.e. $t \in \Omega$.

D e f i n i t i o n 10. Let $F_n \in X_\varphi$ for every $n \in N$. The sequence $\{F_n\}$ fulfils the (C,φ)-condition in X_φ, if $\{\underline{f}(F_n)\}$ and $\{\bar{f}(F_n)\}$ are Cauchy sequences in $L^\varphi(\Omega,\Sigma,\mu)$ and $m(F_n \nsim F_m) \longrightarrow 0$ in $L^\varphi(\Omega,\Sigma,\mu)$ as $n,m \to \infty$.

R e m a r k 12. If $F_n \xrightarrow{\varphi} F$ and $F_n \xrightarrow{\varphi} G$, then it need not to be $F = G$, but $m_1(F(t) \nsim G(t)) = 0$ for μ-a.e. $t \in \Omega$. If in X_φ we admit that $F = G$ iff $m_1(F(t) \nsim G(t)) = 0$ for μ-a.e. $t \in \Omega$, then the assumptions b) in definition 9 may not to be fulfiled, but it is very important in many problems of mathematical programming and optimal control.

R e m a r k 13. X_φ need not to be (C,φ)-complete.

E x a m p l e . Let $\Omega = [0,1]$, $\varphi(t) = |t|$ and let $\{p_n\}$ be the sequence of rational numbers from the interval $[0,1]$ such that $p_1 = 0$ and $p_2 = 1$. Let $F_n(t) = \{p_1\} \cup \{p_2\} \cup \ldots \cup \{p_n\}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for every $t \in \Omega$ and all $n \in N$. The sequence $\{F_n\}$ fulfils the (C,φ)-condition in X_φ, but it is not φ-convergent in X_φ. The set of all rational numbers from the interval $[0,1]$ is not compact. If we take $F(t) = [0,1]$ for every $t \in [0,1]$, then $a_1(F_n(t) \setminus F(t)) = 1$ for every $t \in [0,1]$.

Remark 14. If we replaced the φ-function φ by a φ-function with parameter $\varphi(t,u)$ and L^φ by a generalized Orlicz space (see [2]), then we obtain the same results as in this paper.

BIBLIOGRAPHY

[1] A. Kasperski : The modular spaces of multifunctions, Demonstratio Math. 22 (1989), 419-439.

[2] J. Musielak : Orlicz spaces and modular spaces, Springer Verlag, Berlin-Heidelberg-New York-Tokyo 1983.

INSTITUTE OF MATHEMATICS, SILESIAN TECHNICAL UNIVERSITY,
44-100 GLIWICE, POLAND

Received June 22, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Jan Straburzyński

THE MULTI-VALUED GOURSAT PROBLEM FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

0. Introduction

Given positive numbers α, β, a, b and non-decreasing functions $y = g(x)$, $x = h(y)$ of C' class defined on $[0,a]$, $[0,b]$ let $g(0) = h(0)$, $0 < g(x) \leqslant b$, $0 \leqslant h(y) \leqslant a$, $P = [-\alpha,a]\,[-\beta,b]$ and the curves $y = g(x)$, $x = h(y)$ have only one common point at zero.

Furthermore let $D = \{(s,t) : h(t) < s \leqslant a,\ g(s) < t \leqslant b\}$, $D_{xy} = \{(s,t) : h(t) < s \leqslant x,\ g(s) < t \leqslant y \text{ for } x \in [0,a] \text{ and } y \in [0,b]\}$, $\bar{D}_x = \{y \in [g(x),b] : (x,y) \in \bar{D}\}$, $\bar{D}_y = \{x \in [h(y),a] : (x,y) \in \bar{D}\}$ and $G = P \setminus D$.

In what follows we denote by $C_0(P,R^n)$ and $L^2(P,R^n)$ the space of all continuous functions with supremum norm $\|\cdot\|_0$ and square Lebesgue integrable functions with usual norm $\|\cdot\|_p$, respectively.

Moreover, we use the space $C_1(P,R^n)$ of all functions v such that $v(\cdot,y) : [-\alpha,a] \ni x \longrightarrow v(x,y) \in R^n$ is measurable and $v(x,\cdot) : [-\beta,b] \ni y \longrightarrow v(x,y) \in R^n$ is continuous for a.a $x \in [-\alpha,a]$ and such that

$$\|v\|_1 = \int_{-\alpha}^{a} \max\{|v(x,y)| : y \in [-\beta,b]\}\,dx < \infty.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Similarly we define the space $C_2(P,R^n)$ with the norm $\|\cdot\|_2$. As in [1] we can verify that $(C_1(P,R^n),\|\cdot\|_1)$ and $(C_2(P,R^n),\|\cdot\|_2)$ are Banach spaces.

Denote by $\Phi(P,R^n)$ the set of all absolutely continuous functions such that $\varphi_x \in C_1(P)$, $\varphi_y \in C_2(P)$ and $\varphi_{xy} \in L^2(P)$.

In this paper we will study the following Goursat problem for a functional-differential inclusion in the form

$$(1) \quad z_{xy}(x,y) \in F(x,y,z,z_x,z_y,z_{xy}) \text{ for almost all } (x,y) \in \bar{D}$$

with boundary conditions

$$(2) \quad z(x,y) = \varphi(x,y) \quad \text{for} \quad (x,y) \in G$$

where $F : \bar{D} \times C_0(P) \times C_1(P) \times C_2(P) \quad L^2(P) \longrightarrow \mathrm{Conv}(R^n)$, $\varphi \in \Phi$ is a fixed and $\mathrm{Conv}(R^n)$ denotes the family of all nonempty compact and convex subset of R^n with the Hausdorff metric h.

A function $z:P \longrightarrow R^n$ absolutely continuous satisfying the above conditions will be called a solution of problem (1)-(2).

Generalized functional-differential equations of hyperbolic type have been considered in the author's paper [2] and [3]. In this paper we give sufficient conditions for existence of solutions for problem (1)-(2). This result extends the above, in particular the results given in [6], [7].

1. Assumptions and lemma

We shall consider (1)-(2) with F having the Volterra's property. Namely, we shall say that $F:D \times C_0(P) \times C_1(P) \times C_2(P) \times L^2(P) \longrightarrow \mathrm{Conv}(R^n)$ has the Volterra's property if for $z_1,z_2 \in C_0(P)$, $p_1,p_2 \in C_1(P)$, $q_1,q_2 \in C_2(P)$ and $r_1,r_2 \in L^2(P)$ such that $z_1(s,t) = z_2(s,t)$, $p_1(s,t) = p_2(s,t)$, $q_1(s,t) = q_2(s,t)$ and $r_1(s,t) = r_2(s,t)$ for $(s,t) \in P(x,y)$ it follows that

$$F(x,y,z_1,p_1,q_1,r_1) = F(x,y,z_2,p_2,q_2,r_2),$$

where $P(x,y) = [-\alpha,x] \times [-\beta,y]$ for $(x,y) \in \bar{D}$.

Furthermore, we introduce the following Carathéodory type conditions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $F(\cdot,\cdot,z,p,q,r)$ is measurable for fixed $(z,p,q,r) \in C_0 \times C_1 \times C_2 \times L^2$,

(ii) $F(x,y,\cdot,\cdot,\cdot,\cdot)$ is continuous for fixed $(x,y) \in \bar{D}$,

(iii) there exists a square Lebesgue integrable function $m: \bar{D} \to R$ such that $h(F(x,y,z,p,q,r),\{0\}) \leqslant m(x,y)$ for $(z,p,q,r) \in C_0 \times C_1 \times C_2 \times L^2$ and a.a $(x,y) \in \bar{D}$.

Moreover, it will be assumed that $F(x,y,z,\cdot,\cdot,\cdot)$ is Lipschitz continuous, uniformly with respect to $z \in C_0$ i.e. that

(iv) there exists a square Lebesgue integrable function $k: \bar{D} \to R$ such that

$$h(F(x,y,z,p,q,r),\ F(x,y,z,\bar{p},\bar{q},\bar{r})) \leqslant k(x,y)(\|p-\bar{p}\|_1+\|q-\bar{q}\|_2+\|r-\bar{r}\|$$

In the present paper will be used the following result [4].

L e m m a . Suppose A is a nonempty, closed, bounded and convex subset of a Hilbert space $(Y,\|\cdot\|)$ and let Γ be an operator with domain A and range in a Banach space $(X,|\cdot|)$. Suppose further that $G : A \times \Gamma(A) \to Cl(A)$ is such that

1^0 $G(\cdot,y)$ is a contraction, uniformly with respect to $y \in \Gamma(A)$,

2^0 $G(x,\cdot)$ is continuous on $\Gamma(A)$ in the relative topology.

If Γ is complety continuous, then there exists $x \in A$ such that $x \in G(x,\Gamma(x))$.

2. Existence theorem

T h e o r e m . Suppose $F : \bar{D} \times C_0 \times C_1 \times C_2 \times L^2 \to \mathrm{Conv}\, R^n$ satisfies the Carathéodory conditions, the Lipschitz condition (iv) and let has the Volterra´s property. Then problem (1)-(2) has at least one solution.

P r o o f . Let $A = \{ w \in L^2(P,R^n) : w|_G = \varphi_{xy}, |w(x,y)| \leqslant m(x,y)$ for a.a $(x,y) \in D\}$. It is easy to see that A is nonempty, closed, bounded and convex subset of $L^2(P)$. Define now a mapping $\Gamma : A \to C_0(P)$ by setting ([6])

$$\Gamma(w)(x,y) = \begin{cases} \varphi(x,y) & \text{for } (x,y) \in G, \\ \varphi(x,g(x)) + \varphi(h(y),y) - \varphi(0,0) + \iint\limits_{\bar{D}_{xy}} w(s,t)\,ds\,dt, \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $(x,y) \in D$. We observe that Γ is a completely continuous.

Moreover, let

$$\Gamma_1(w)(x,y) = \begin{cases} \varphi_x(x,y) & \text{for } (x,y) \in G \\ \varphi_x(x,g(x)) + \int_{g(x)}^{y} w(x,t)dt & \text{for a.a } x \in [0,a],\ y \in \bar{D}_x \end{cases}$$

and

$$\Gamma_2(w)(x,y) = \begin{cases} \varphi_y(x,y) & \text{for } (x,y) \in G \\ \varphi_y(h(y),y) + \int_{h(y)}^{x} w(s,y)ds & \text{for } x \in \bar{D}_y \text{ and a.a } y \in [0,b] \end{cases}$$

Writing

(3) $w(x,y) \in F(x,y,\Gamma(w),\Gamma_1(w),\Gamma_2(w),w)$ for a.a $(x,y) \in \bar{D}$

we observe that for each $w \in A$ satisfying (3) the function $z = \Gamma(w)$ satisfies (1)-(2).

As in [4] let $G(w,z)$ denote for fixed $(w,z) \in A \times C_0(P)$ a subset of $L^2(P)$ containing all measurable selectors $v: \bar{D} \longrightarrow R^n$ a mapping $H: \bar{D} \longrightarrow \mathrm{Conv}(R^n)$ defined by setting $H(x,y) = F(x,y,z,\Gamma_1(w),\Gamma_2(w),w)$ for $(x,y) \in \bar{D}$. By a general selector theorem of Kuratowski and Ryll-Nardzewski [5], we have $G(w,z) \neq \emptyset$ for each $(w,z) \in A \times C_0(P)$. Furthermore, from assumptions concerning the mappings F we see that $G(w,z)$ is a closed and convex subset of A. Then, $G: A \times C_0(P) \longrightarrow Cl(A)$.

We shall show now that G satisfies the assumptions of Lemma. Let $z \in C_0(P)$ be fixed and let $w_1, w_2 \in A$. For each $\lambda \in G(w_1,z)$ we have $\lambda(x,y) \in F(x,y,z,\Gamma_1(w_1),\Gamma_2(w_1),w_1)$ for almost all $(x,y) \in \bar{D}$.

Observe that we can find (x_0,y_0) and k such that

(x) $$\left(2\sqrt{D_{x_0y_0}} + 1\right)\|k\|_{D_{x_0y_0}} < 1.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since

$$h(F(x,y,z,\Gamma_1(w_1),\Gamma_2(w_1),w_1),\ F(x,y,z,\Gamma_1(w_2),\Gamma_2(w_2),w_2)) \leqslant$$

$$\leqslant k(x,y)(|\Gamma_1(w_1) - \Gamma_1(w_2)|_1 + |\Gamma_2(w_1) - \Gamma_2(w_2)|_2 + \|w_1 - w_2\|_P) =$$

$$= k(x,y)\left\{\int_0^a \max_{y\in[0,b]} \left(\left|\int_{g(x)}^{y} w_1(x,t)dt - \int_{g(x)}^{y} w_2(x,t)dt\right|\right)dx +\right.$$

$$\left. + \int_0^b \max_{x\in[0,a]} \left(\left|\int_{h(y)}^{x} w_1(x,y)ds - \int_{h(y)}^{x} w_2(s,y)ds\right|\right)dy + \|w_1 - w_2\|_P\right\} \leqslant$$

$$\leqslant k(x,y)\left(2\iint_D |w_1(x,y) - w_2(x,y)|\,dxdy + \|w_1 - w_2\|_P\right) \leqslant$$

$$\leqslant k(x,y)(2\sqrt{|D|} + 1)\,\|w_1 - w_2\|_P$$

for a.a $(x,y) \in D$, then as in [4] there exists $\gamma \in G(w_2,z)$ such that

$$|\lambda(x,y) - \gamma(x,y)| \leqslant k(x,y)(2\sqrt{|D|} + 1)\,\|w_1 - w_2\|_P$$

for a.a $(x,y) \in \bar{D}$. Consequently,

$$\|\lambda - \gamma\|_P \leqslant (2\sqrt{|D|} + 1)\|k\|_D\|w_1 - w_2\|_P.$$

Hence it follows that

$$h_1(G(w_1,z),G(w_2,z)) \leqslant (2\sqrt{|D|} + 1)\|k_D\|w_1 - w_2\|_P$$

where h_1 denotes the Hausdorff metric in Cl(A) generated by the norm $\|\cdot\|_P$. Now, by assumption that F has Volterra property and (x) it follows that $(2\sqrt{|D|}+1)\|k\|_D < 1$. Hence, $G(\cdot,z)$ is a contraction. Similarly we can verify that $G(w,\cdot)$ is continuous for fixed $w \in A$. Now, in virtue of Lemma there exists $w \in A$ such that $w \in G(w,\Gamma(w))$ which means that $w(x,y) \in F(x,y,\Gamma(w),\Gamma_1(w),\Gamma_2(w),w)$ for a.a $(x,y) \in \bar{D}$. Hence, it follows that $z = \Gamma(w)$ is absolutely continuous on P and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$z(x,y) = \varphi(x,y)$ for $(x,y) \in G$

$z_{xy}(x,y) \in F(x,y,z,z_x,z_y,z_{xy})$ for a.a $(x,y) \in \overline{D}$

which completes the proof.

REFERENCES

[1] K. Deimling: A Carathéodory theory for systems of integral equations, Ann. Mat. Pura Appl. (4) 86 (1970) 217-260.

[2] T. Pruszko: Some applications of the topological degree theory to multivalued boundary value problems, Dissertationes Math. 129 (1984), 36-39.

[3] W. Sosulski: Properties of the solution of generalized differential equation of hyperbolic type, W.S-Inż. Zielona Góra (1982).

[4] M. Kisielewicz: Generalized functional-differential equations of neutral type, Ann. Polon. Math. 42 (1983), p.139-148.

[5] K. Kuratowski, C. Ryll-Nardzewski: A general theorem on selectors, Bull. Pol. Acad. Sci., 13 (1965), 397-403.

[6] J. Straburzyński: Existence of solutions of the Goursat problem for some functional-differential equations, Demonstratio Math. 15 (1982), 883-897.

[7] J. Straburzyński: On the convergence of successive approximations for functional-differential equations of hyperbolic type, Demonstratio Math. (1988) (to appear)

INSTITUTE OF MATHEMATICS AND PHYSICS, ENGINEERING COLLEGE, 65-246 ZIELONA GÓRA, POLAND

Received July 4, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Nguyen Van Mau

INTERPOLATION PROBLEMS INDUCED BY RIGHT AND LEFT INVERTIBLE OPERATORS AND ITS APPLICATIONS TO SINGULAR INTEGRAL EQUATIONS

The interpolation problems induced by right invertible operators were investigated by Przeworska-Rolewicz [1]. In the present paper we obtain necessary and sufficient conditions for general interpolation problems induced by right and left invertible operators to have a unique solution. As applications, we deal with interpolation problems from theory of singular integral operators.

1. Characterization of initial operators

Let X be a linear space over a field $\mathcal{F}$ of scalars (where $\mathcal{F}=R$ or $\mathcal{F}=C$). Denote by $L(X)$ the set of all linear operators with domains and ranges in X and let $L_0(X) = \{A \in L(X): \operatorname{dom} A = X\}$. Denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$. For a $D \in R(X)$ we write

$$\mathcal{R}_D = \{R \in L_0(X): DR = I\},$$

$$\mathcal{I}_D = \{P \in L_0(X): P^2 = P,\ PX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D} FR = 0\}.$$

In the sequel we assume that $\dim \ker D \neq 0$, i.e. D is not left invertible.

Definition 1.1 (cf. [2]). (i) Every operator $P \in \mathcal{I}_D$ is said to be an initial operator for D corresponding to R.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Elements of the form $z_0 + Rz_1 + \dots + R^m z_m$, where $z_k \in \ker D$, are called D-polynomials.

D e f i n i t i o n 1.2. (cf. [1]). (i) Let $D \in R(X)$. An initial operator F_0 for D has the property c(R) for an $R \in \mathcal{R}_D$, if there exist scalars c_k such that

$$F_0 R^k z = \frac{c_k}{k!} z, \quad \text{for all } z \in \ker D, \quad k \in \mathbb{N}.$$

(We admit $c_0 = 1$, since $F_0 z = z$ for all $z \in \ker D$).

(ii) A set $\mathcal{F}_D^0 \subset \mathcal{F}_D$ has the property (c), if for every $F_0 \in \mathcal{F}_D^0$ there exists an $R \in \mathcal{R}_D$ such that $F_0 \in c(R)$.

The following result has been obtained in [1].

T h e o r e m 1.1. Let $D \in R(X)$. The set $\mathcal{F}_D$ of all initial operators has the property (c), if and only if dim ker D=1.

Clearly, if the system $\{F_0, F_1, \dots, F_{N-1}\} \in \mathcal{F}_D$ has the property c(R) with constants d_{ik}, i.e.

$$F_i R^k z = \frac{d_{ik}}{k!} z \quad \text{for} \quad i = 0,1,\dots,N-1, \quad k \in \mathbb{N}_0, \tag{1.0}$$

and $F_0, \dots, F_{N-1}$ are linearly dependent, then

$$V_N = \det(d_{ik})_{i,k=0}^{N-1} = 0. \tag{1.1}$$

The following question was stated in [1]: does the determinant V_n of the form (1.1) not vanish for any system $\{F_0, F_1, \dots, F_{N-1}\}$ of linearly independent initial operators having the property c(R)?

It is easy to see that the answer is positive for N = 1 (since by definition, $d_{00} = 1$). However, we shall show that, in general, the answer is negative for $N \geq 2$.

E x a m p l e 1.1. Let $X = C(R)$, $D = \frac{d}{dt}$, $R = \int_0^t$. Write

$$(F_0 x)(t) = x(0), \quad (F_1 x)(t) = \frac{1}{2}(x(1) + x(-1)),$$

$$(F_2 x)(t) = \frac{1}{2}(x(2) + x(-2)).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Obviously, F_0, F_1, F_2 are initial operators for D having the property (c), since dim ker D = 1. We prove that F_0, F_1, F_2 are linearly independent. Indeed, let $a_0F_0 + a_1F_1 + a_2F_2 = 0$, i.e.

$$(1.2)\quad a_0(F_0x)(t) + a_1(F_1x)(t) + a_2(F_2x)(t) = 0,\ a_0,a_1,a_2 \in \mathcal{F}.$$

(i) If $x(t) = 1$, then (1.2) implies $a_0 + a_1 + a_2 = 0$.

(ii) If $x(t) = \exp\left(\frac{\pi i t}{2}\right)$, then (1.2) implies $a_0 - a_2 = 0$.

(iii) If $x(t) = \exp(\pi i t)$, then (1.2) implies $a_0 - a_1 + a_2 = 0$.

From (i)-(iii) we get $a_0 = a_1 = a_2 = 0$. Hence F_0, F_1, F_2 are linearly independent.

On the other hand, it is easy to see that

$F_0R^0z = F_1R^0z = F_2R^0z = z$, hence $d_{00} = d_{10} = d_{20} = 1$,

$F_0Rz = F_1Rz = F_2Rz = 0$, i.e. $d_{01} = d_{11} = d_{21} = 0$,

$F_0R^2z = 0$, $F_1R^2z = z$, $F_2F^2z = 4z$, i.e.

$d_{02} = 0$, $d_{12} = 1$, $d_{22} = 8$.

Hence

$$(d_{ik})^2_{i,k=0} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 8 \end{pmatrix}.$$

It is easy to see that $V_3 = \det(d_{ik})^2_{i,k=0} = 0$.

Observe that the vectors $d_0 = (1,0,0)$, $d_1 = (1,0,2)$, $d_2 = (1,0,8)$ are linearly dependent, i.e.

$$3d_0 + d_2 - 4d_1 = 0.$$

Thus, in general, linearly independent initial operators do not imply linearly independent vectors $d_0 = (d_{00},d_{01},\ldots,d_{0N-1}),\ldots,d_{N-1} = (d_{N-1,0},d_{N-1,1},\ldots,d_{N-1,N-1})$.

The following question arises: does exist a set $X_0 \subset X$ such that $V_n \neq 0$ (cf. (1.1)), if and only if the restrictions of initial operators $F_0,F_1,\ldots,F_{N-1}$ to X_0 are linearly in-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

dependent? The following theorem shows that the answer to this question is positive.

T h e o r e m 1.2. If $F_0,F_1,\dots,F_{N-1} \in \mathcal{F}_D$ have the property (c), then a necessary and sufficient condition for $V_N \neq 0$, where V_N is defined by formula (1.1), is that $F_0,F_1,\dots,F_{N-1}$ are linearly independent on $P_N(R)$ defined as follows

$$P_N(R) = \operatorname{lin}\{R^k z : z \in \ker D, \quad k=0,\dots,N-1\}. \tag{1.3}$$

The proof is based on the following additional lemma.

L e m m a 1.1. Let $F_0,F_1,\dots,F_{N-1} \in \mathcal{F}_D$ possess the property (c). Write

$$\hat{F}_i = (F_i, F_iR,\dots,F_iR^{N-1}) \text{ for } i=0,\dots,N-1, \tag{1.4}$$

$$d_i = (d_{i0}, d_{i1},\dots,d_{iN-1}) \text{ for } i=0,\dots,N-1, \tag{1.5}$$

where d_{ik} are defined by formula (1.0).

Then the vectors $\hat{F}_0,\hat{F}_1,\dots,\hat{F}_{N-1}$ of the form (1.4) are linearly independent on ker D, i.e. the equality

$$\sum_{i=0}^{N-1} a_i\hat{F}_i z = 0 \quad \text{for all } z \in \ker D, \quad a_i \in \mathcal{F}, \quad i = 0,1,\dots,N-1,$$

implies $a_0 = a_1 = \dots = a_{N-1} = 0$, if and only if the vectors $d_0,d_1,\dots,d_{N-1}$ of the form (1.5) are linearly independent.

P r o o f . Indeed, the operators $\hat{F}_0,\hat{F}_1,\dots,\hat{F}_{N-1}$ are linearly independent on ker D, if and only if for every fixed index j $(0 \leqslant j \leqslant N-1)$, the operators $F_0R^j,F_1R^j,\dots,F_{N-1}R^j$ are independent on ker D. Hence, if $d_0,d_1,\dots,d_{N-1}$ are linearly independent and

$$\sum_{i=0}^{N-1} a_iF_iR^jz = 0 \quad \text{for all } z \in \ker D \text{ and } a_i \in \mathcal{F} \ (i=0,1,\dots,N-1)$$

then, by assumptions (1.1), we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\sum_{i=0}^{N-1} a_i \frac{d_{ij}}{j!} z = 0$ for all $z \in \ker D$ and for every j $(0 \leqslant j \leqslant N-1)$.

The arbitrariness of $z \in \ker D$ implies that

$$\sum_{i=0}^{N-1} a_i d_{ij} = 0 \quad \text{for each} \quad j \quad (0 \leqslant j \leqslant N-1).$$

This system can be written as

$$\sum_{i=0}^{N-1} a_i (d_{i0}, d_{i1}, \ldots, d_{iN-1}) = 0 \quad \text{i.e.} \quad \sum_{i=0}^{N-1} a_i d_i = 0.$$

Our assumption implies $a_0 = a_1 = \ldots = a_{N-1} = 0$.

Conversely, suppose that for every fixed index j $(0 \leqslant j \leqslant N-1)$ the operators $F_0 R^j$, $F_1 R^j, \ldots, F_{N-1} R^j$ are linearly independent on $\ker D$ and that

$$\sum_{i=0}^{N-1} b_i d_i = 0, \text{ where } d_i \text{ are defined by (1.5).}$$

The last equality can be written in the form

$$\sum_{i=0}^{N-1} b_i d_{ij} z = 0 \quad \text{for every} \quad z \in \ker D, \quad j = 0, \ldots, N-1.$$

This and assumption (1.0) together imply

$$\sum_{i=0}^{N-1} b_i F_i R^j z = 0 \quad \text{for every} \quad z \in \ker D, \quad j = 0, \ldots, N-1,$$

i.e. $b_0 = b_1 = \ldots = b_{N-1} = 0$, which was to be proved.

C o r o l l a r y 1.1. Let $D \in R(X)$, $R \in \mathcal{R}_D$ and $F_0, F_1, \ldots, F_{N-1} \in c(R)$. Then $V_N \neq 0$, if and only if $F_0 R^k, F_1 R^k, \ldots, F_{N-1} R^k$ are linearly independent on $\ker D$ for every fixed index k $(0 \leqslant k \leqslant N-1)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . Indeed, if $F_0R^k, F_1R^k,\dots,F_{N-1}R^k$ are linearly independent on ker D for every fixed k, then, by Lemma 1.1, the vectors $d_0,d_1,\dots,d_{N-1}$ of the form (1.5) are linearly independent i.e. $\operatorname{rank}(d_{ik})_{i,k=0}^{N-1} = N$. This implies that $V_N \neq 0$.

Conversely, if $V_N \neq 0$, then $\operatorname{rank}(d_{ik}) = N$. Hence, $d_0,d_1,\dots,d_{N-1}$ are linearly independent. By Lemma 1.1, $F_0R^k, F_1R^k,\dots,F_{N-1}R^k$ are linearly independent.

P r o o f of Theorem 1.2. Suppose that $V_N \neq 0$. Then, by Lemma 1.1, the vectors $\hat{F}_0,\hat{F}_1,\dots,\hat{F}_{N-1}$ of the form (1.4) are linearly independent on ker D. This means that the vectors $F_0R^j, F_1R^j,\dots,F_{N-1}R^j$ are linearly independent on ker D for every fixed index j ($0 \leqslant j \leqslant N-1$), i.e. $F_0,F_1,\dots,F_{N-1}$ are linearly independent on

$$\bigcup_{j=0}^{N-1} R^j(\ker D) = P_N(R).$$

Conversely, if $F_0,F_1,\dots,F_{N-1} \in c(R)$ are linearly independent on $P_N(R)$, then $V_N \neq 0$. By Corollary 1.1, it is enough to show that the system of vector-operators $\{\hat{F}_i = (F_i,F_iR,\dots,F_iR^{N-1}), i = 0,1,\dots,N-1\}$ is linearly independent on ker D. Suppose that

$$\sum_{i=0}^{N-1} a_i\hat{F}_iz = 0 \quad \text{for all } z \in \ker D, \quad a_i \in \mathcal{F}, \quad i = 0,1,\dots,N-1.$$

This means that

$$\sum_{i=0}^{N-1} a_iF_iR^jz = 0 \quad \text{for all } z \in \ker D, \quad j = 0,1,\dots,N-1.$$

The arbitrariness of j, $j = 0,1,\dots,N-1$, and $z \in \ker D$ implies that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sum_{j=0}^{N-1} b_j \sum_{i=0}^{N-1} a_i F_i R^j z = 0 \quad \text{for all } z \in \ker D,\ b_j \in \mathcal{F},$$

i.e.

$$\sum_{i=0}^{N-1} a_i F_i \sum_{j=0}^{N-1} b_j R^j z = 0.$$ This means that

$$\sum_{i=0}^{N-1} a_i F_i x = 0 \quad \text{for every} \quad x \in P_N(R).$$

Our assumption implies $a_0 = a_1 = \ldots = a_{N-1}$. The proof is complete.

E x a m p l e 1.2. Let X, D, R, F_i, i = 0,1,2, be defined as in Example 1.1. The set $P_3(R)$ is of the form

$$P_3(R) = \{z(t) : z(t) = a_0 + a_1 t + a_2 t^2;\ a_0, a_1, a_2 \in \mathcal{F}\}.$$

If $z \in P_3(R)$ and $z(t) = a_0 + a_1 t + a_2 t^2$, then

$$F_0 z = a_0, \quad F_1 z = a_0 + a_2, \quad F_2 z = a_0 + 4a_2.$$

It is easy to see that $3F_0 + F_2 - 4F_1 = 0$, i.e. F_0, F_1, F_2 are linearly dependent over $P_3(R)$. Theorem 1.2 shows that $V_3 = 0$.

E x a m p l e 1.3. Let X, D, R be defined as in Example 1.1 and $(Fx)(t) = x(0)$. Write:

$$(S_h x)(t) = x(t + h) \quad \text{for} \quad t, h \in \mathbb{R}, \quad x \in X,$$

thus

$$(F_h x)(t) = x(t), \quad R_h = \int_h^t .$$

$$F_h R^k c = c \frac{h^k}{k!} \quad \text{for} \quad c, h, t \in \mathbb{R}, \quad x \in X, \ k \in \mathbb{N}.$$

In this case, $d_{ik} = i^k$ and $V_N \neq 0$. The set $P_N(R)$ is of the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$P_N(R) = \{ z : z(t) = a_0 + a_1 t + \dots + a_{N-1} t^{N-1} \}.$$

Hence

$$F_i z = a_0 + a_1 i + \dots + a_{N-1} i^{N-1} \qquad (i = 0,1,\dots,N-1).$$

By Theorem 1.2, $F_0, F_1, \dots, F_{N-1}$ are linearly independent on $P_N(R)$.

2. General interpolation problem

Now we consider the following general interpolation problem:

Given n finite sets I_i of positive integers. Denote by r_i the power of the set I_i i.e. $\# I_i = r_i$ for $i = 1,2,\dots,n$. Let $N = r_1 + r_2 + \dots + r_n$. We are looking for a D.- polynomial u of degree $N-1$ satisfying for n given different initial operators $F_1, F_2, \dots, F_n$ the conditions

$$F_i D^k u = u_{ik}, \quad k \in I_i, \ i = 1,2,\dots,n, \tag{2.0}$$

where $u_{ik} \in \ker D$ are given, $u = z_0 + Rz_1 + \dots + R^{N-1} z_{N-1}$ for an $R \in \mathcal{R}_D$ and $z_0, z_1, \dots, z_{N-1} \in \ker D$ are to be determined.

Suppose that $F_1, F_2, \dots, F_n \in c(R)$, i.e. there exist scalars d_{ik} satisfying condition (1.0):

$$F_i R^k z = \frac{d_{ik}}{k!} z \quad \text{for all elements} \quad z \in \ker D.$$

In the sequel we assume that all elements of the set I_i are ordered, i.e.

$$0 \leqslant k_1 < k_2 < \dots < k_{r_i}, \quad k_j \in I_i \quad (j = 1,2,\dots,r_i).$$

Hence, the condition (2.0) is of the form

$$F_i D^{k_j} u = u_{ik_j}, \qquad i = 1,\dots,n; \ j = 1,\dots,r_i. \tag{2.1}$$

Let

$$u = \sum_{m=0}^{N-1} R^m z_m, \quad z_m \in \ker D, \quad m = 0,\dots,N-1.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then

$$F_i D^{k_j} u = \sum_{m=0}^{N-1} F_i D^{k_j} R^m z_m = \sum_{m=0}^{k_j-1} F_i D^{k_j-m} z_m + \sum_{m=k_j}^{N-1} F_i R^{m-k_j} z_m =$$

$$= \sum_{m=k_j}^{N-1} F_i R^{m-k_j} z_m = \sum_{m=k_j}^{N-1} \frac{d_{i,m-k_j}}{(m-k_j)!} z_m, \quad i = 1,\dots,n;\ j = 1,\dots,r_i.$$

Define the vectors

$$G_i^{(k_j)} = \left(\underbrace{0,\dots,0}_{k_j}, \frac{d_{i,0}}{0!}, \frac{d_{i,1}}{1!}, \dots, \frac{d_{i,N-1-k_j}}{(N-1-k_j)!} \right), \tag{2.2}$$

$$i = 1,\dots,n; \quad j = 1,\dots,r_i,$$

the matrix of dimension $r_i \times N$

$$G_i = \begin{pmatrix} G_i^{(k_1)} \\ G_i^{(k_2)} \\ \vdots \\ G_i^{(k_{r_i})} \end{pmatrix}, \tag{2.3}$$

and the square matrix of dimension N

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix}. \tag{2.4}$$

Consider the following vector-operators of dimension N

$$\hat{F}_i^{(k_j)} = \left(F_i D^{k_j}, F_i D^{k_j-1}, \dots, F_i, F_i R, \dots, F_i R^{N-1-k_j} \right), \tag{2.5}$$

$$i = 1,\dots,n; \ j = 1,\dots,r_i.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

L e m m a 2.1. The system or vector-operators (2.5) is linearly independent on ker D, if and only if rank G = N, where G is defined by formulae (2.2)-(2.4).

P r o o f . Suppose that the system $\left\{G_i^{(k_j)}\right\}$ $(i=1,\dots,n;$ $j=1,\dots,r_i)$ is linearly independent and that

$$(2.6)\qquad \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\hat{F}_i^{(k_j)} z = 0 \text{ for all } z \in \ker D, \quad a_{ij} \in \mathcal{F}.$$

Hence, for every fixed index m $(0 \leqslant m \leqslant N-1)$, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i D^{k_j} R^m z = 0 \quad \text{for all} \quad z \in \ker D,$$

which implies, by (1.0), that

$$\sum_{m=0}^{N-1} b_m \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i D^{k_j} R^m z =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i \sum_{m=0}^{N-1} b_m D^{k_j} R^m z =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i \sum_{m=k_j}^{N-1} b_m R^{m-k_j} z =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij} \sum_{m=k_j}^{N-1} b_m F_i R^{m-k_j} z =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij} \sum_{m=k_j}^{N-1} b_m \frac{d_{i,m-k_j}}{(m-k_j)!} z = 0$$

for all $z \in \ker D$, $b_m \in \mathcal{F}$ $(m = 0,1,\dots,N-1)$. The arbitrariness of $z \in \ker D$ and $b_m \in \mathcal{F}$ implies that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}G_i^{(k_j)} = 0$$

i.e. $a_{ij} = 0$ $(i = 1,\dots,n;\ j = 1,\dots,r_i)$.

Conversely, suppose that the system $\left\{\hat{F}_i^{(k_j)}\right\}_{(i=1,\dots,n;j=1,\dots,r_i)}$ is linearly independent on $\ker D$ and that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}G_i^{(k_j)} = 0,\quad a_{ij}\in\mathcal{F},\quad i=1,\dots,n;\ j=1,\dots,r_i.$$

By formula (2.2), this means that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\frac{d_{i,m-k_j}}{(m-k_j)!} = 0,\qquad m=0,1,\dots,N-1,$$

(where we admit $d_{i,q} = 0$ for $q < 0$).

Since $\dim \ker D \neq 0$, then the last equalities are equivalent to the following ones

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\frac{d_{i,m-k_j}}{(m-k_j)!}z = 0 \quad \text{for all} \quad z\in\ker D,\ m=0,\dots,N-1.$$

which, by (1.0), can be written in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\hat{F}_i^{(k_j)}z = 0 \quad \text{for all} \quad z\in\ker D.$$

Our assumption implies $a_{ij} = 0$, $i=1,\dots,n$; $j=1,\dots,r_i$, which was to be proved.

It is easy to see (cf. (1.4)) that the vector-operators $\hat{F}_i^{(k_j)}$ can be written as

(2.7)
$$\hat{F}_i^{(k_j)} = \left(F_{ik_j}, F_{ik_j}R,\dots,F_{ik_j}R^{N-1}\right),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where

$$F_{ik_j} = F_i D^{k_j} \quad (i=1,\dots,n;\ j=1,\dots,r_i-1). \tag{2.8}$$

T h e o r e m 2.1. A necessary and sufficient condition for $\det G \neq 0$ is that all operators F_{ik_j} of the form (2.8) are linearly independent on $P_N(R)$, where $P_N(R)$ is defined by formula (1.3).

P r o o f . Suppose that $\det G \neq 0$. Then, by Lemma 2.1, the vector-operators $\left\{\hat{F}_i^{(k_j)}\right\}$ $(i=1,\dots,n;\ j=1,\dots,r_i)$ are linearly independent on $\ker D$. Hence, for every fixed index m $(0 \leqslant m \leqslant N-1)$ the operators $F_{ik_j} R^m$, $i = 1,\dots,n;\ j = 1,\dots,r_i$, are linearly independent on $\ker D$. This means that the system of operators F_{ik_j} of the form (2.8) is linearly independent on

$$\bigcup_{m=0}^{N-1} R^m(\ker D) = P_N(R).$$

Conversely, suppose that F_{ik_j} $(i=1,\dots,n;\ j=1,\dots,r_i)$ are linearly independent on $P_N(R)$. By Lemma 2.1, to prove $\det G \neq 0$, it is enough to show that the system $\left\{\hat{F}_i^{(k_j)}\right\}_{(i=1,\dots,n;\,j=1,\dots,r_i)}$ is linearly independent on $\ker D$.

Let

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} a_{ij} \hat{F}_i^{(k_j)} z = 0 \quad \text{for all} \quad z \in \ker D,\ a_{ij} \in \mathcal{F},$$

i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} a_{ij} F_i D^{k_j} R^m z_m = 0 \text{ for every fixed index } m \ (0 \leqslant m \leqslant N-1)$$

and for all $z_m \in \ker D$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence

$$\sum_{m=0}^{N-1} b_m \sum_{i=1}^{n} \sum_{j=1}^{r_i} a_{ij} F_i D^{k_j} R^m z_m = 0 \quad \text{for all} \quad z_m \in \ker D,$$

$$b_m \in \mathcal{F}, \quad m = 0,1,\ldots,N-1, \quad \text{i.e.}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} a_{ij} F_i D^{k_j} \sum_{m=0}^{N-1} b_m R^m z_m = 0$$

for all $z_m \in \ker D$, $b_m \in \mathcal{F}$. This means that

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} a_{ij} F_i D^{k_j} x = 0 \quad \text{for all} \quad x \in P_N(R).$$

Thus, by our assumption, we get $a_{ij} = 0$, $i=1,\ldots,n$; $j=1,\ldots,r_i$.

Now we can formulate the main result for the generalized interpolation problem induced by right invertible operators.

T h e o r e m 2.2. The generalized interpolation problem has a unique solution, if and only if the system of operators $\{F_i D^{k_j}\}$ $(i=1,\ldots,n; j=1,\ldots,r_i)$ is linearly independent on $P_N(R)$.

P r o o f . By assumptions, for every pair of indices (i,j), $i=1,\ldots,n$; $j=1,\ldots,r_i$, we have

$$u_{ik_j} = F_i D^{k_j} u = \sum_{m=0}^{N-1} F_i D^{k_j} R^m z_m = \sum_{m=k_j}^{N-1} \frac{d_{i,m-k_j}}{(m-k_j)!} z_m.$$

We obtain a system

$$\sum_{m=k_j}^{N-1} \frac{d_{i,m-k_j}}{(m-k_j)!} z_m = u_{ik_j} \quad (i=1,\ldots,n;\ j=1,\ldots,r_i) \tag{2.9}$$

of N equations with N unknowns (cf. [1], Theorem 3.2).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Observe that the matrix of coefficients of (2.9) is the matrix G defined by formulae (2.2)-(2.4). By Theorem 2.1, the system (2.9) has a unique solution, if and only if the operators $F_i D^{k_j}$, $i=1,\dots,n$; $j=1,\dots,r_i$, are linearly independent on $P_N(R)$. Hence, the proof immediately follows from Theorem 2.1.

As applications of Theorem 2.2 we deal with classical interpolation problems for right invertible operators.

a. Hermite interpolation problem

The interpolation problem (H) is to find a D-polynomial u of degree N-1 which admits, for $n \leqslant N$ given different initial operators $F_1, F_2, \dots, F_n$, the given values together with $D^k u$ up to the order $r_j - 1$, i.e. for a solution of the following problem

$$F_i D^j u = u_{ij}, \quad i=1,\dots,n;\ j=1,\dots,r_i-1, \tag{2.10}$$

where $r_1 + r_2 + \dots + r_n = N$, $u_{ij} \in \ker D$ are given, $u = z_0 + Rz_1 + \dots + R^{N-1} z_{N-1}$ for $R \in \mathcal{R}_D$ and $z_0, z_1, \dots, z_{N-1}$ are to be determined (cf. [1]).

Note that, in this case, $I_i = \{0,1,\dots,r_i-1\}$ for $i = 1,2,\dots,n$. Hence from Theorem 2.2 we have

T h e o r e m 2.3. Suppose that $D \in R(X)$ and that $F_1, F_2, \dots, F_n \in c(R)$ for an $R \in \mathcal{R}_D$. Then the interpolation problem (H) has a unique solution, if and only if the system of operators $\{F_i D^j\}$ $(i=1,\dots,n; j=1,\dots,r_i-1)$ is linearly independent on $P_N(R)$. If this condition is satisfied, then a unique solution of (2.2) is of the form

$$u = \sum_{j=0}^{N-1} V_{Nj}(R) u_j, \tag{2.11}$$

where

$$V_{Nj}(R) = (\det G)^{-1} \sum_{k=0}^{N-1} (-1)^{k+j} V_{Njk} R^k \quad (j = 0,\dots,N-1),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

with the minor determinant V_{Njk} obtained by cancelling in G the k-th column and the j-th row, $j,k = 0,\dots,N-1$, and elements $u_0,u_1,\dots,u_{N-1} \in \ker D$ are defined as follows (cf. [1])

$$u_q = u_{1q}, \quad \text{if} \quad q = 0,1,\dots,r_1-1,$$

$$u_{r_1+q} = u_{2q}, \quad \text{if} \quad q = 0,1,\dots,r_2-1,$$

$$u_{r_1+r_2+q} = u_{3q}, \quad \text{if} \quad q = 0,1,\dots,r_3-1,$$

$$\dots\dots\dots$$

$$u_{r_1+r_2+\dots+r_{n-1}+q} = u_{nq}, \quad \text{if} \quad q = 0,1,\dots,r_n-1.$$

b. Lagrange interpolation problem

If $I_i = \{0\}$, $i = 1,2,\dots,n$, then we obtain the interpolation problem (L): to find a D-polynomial of degree n-1 which admits, for given different initial operators $F_1,F_2,\dots,F_n$, the given values

(2.12) $\qquad F_i u = u_i$, where $u_i \in \ker D$ are given,

$u = z_0 + Rz_1 + \dots + R^{n-1}z_{n-1}$.

Theorem 2.2 immediately implies the following one.

T h e o r e m 2.4. A necessary and sufficient condition for the interpolation problem (L) to have a unique solution is that the system $\{F_1,F_2,\dots,F_n\}$ is linearly independent on $P_n(R)$.

E x a m p l e 2.1. Let $X = C(\mathbb{R})$, $D = \frac{d}{dt}$, $R = \int_0^t$, $(Rx)(t) = x(0)$ and write

$$(F_h x)(t) = x(h), \quad x \in X.$$

Then

$$F_h R^k c = c\frac{h^k}{k!} \quad \text{for} \quad c,h \in \mathbb{R},\ k \in \mathbb{N}_0.$$

Hence $d_{hk} = h^k$.

It is well-known that, for the Lagrange, Hermite and Newton interpolation problems with $F_{h_1},F_{h_2},\dots,F_{h_n}$, the corres-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ponding determinants do not vanish. The following question arises: Is $\det G \neq 0$? (cf. [1]). In general, the answer is negative. Indeed, if $0 \notin \bigcup_{i=1}^{n} I_i$, then the first column of the matrix G consists only zeros, i.e. rank $G \leqslant N-1 < N$.

3. General interpolation problem induced by left invertible operators

D e f i n i t i o n 3.1 [2]: An operator $\Delta \in L_o(X)$ is said to be left invertible, if there is an operator $L \in L(X)$ such that $\Delta X \subset \text{dom } L$ and $L\Delta = I$.

Denote by $\Lambda(X)$ the set of all left invertible operators and by $\mathcal{L}_\Delta$ the set of all left inverses of a $\Delta \in \Lambda(X)$.

D e f i n i t i o n 3.2 [2]: An operator $G \in L(X)$ is said to be co-initial operator corresponding to a left inverse L of Δ, if it satisfies the conditions

$$(3.1) \quad G^2 = G, \{x \in \text{dom } L : Gx = x\} = \ker L, \ker G = \Delta X.$$

For a $\Delta \in \Lambda(X)$ we write

$$(3.2) \quad \mathcal{G}_\Delta = \{G \in L(X) : G^2 = G, \{x \in \text{dom } L : Gx = x\} = \ker L, \ker G = \Delta X\}.$$

In the sequel, we assume that $\dim \operatorname{coker} \Delta \neq 0$, i.e. Δ is not right invertible.

D e f i n i t i o n 3.3. Elements of the form $z_o + \Delta z_1 + \ldots + \Delta^m z_m$, where $z_k \in \ker L$, are called Δ-polynomials.

D e f i n i t i o n 3.4. Let $\Delta \in \Lambda(X)$. A co-initial operator G_o for Δ has the property $c(\Delta)$ for an $L \in \mathcal{L}_\Delta$, if there exist scalars c_k such that

$$(3.3) \quad G_o \Delta^k z = \frac{c_k z}{k!} \quad \text{for all } z \in \ker L, \ k \in \mathbb{N}.$$

We shall write $G_o \in c(\Delta)$.

Now we consider the following general interpolation problem for a left invertible operator Δ.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Given n finite sets I_i of positive integers, suppose that all elements of I_i are ordered, i.e.

$$0 \leqslant k_1 < k_2 < \ldots < k_{r_i}, \quad k_j \in I_i \ (j = 1,2,\ldots,r_i).$$

Let $N = r_1 + r_2 + \ldots + r_n$. We are looking for a Δ-polynomial $u = z_0 + \Delta z_1 + \ldots + \Delta^{N-1} z_{N-1}$ of degree N-1 satisfying, for n given different co-initial operators $G_1, G_2, \ldots, G_n \in c(\Delta)$, the conditions

$$G_i L^k u = u_{ik}, \quad k \in I_i;\ i = 1,\ldots,n, \tag{3.4}$$

where $u_{ik} \in \ker L$ are given.

By our assumptions, there exist scalars d_{ik} satisfying condition (3.3) and we have

$$G_i \Delta^k z = \frac{d_{ik}}{k!} z \quad \text{for all} \quad z \in \ker L, \quad k \in I_i, \quad i = 1,2,\ldots,n,$$

$$G_i L^{k_j} u = \sum_{m=0}^{N-1} G_i L^{k_j} \Delta^m z_m = \sum_{m=0}^{k_j-1} G_i L^{k_j-m} z_m +$$

$$+ \sum_{m=k_j}^{N-1} G_i \Delta^{m-k_j} z_m = \sum_{m=k_j}^{N-1} G_i \Delta^{m-k_j} z_m = \sum_{m=k_j}^{N-1} \frac{d_{i,m-k_j}}{(m-k_j)!} z_m,$$

$i = 1,2,\ldots,n;\ j = 1,2,\ldots,r_i$.

As in Section 2, denote by $\hat{G}$ the matrix of the forms (2.2)-(2.4). Consider the following vector-operators of dimension N

$$G_i^{(k_j)} = \left(G_i L^{k_j}, G_i L^{k_j-1}, \ldots, G_i, G_i \Delta, \ldots, G_i \Delta^{N-1-k_j} \right), \tag{3.5}$$

$i = 1,2,\ldots,n;\ j = 1,2,\ldots,r_i$, and let

$$P_N(\Delta) = \operatorname{lin}\{\Delta^k z : z \in \ker L;\ k = 0,1,\ldots,N-1\}.$$

Theorems 2.1 and 2.2 together imply the following result.

T h e o r e m 3.1. The general interpolation problem for the left invertible operator Δ has a unique solution, if

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and only if the system of vectors $\left\{G_i L^{k_j}\right\}_{(i=1,\dots,n;\, j=1,\dots,r_i)}$ is linearly independent on $P_N(\Delta)$.

Similarly, we can formulate a necessary and sufficient condition for the Hermite and Lagrange interpolation problems to have a unique solution.

4. Interpolation problems for singular integral operators

Let Γ be a closed regular arc in the complex plane. Let $X = H^{\mu}(\Gamma)$ $(0 < \mu \leqslant 1)$ with the norm:

$$\|x\|_X = \sup_{t\in\Gamma} |x(t)| + \sup_{t_1,t_2\in\Gamma} \left\{|x(t_1) - x(t_2)|\, |t_1 - t_2|^{-\mu}\right\}.$$

Consider the following operator

$$(Kx)(t) = a(t)x(t) + \frac{b(t)}{\pi i}\int_{\Gamma}\frac{x(u)du}{u-t}, \tag{4.1}$$

where $a,b \in X$, $a^2 - b^2 = 1$, $b(t) \not\equiv 0$. Denote by $\varkappa$ the index of the operator K, i.e.

$$\varkappa = \operatorname{Ind} K = \frac{1}{2\pi}\left\{\frac{a(t) - b(t)}{a(t) + b(t)}\right\}_{\Gamma}$$

It is well-known that K is right invertible for $\varkappa > 0$ and is left invertible for $\varkappa < 0$. If $\varkappa = 0$, then K is invertible (cf. [3]-[5]).

a. Suppose that $\varkappa = 1$, i.e. K is right invertible. Let

$$(Rx)(t) = a(t)x(t) - \frac{b(t)Z(t)}{\pi i}\int_{\Gamma}\frac{x(u)du}{(u-t)Z(u)}, \tag{4.2}$$

$$z(t) = b(t)Z(t),$$

where

$$Z(t) = t^{-\frac{1}{2}}\, e^{V(t)},$$

$$V(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{\ln\left[u^{-1}(a(u) - b(u))(a(u) + b(u))^{-1}\right]}{u - t}du.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is easy to check that $R \in \mathcal{R}_K$ and

$$\ker K = \{z \in X : z(t) = b(t)Z(t)c,\ c \in \mathbb{C}\}.$$

Hence, dim ker K = 1. Every right inverse of K belongs to the set

$$\mathcal{R}_K = \{\hat{R} \in L_o(X) : \hat{R} = A - RK\,A + R,\quad A \in L_o(X)\}.$$

The set of all initial operators of K is defined by the formula

$$\mathcal{F}_K = \{\hat{F} \in L_o(X) : \hat{F} = I - \hat{R}\,K,\ \hat{R} \in \mathcal{R}_K\}.$$

By Theorem 1.1, every initial operator $\hat{F} \in \mathcal{F}_K$ has the property (c).

From Theorem 2.3 we obtain the following

C o r o l l a r y 4.1. A necessary and sufficient condition for the interpolation problem L (i.e. the problem: to find an element $u = z_o + Rz_1 + \ldots + R^{n-1}z_{n-1}$ ($z_k = b\,Z\,c_k$, $c_k \in \mathbb{C}$) which admits for given initial operators $F_o, F_1, \ldots, F_{n-1}$ the given values: $F_i u = u_i$, where $u_i = b\,Z\,e_i$, $e_i \in \mathbb{C}$) to have a unique solution is that the system $\{F_o, F_1, \ldots, F_{n-1}\}$ is linearly independent on $P_n(R)$, where

$$P_n(R) = \operatorname{lin}\{R^k(bZ),\quad k = 0,1,\ldots,n-1\}. \tag{4.3}$$

Similarly, we can formulate a necessary and sufficient condition for general interpolation problem to have a unique solution.

b. Consider now the case $\varkappa = -1$, i.e. K is left invertible. For this case, it is easy to see that $R \in \Lambda_K$, where R is defined by formula (4.2) and

$$\ker R = \{z \in X : z(t) = b(t)\,Z(t)c,\quad c \in \mathbb{C}\}.$$

Hence, dim ker R = 1. By Theorem 1.1, every co-initial operator $G \in \mathcal{G}_K$ has the property (c).

From Theorem 2.3 we get the following

C o r o l l a r y 4.2. Let K be of the form (4.1) and let $\varkappa = -1$. Suppose that $G_1, G_2, \ldots, G_n$ are co-initial opera-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

tors for K. Then the interpolation problem (H) has a unique solution, if and only if the system of operators $\{G_i K^k\}_{(i=1,\ldots,n;k=0,\ldots,r_i-1)}$ is linearly independent on $P_N(K)$, where $P_N(K)$ is defined by formula (4.3).

Similarly, we can formulate the general interpolation problem for singular integral operators with a regular part.

Let the operator $M = K + T$, where K is defined by (4.1) and $(Tx)(t) = \int_\Gamma T(u,t)x(t)du$, be compact in X.

If $\varkappa > 0$ and the operator $I + TR$ is invertible, then M is right invertible. Indeed, in this case $M = K(I + RT)$. The right inverse $\hat{R}$ of M is of the form $\hat{R} = (I + RT)^{-1}R$.

If $\varkappa < 0$ and the operator $I + TR$ is invertible, then M is left invertible. In this case $M = (I + TR)K$, and the left inverse $\hat{L}$ of M is of the form: $\hat{L} = R(I + TR)^{-1}$.

If $\varkappa = 0$ and the operator $I + RT$ (or $I + TR$) is invertible, then M is invertible.

Hence, for those cases we can give conditions for the uniqueness of solutions to general interpolation problems.

The following question arises.

Does it exist a system of initial operators for K such that all corresponding classical interpolation problems always have a unique solution?

We shall give the positive answer to this question.

To begin with we need some notions. Suppose that $\varkappa = \mathrm{Ind}\, K > 0$, where K is of the form (4.1). Then K is bounded in X, $\mathrm{dom}\, K = \mathrm{dom}\, K^j = X$ for all $j \in \mathbb{N}$ and K^j are right invertible.

Let $h \in \mathbb{C}$. Write

$$S_h = \sum_{k=0}^{\infty} \frac{h^k}{k!} K^k \qquad S_0 = I, \tag{4.4}$$

(obviously the series is convergent in the norm). It is easy to see that S_h is K-shift (cf. [2]).

For given initial operator $F = I - RK$ we write $F_h = F S_h$ for $h \in \mathbb{C}$. Then F_h is an initial operator for K corresponding to right inverse $R_h = R - F_h R$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is easy to check that

$$\ker K = \left\{ z \in X : z(t) = \sum_{k=1}^{\varkappa} c_k b(t) Z(t) t^{k-1},\ c_k \in \mathbb{C} \right\},$$

and

$$F_h R^k z = \frac{h^k}{k!} z \quad \text{for all} \quad z \in \ker K,\ k \in \mathbb{N}_0. \tag{4.5}$$

Similarly, if $\varkappa = \operatorname{Ind} K < 0$, we can construct a co-initial operators for K by the same way, i.e.

$$G_h = F Q_h,$$

where

$$Q_h = \sum_{k=0}^{\infty} \frac{h^k}{k!} R^k, \quad Q_0 = I, \quad h \in \mathbb{C}.$$

Then

$$Q_h K^k z = \frac{h^k}{k!} z \quad \text{for all} \quad z \in \ker R,\ k \in \mathbb{N}_0. \tag{4.6}$$

An immediate consequence of Theorem 2.3 is as follows.

C o r o l l a r y 4.3. Let K be of the form (4.1) and let $\operatorname{Ind} K > 0$. Then there exists a unique K-polynomial

$$u = \sum_{m=0}^{N-1} R^m z_m, \quad z_m \in \ker K \quad (m = 0,1,\ldots,N-1),$$

which admits, for n ($\leqslant N$) given different initial operators $F_{h_0}, F_{h_1}, \ldots, F_{h_{n-1}}$, the given values together with the $K^k u$ up to the order r_j, i.e. the following system has a unique solution

$$F_{h_j} K^k u = u_{jk}, \quad j=1,\ldots,n;\ k=0,\ldots,r_j-1;\ r_1+\ldots+r_n=N, \tag{4.7}$$

where $u_{jk} \in \ker K$ are given.

Indeed, in this case, the matrix induced by system (4.7) is the classical Hermite matrix.

Similarly, from Theorem 2.3 we get the following result.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Corollary 4.4. Let K be of the form (4.1) and let Ind $K < 0$. Then there exists a unique polynomial

$$u = z_0 + Kz_1 + \ldots + K^{N-1}z_{N-1}, \quad z_m \in \ker R \quad (m=0,1,\ldots,N-1),$$

which admits, for n given different co-initial operators $G_{h_0},\ldots,G_{h_{n-1}}$, the given values together with $R^k u$ up to the order r_j.

REFERENCES

[1] D. Przeworska-Rolewicz: Property (c) and interpolation formulae induced by right invertible operators. Demonstratio Math. 21 (1988) 1023-1044.

[2] D. Przeworska-Rolewicz: Algebraic Analysis. PWN - Polish Scientific Publishers and D. Reidel, Warszawa-Dordrecht, 1987.

[3] D. Przeworska-Rolewicz, S. Rolewicz: Equations in linear spaces. Monografie Matematyczne 47, Warszawa 1968.

[4] F.D. Gakhov: Boundary value problems. Pergamon Press, Oxford 1966.

[5] N.I. Muskhelisvili: Singular integral equations. Nordhoff-Groningen, 1953.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received July 7, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Valeriu Popa

THEOREMS OF UNIQUE FIXED POINT FOR EXPANSION MAPPINGS

In [1]-[4] some fixed point theorems of expansion mappings are proved. The majority of these theorems do not ensure the uniqueness of the fixed point. In this note we propose to establish a supplementary condition, in which the fixed point is unique, and to demonstrate some theorems meant to improve some of the results from [1]-[4].

Let R_+ be the set of all nonnegative real numbers and let $\psi: R_+^3 \to R_+$ be a real-valued function.

D e f i n i t i o n 1. A real-valued function $\psi: R_+^3 \to R_+$ satisfies property (B) if for every $u, v \in R_+$ such that $u \geq \psi(v,u,v)$, then $u \geqslant hv$, where $\psi(1,1,1) = h \geqslant 1$ [1].

D e f i n i t i o n 2. A real-valued function $\psi: R_+^3 \to R_+$ satisfies property (C) if ψ is continuous in the first and second variable [1].

D e f i n i t i o n 3. A real-valued function $\psi: R_+^3 \to R_+$ satisfies property (U) if $\psi(u,0,0) > u$, $\forall u \neq 0$.

T h e o r e m 1. Let (X,d) be a metric space and f a self mapping of X satisfying the inequality

(1) $$d(fx,fy) \geqslant \psi(d(x,y),d(x,fx),d(y,fy))$$

for all x,y in X with $x \neq y$, where ψ satisfies property (U). If f has a fixed point z, then z is unique fixed point for f.

P r o o f . Suppose that f has a second fixed point $z' \neq z$. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$d(z,z') = d(fz,fz') \geqslant \psi(d(z,z'),d(z,fz),d(z',fz')) =$$

$$= \psi(d(z,z'),0,0)) > d(z,z')$$

which implies $z = z$.

Theorem 2. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$ a surjective mapping satisfying the inequality (1) for all x,y in X with $x \neq y$, where ψ satisfies property (B) with $h > 1$. If properties (C) and (U) hold, then f has a unique fixed point.

Proof. Follows from Theorem 1 and Theorem 3 of [1].

Corollary 1. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$. If there is a constant $a > 1$ such that

$$d(fx,fy) \geqslant a\, d(x,y) \tag{2}$$

for any x,y in X and f is onto, then f has a unique fixed point.

(Theorem 1; [4]).

Corollary 2. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$ a surjective mapping. If there exists non-negative reals a,b,c with $a < 1$ and $a + b + c > 1$ such that

$$d^2(fx,fy) \geqslant a\, d^2(x,fx) + b\, d^2(y,fy) + c\, d^2(x,y) \tag{3}$$

for each $x \neq y$ in X, then f has a fixed point. Further, if $c > 1$ then the fixed point is unique. (Theorem 1; [3]).

Theorem 3. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$ a surjective mapping. If there exist non-negative reals a,b,c with $2a + b + c > 3$ such that

$$d(fx,fy) \geqslant \tag{4}$$

$$\geqslant \frac{a[d(x,fx)d(x,y)+d(y,fy)d(x,y)]+bd(x,fx)d(y,fy)+cd^2(x,y)}{d(x,fx)+d(y,fy)+d(x,y)}$$

for each $x \neq y$ in X, then f has a fixed point. Further, if $c > 1$ then the fixed point is unique.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

p r o o f . Let $x_0 \in X$. Since f is surjective, there exists an element $x_1 \in X$ satisfying $x_1 \in f^{-1}(x_0)$. In the same way we can take $x_n \in f^{-1}(x_{n-1})$, $n = 1,2,\dots$. If $x_m = x_{m-1}$ for some m, then x_m is a fixed point of f. Without loss of generality, we can suppose $x_{n-1} \neq x_n$ for every n. Then from (4)

$$d(x_{n-1},x_n) = d(fx_n,fx_{n+1}) \geqslant$$

$$\geqslant \frac{ad(fx_n,x_n)d(x_n,x_{n+1})+d(x_{n+1},fx_{n+1})\cdot d(x_n,x_{n+1})}{d(x_n,fx_n)+d(x_{n+1},fx_{n+1})+d(x_n,x_{n+1})} +$$

$$+ \frac{bd(x_n,fx_n)d(x_{n+1},fx_{n+1}) + cd^2(x_n,x_{n+1})}{d(x_n,fx_n)+d(x_{n+1},fx_{n+1})+d(x_n,x_{n+1})} =$$

$$= \frac{a[d(x_n,x_{n-1})d(x_n,x_{n+1})+d^2(x_n,x_{n+1})]}{d(x_n,x_{n-1})+2d(x_n,x_{n+1})} +$$

$$+ \frac{bd(x_n,x_{n-1})d(x_n,x_{n+1}) + cd^2(x_n,x_{n+1})}{d(x_n,x_{n-1})+2d(x_n,x_{n+1})} .$$

Thus $(a+c)t^2 + (a-b-2)\,t - 1 \leqslant 0$ where $t = d(x_{n+1},x_n)/d(x_n,x_{n-1})$.

Let $g: [0,\infty) \longrightarrow R$ be the function $g(t) = (a+c)t^2 + (a+b-2)t-1$. Then $g(0) = -1$ and $g(1)\ 2a+b+c-3 > 0$ from the hypothesis. Let $k \in (0,1)$ be the root of the equation $g(t) = 0$, then $g(t) \leqslant 0$ for $t < k$ and thus

$$d(x_{n+1},x_n) \leqslant kd(x_n,x_{n-1}) \leqslant \dots \leqslant k^n d(x_1,x_0).$$

Therefore, by the well known way, $\{x_n\}$ converges to some $x \in X$. Let $y \in f^{-1}(x)$ be. Then we have

$$d(x_n,x) = d(fx_{n+1},fy) \geqslant$$

$$\geqslant \frac{a[d(x_{n+1},fx_{n+1})d(x_{n+1},y)}{d(x_{n+1},fx_{n+1})+d(y,fy)+d(x_{n+1},y)} +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\frac{d(y,fy)d(x_{n+1},y)]+bd(x_{n+1},fx_{n+1})d(y,fy)+cd^2(x_{n+1},y)}{d(x_{n+1},fx_{n+1})+d(y,fy)+d(x_{n+1},y)}.$$

$$=\frac{a[d(x_{n+1},x_n)d(x_{n+1},y)+d(y,fy)d(x_{n+1},y)]}{d(x_{n+1},x_n)+d(y,fy)+d(x_{n+1},y)}+$$

$$+\frac{bd(x_{n+1},x_n)d(y,fy)+cd^2(x_{n+1},y)}{d(x_{n+1},x_n)+d(y,fy)+d(x_{n+1},y)}$$

and on letting n tend to infinity we have if $x \neq y$, $0 \geqslant \frac{(a+c)}{2} d(x,y)$, which implies $x = y$. Since $x = f(y)$, then $x = f(x)$. So f has a fixed point.

Let

$$\psi(d(x,y),d(x,fx),d(y,fy)) =$$

$$=\frac{a[d(x,fx)d(x,y)+d(y,fy)d(x,y)]+bd(x,fx)d(y,fy)+cd^2(x,y)}{d(x,y)+d(x,fx)+d(y,fy)}$$

then if $x \neq y$, $\psi(d(x,y),0,0) = cd(x,y) > d(x,y)$.

From Theorem 1, f has a unique fixed point.

T h e o r e m 4. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$ a surjective mapping. If there exist non-negative reals a,b with $a < 1$ and $2a+b-3 > 0$ such that

$$(5) \qquad d(fx,fy) \geqslant \frac{a[\{d(x,fx)\}^2+\{d(y,fy)\}^2]+b\{d(x,y)\}^2}{d(x,fx)+d(y,fy)+d(x,y)}.$$

P r o o f . It is similar to the proof of Theorem 3.

T h e o r e m 5. Let (X,d) be a complete metric space and $f:(X,d) \longrightarrow (X,d)$ a surjective mapping. If there exists non-negative reals a,b,c,d with $a+b+c+d > 1$ such that

$$(6) \qquad d^k(fx,fy) \geqslant ad^q(x,fx)d^{k-q}(x,y)+bd^m(y,fy)d^{k-m}(x,y)+$$

$$+cd^p(x,fx)d^{k-p}(y,fy)+dd^k(x,y)$$

where $k \geqslant 1$, $q \geqslant 0$, $m \geqslant 0$, $p \geqslant 0$ and $q \leqslant k$, $m \leqslant k$, $p \leqslant k$ for each x,y in X with $x \neq y$, then f has a fixed point. Further, if $d > 1$ then the fixed point is unique.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f . As in Theorem 3, there is a sequence $\{x_n\}$ with $x_{n-1} \neq x_n$ and $x_{n-1} = f(x_n)$. This implies

$$d^k(x_{n-1},x_n) = d^k(fx_n,fx_{n+1}) \geqslant ad^q(x_n,fx_n)d^{k-q}(x_n,x_{n+1}) +$$

$$+ bd^m(x_{n+1},fx_{n+1})d^{k-m}(x_n,x_{n+1}) + cd^p(x_n,fx_n)d^{k-p}(x_{n+1},fx_{n+1}) +$$

$$+ dd^k(x_{n+1},x_n) = ad^q(x_n,x_{n-1})d^{k-q}d(x_n,x_{n+1}) + bd^k(x_n,x_{n+1}) +$$

$$+ cd^p(x_n,x_{n-1})d^{k-p}(x_{n+1},x_n) + dd^k(x_{n+1},x_n)$$

and thus $(b+d)t^k + ct^{k-p} + ad^{k-q} - 1 \leqslant 0$ where $t = d(x_{n+1},x_n)/d(x_n,x_{n-1})$.

Let $g: [0,\infty) \longrightarrow R$ be the function $g(t) = (b+d)t^k + ct^{k-p} + at^{k-q} - 1$. Then $g'(t) > 0$ for any $t > 0$, $f(0) < 0$ and $g(1) = a + b + c + d - 1 > 0$. Let $r \in (0,1)$ be the root of the equation $g(t) = 0$, then $f(t) \leqslant 0$ for $t \leqslant r$ and thus

$$d(x_{n+1},x_n) \leqslant rd(x_n,x_{n-1}) \leqslant \ldots \leqslant r^n d(x_1,x_0).$$

Therefore, $\{x_n\}$ converges to some $x \in X$. Let $y \in f^{-1}(x)$. Then

$$d(x_n,x) = d(fx_{n+1},fy) \geqslant ad^q d(x_{n+1},fx_{n+1})d^{k-q}(x_{n+1},y) + bd^m(y,fy) \cdot$$

$$\cdot d^{k-m}(x_{n+1},y) + cd^p(x_{n+1},fx_{n+1})d^{k-p}(y,fy) + dd^k(x_{n+1},y) =$$

$$= ad^q(x_{n+1},x_n)d^{k-q}(x_{n+1},y) + bd^m(y,x)d^{k-m}(x_{n+1},y) +$$

$$+ cd^p(x_{n+1},x_n)d^{k-p}(y,fy) + dd^k(x_{n+1},y).$$

If $x \neq y$ letting n tend to infinity we have

$$0 \geqslant (b+d)d^k(x,y)$$

which implies $y = x$. Since $x = f(y)$, then $x = f(x)$. So f has a fixed point. If $d > 1$ let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\psi = [ad^q(x,fx)d^{k-q}(x,y) + bd^m(y,fy)d^{k-m}(x,y) +$$
$$+ cd^p(x,fx)d^{k-p}(x,y) + dd^k(x,y)]^{1/k}$$

where y is a fixed point. Then if $x \neq y$ we have

$$\psi(d(x,y),0,0) = d^{1/k}d(x,y) > d(x,y).$$

From Theorem 1, f has a unique fixed point.

REFERENCES

[1] M.A. Khan, M.S. Khan, S. Sessa: Some theorems on expansion mappings and their fixed points, Demonstratio Math. 19 (1986) 673-683.

[2] V. Popa: Some fixed point theorems of expansion mappings, Demonstratio Math. 19, 3 (1986) 699-702.

[3] V. Popa: Fixed point theorems for expansion mappings, Preprint Babes-Bolyai Univ., Fac. Math. Phys. Res. Semin. 3 (1987) 25-30.

[4] S.Z. Wang, B.Y. Li, Z.M. Gao, K. Iséki: Some fixed point theorems on expansion mappings, Math. Japonica, 29, 4 (1984), 631-636.

HIGHER EDUCATION INSTITUTE, 5500 BAÇAU, ROMANIA

Received July 12, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Jacek Dąbrowski

ASYMPTOTIC NORMALITY OF A CERATIN ESTIMATOR IN PARTIALLY OBSERVED LINEAR SYSTEMS

1. Introduction

In this paper we are concerned with asymptotic behaviour of the estimator of a parameter "a" appearing in the following one dimensional linear system defined on the probability space $(\Omega, \mathcal{F}, P)$:

$$(1.1)\qquad \begin{cases} dX_t = a\,X_t dt + dW_t, & X_0 = 0, \quad a < 0, \\ dY_t = X_t dt + dV_t, & Y_0 = 0, \quad t \geqslant 0, \end{cases}$$

where $(W_t: t \geqslant 0)$ and $(V_t: t \geqslant 0)$ are independent one dimensional Wiener processes. The process (Y_t) is assumed to be observed. A. Bagchi and V. Borkar in [1] studied similar systems, where (X_t), (Y_t), (W_t) and (V_t) are Hilbert space valued processes.

The estimator of "a" created in [2] is based on discrete observations of (Y_t) at the moments $t = k\Delta$, $\Delta > 0$, $k=1,2,\ldots$. This estimator is defined by the formula

$$\hat{f}_n(a) = \frac{1}{n} \sum_{k=1}^{n} (Y_{k\Delta} + Y_{(k-1)\Delta})^2.$$

It has been proved in [2] that $\lim_{n\to\infty} \hat{f}_n(a) = f(a)$ a.s. if $a < 0$,

$$f(a) = a^{-3}\left[1 - \exp(a\Delta)\right] + \Delta a^{-2} + \Delta.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In this paper we shall prove that $\sqrt{n}\,(\hat{f}_n(a) - f(a))$ is asymptotically normally distributed.

2. Stationary process approach

Let us consider the following system

$$(2.1)\qquad \begin{aligned} d\tilde{X}_t &= a\,\tilde{X}_t dt + d\tilde{W}_t, && -\infty < t < +\infty,\ a<0,\\ d\tilde{Y}_t &= \tilde{X}_t dt + dV_t, && t \geqslant 0,\ \tilde{Y}_0 = 0, \end{aligned}$$

where $(W_t,\ -\infty < t < \infty)$ and $(V_t,\ t \geqslant 0)$ are independent Wiener processes defined on the same probability space $(\Omega, \mathcal{F}, P)$ (see [5]). It is well known that there is the unique stationary solution $\tilde{X}_t$ to (2.1) given by the formula

$$(2.2)\qquad X_t = \int_{-\infty}^{t} \exp\left[a(t-s)\right] d\tilde{W}_s, \quad \text{see } [4] \text{ or } [6] \text{ p.271.}$$

According to (1.1) we can assume that $\tilde{W}_t = W_t$ a.s. if $t \geqslant 0$. Let us make the notations

$$(2.3)\qquad \xi_k = (Y_{k\Delta} - Y_{(k-1)\Delta})^2,$$

$$(2.4)\qquad \tilde{\xi}_k = (\tilde{Y}_{k\Delta} - \tilde{Y}_{(k-1)\Delta})^2, \quad k=1,2,\ldots,\ \Delta>0.$$

For fixed $\Delta > 0$ we also define

$$(2.5)\qquad g_k(t) = \begin{cases} a^{-1}\left(\exp\left[a(k\Delta - t)\right] - \exp\left[a((k-1)\Delta - t)\right]\right) & \text{for } t < (k-1)\Delta \\ a^{-1}\left(\exp\left[a(k\Delta - t)\right] - 1\right) & \text{for } (k-1)\Delta \leqslant t < k\Delta \\ 0 & \text{for } t \geqslant k\Delta, \end{cases}$$

$$(2.6)\qquad e_k = \sum_{i=0}^{\infty}\left[\int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)\,d\tilde{W}_t \left(2\int_{-\infty}^{(k-1)\Delta} g_{i+k}(t)\,d\tilde{W}_t + \int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)\,d\tilde{W}_t\right) - \int_{(k-1)\Delta}^{k\Delta} g_{i+k}^2(t)\,dt\right].$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is easy to see that the above series converges absolutely in L^1. We are able now to formulate the main result of this paper.

T h e o r e m 1. If $a<0$, then $\sqrt{n}\,(\hat{f}_n(a)-f(a))$ converges weakly to the normal distribution $N(0; E(e_1^2))$ of mean zero and variance $E(e_1^2)$.

Before proving the theorem, we need some preliminary results. Let (1.1) and (2.1) be defined on the same probability space $(\Omega, \mathcal{F}, P)$.

L e m m a 1. If $a<0$ and $0<\lambda<-\frac{a}{2}$, then we have

$$\lim_{k\to\infty} (\tilde{\xi}_k - \xi_k)\cdot\exp(k\lambda\Delta) = 0 \quad \text{a.s.} \tag{2.7}$$

P r o o f . Let us use the analogous procedure as in the proof of Lemma 1 in [2]. The factor $\exp(k\lambda\Delta)$ (which does not appear in Lemma 1 of [2]) increases the upper bound of $P(B_k)$ appearing in the mentioned proof, but the new upper bound is still tight enough to allow use of the Borel-Cantelli lemma and one can easily obtain the assertion of Lemma 1.

L e m m a 2. The sequence $\{\tilde{\xi}_k,\ k=1,2,\ldots\}$ is a stationary and ergodic process.

P r o o f . See [2].

L e m m a 3. Let $\mathcal{F}_k = \sigma(\tilde{W}_t,\ t\leqslant k\Delta)$ be the σ-field generated by $(\tilde{W}_t,\ t\leqslant k\Delta)$. Then, the sequence $\left\{E\left[e_k^2\middle|\mathcal{F}_{k-1}\right],\ k=\ldots,-2,-1,0,1,2,\ldots\right\}$ is a stationary and ergodic process.

P r o o f . Since $E(e_k^2)<\infty$ we have

$$E\left[e_k^2\middle|\mathcal{F}_{k-1}\right] = E\left\{\sum_{i,j=0}^{\infty}\left[\left(2\int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)\,d\tilde{W}_t \int_{-\infty}^{(k-1)\Delta} g_{i+k}(t)\,d\tilde{W}_t + \right.\right.\right. \tag{2.8}$$

$$\left.\left.\left. + \left(\int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)\,d\tilde{W}_t\right)^2 - \int_{(k-1)\Delta}^{k\Delta} g_{i+k}^2(t)\,dt\right)\times\right.\right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\times 2\left(\int_{(k-1)\Delta}^{k\Delta} g_{j+k}(t)\,d\tilde{W}_t \int_{-\infty}^{(k-1)\Delta} g_{j+k}(t)\,d\tilde{W}_t + \left(\int_{(k-1)\Delta}^{k\Delta} g_{j+k}(t)\,d\tilde{W}_t\right)^2 + \right.$$

$$\left. - \int_{(k-1)\Delta}^{k\Delta} g_{j+k}^2(t)\,dt\right) \Bigg| \mathcal{F}_{k-1}\Bigg\} =$$

$$= \sum_{i,j=0}^{\infty}\Bigg[4 \int_{-\infty}^{(k-1)\Delta} g_{i+k}(t)\,d\tilde{W}_t \int_{-\infty}^{(k-1)\Delta} g_{j+k}(t)\,d\tilde{W}_t \int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)g_{j+k}(t)\,dt +$$

$$+ E\left[\left(\int_{(k-1)\Delta}^{k\Delta} g_{i+k}(t)\,dW_t\right)^2 \left(\int_{(k-1)\Delta}^{k\Delta} g_{j+k}(t)\,dW_t\right)^2\right] +$$

$$+ \int_{(k-1)\Delta}^{k\Delta} g_{i+k}^2(t)\,dt \int_{(k-1)\Delta}^{k\Delta} g_{j+k}^2(t)\,dt\Bigg].$$

Let us define the two-dimensional process

$$\pi_k = \left(\int_{-\infty}^{k\Delta} \exp\left[a(k\Delta - t)\right] d\tilde{W}_t;\ \tilde{W}_{k\Delta} - \tilde{W}_{(k-1)\Delta}\right)^T.$$

It is easy to see that $\{\pi_k,\ k = \ldots,-2,-1,0,1,2,\ldots\}$ is a stationary process. Besides, π_k is a Gaussian process and applying the property of stochastic integrals it is easy to see that every component of the matrix $E[\pi_0\pi_k^T]$ tends to zero as $k \to \infty$. Therefore $\{\pi_k\}$ is the ergodic process (see [5] Chapter II p.143).

Without loss of generality we can assume that $\{\pi_k\}$ is defined on the canonical probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, where $\tilde{\Omega} = \ldots \times R^2 \times R^2 \times R^2 \times \ldots$ is endowed with Borel σ-field $\tilde{\mathcal{F}}$. We have $\pi_k(\omega) = \omega(k)$, where $\omega = (\ldots,\omega(-2),\omega(-1),\omega(0),\omega(1),\ldots)$. Let us define the $\tilde{P}$-preserving transformation T and the operator U (on the set of random variables) as follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$T\{\omega(k)\}_{k=-\infty}^{+\infty} = \omega(k+1)_{k=-\infty}^{+\infty}, \quad U\,Z(\omega) = Z(T\omega). \tag{2.9}$$

According to the formula (2.8) and (2.5) it is easy to verify that there exists $\varphi : \Omega \to R$, such that φ is $\mathcal{F}$-measurable and we have

$$E\left[e_k^2 \middle| \mathcal{F}_{k-1}\right] = \varphi(\pi_{k-1}, \pi_k, \pi_{k+1}, \ldots) = \varphi(T^k\omega).$$

From the very definition of g_k it also follows that the series appearing in (2.8) converges absolutely in $L^1(\Omega,P)$ and $E\left[e_k^2 \middle| \mathcal{F}_{k-1}\right]$, $E\left[e_k^2\right]$ are well defined $(E(e_k^2) < \infty)$. Using the general theory of stationary processes (see [5]) we obtain the assertion of Lemma 3.

3. Martingale approach

It is easy to see that

$$S_n = \sum_{k=1}^{n} e_k \tag{3.1}$$

is $\mathcal{F}_n$-martingale. Using the formula (2.2) we have

$$\tilde{Y}_{k\Delta} - \tilde{Y}_{(k-1)\Delta} = \frac{1}{a}\int_{(k-1)\Delta}^{k\Delta} a\tilde{X}_t\,dt + V_{k\Delta} - V_{(k-1)\Delta} =$$

$$= \frac{1}{a}\left[\tilde{X}_{k\Delta} - \tilde{X}_{(k-1)\Delta} + (\tilde{W}_{k\Delta} - \tilde{W}_{(k-1)\Delta})\right] + V_{k\Delta} - V_{(k-1)\Delta} =$$

$$= \int_{-\infty}^{k\Delta} g_k(t)\,d\tilde{W}_t + V_{k\Delta} - V_{(k-1)\Delta}.$$

Therefore we obtain

$$\tilde{\xi}_k = \left(\int_{-\infty}^{k\Delta} g_k(t)\,d\tilde{W}_t + V_{k\Delta} - V_{(k-1)\Delta}\right)^2. \tag{3.2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

L e m m a 4. If $a<0$ and $(\sqrt{n})^{-1}\cdot\sum_{k=1}^{n}(\tilde{\xi}_k - E\tilde{\xi}_k) \xrightarrow{\mathcal{D}} \xrightarrow{\mathcal{D}} N(0,E(e_1^2))$ (converges weakly to the normal distribution), then also $\sqrt{n}\,(\hat{f}_n(a) - f(a)) \xrightarrow{\mathcal{D}} N(0,E(e_1^2))$.

P r o o f . It has been proved in [2] that $E\tilde{\xi}_k = f(a)$. Hence we have

$$\sqrt{n}\,(\hat{f}_n(a) - f(a)) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}(\xi_k - E\tilde{\xi}_k)\right) =$$

$$= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\xi_k - \tilde{\xi}_k) + \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\tilde{\xi}_k - E\tilde{\xi}_k).$$

The factor $\exp(k\lambda\Delta)$ appearing in Lemma 1 plays an essential role in getting the following convergence

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\xi_k - \tilde{\xi}_k) = 0 \quad \text{a.s.}$$

The proof of Lemma 4 is now complete.

To prove Theorem 1 it suffices to show asymptotic normality of the martingale S_n. Namely we have

L e m m a 5. If $a<0$ and $(\sqrt{n})^{-1}\cdot S_n \xrightarrow{\mathcal{D}} N(0,E(e_1^2))$, then $(\sqrt{n})^{-1}\sum_{k=1}^{n}(\tilde{\xi}_k - E\tilde{\xi}_k) \xrightarrow{\mathcal{D}} N(0,E(e_1^2))$.

Using the formula (3.2) for any integer $k \leqslant n$ we have

$$E\left[\tilde{\xi}_n \middle| \mathcal{F}_k\right] = E\left[\left(\int_{-\infty}^{n\Delta} g_n(t)\,d\tilde{W}_t + V_{n\Delta} - V_{(n-1)\Delta}\right)^2 \middle| \mathcal{F}_k\right] =$$

$$= \left(\int_{-\infty}^{k\Delta} g_n(t)\,dW_t\right)^2 + \int_{k\Delta}^{n\Delta} g_n^2(t)\,dt + \Delta,$$

and consequently

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sum_{i=0}^{\infty}\left[\left(\mathrm{E}\left[\tilde{\xi}_{i+k}\middle|\mathcal{F}_k\right]-\mathrm{E}\left[\tilde{\xi}_{i+k}\middle|\mathcal{F}_{k-1}\right]\right)=\right.$$

$$=\sum_{i=0}^{\infty}\left[\left(\int_{-\infty}^{k\Delta} g_{i+k}(t)\,d\tilde{W}_t\right)^2-\left(\int_{-\infty}^{(k-1)\Delta} g_{i+k}(t)\,dW_t\right)^2+\right.$$

$$\left.+\int_{k\Delta}^{(i+k)\Delta} g_{i+k}^2(t)\,dt-\int_{(k-1)\Delta}^{(i+k)\Delta} g_{i+k}^2(t)\,dt\right]=e_k.$$

It follows from the definition of g_k that the above series are absolutely convergent in L^1.

Note also that

$$\lim_{k\to-\infty}\mathrm{E}\left[\tilde{\xi}_n\middle|\mathcal{F}_k\right]=\int_{-\infty}^{n\Delta} g_n^2(t)\,dt+\Delta=\mathrm{E}(\tilde{\xi}_n)=f(a).$$

Applying the similar procedure to the one given in [7] p.129 we get

$$\tilde{\xi}_k-\mathrm{E}(\tilde{\xi}_k)=\tilde{\xi}_k-\mathrm{E}\left[\xi_k\middle|\mathcal{F}_{k-1}\right]+$$

$$+\sum_{i=1}^{\infty}\left(\mathrm{E}\left[\tilde{\xi}_k\middle|\mathcal{F}_{k-i}\right]-\mathrm{E}\left[\tilde{\xi}_k\middle|\mathcal{F}_{k-i-1}\right]\right)=$$

$$=\sum_{i=0}^{\infty}\left(\mathrm{E}\left[\tilde{\xi}_{i+k}\middle|\mathcal{F}_k\right]-\mathrm{E}\left[\tilde{\xi}_{i+k}\middle|\mathcal{F}_{k-1}\right]\right)+$$

$$-U(U^k\eta)+U^k\eta=e_k-U(U^k\eta)+U^k\eta,$$

where

$$\eta=\sum_{i=1}^{\infty}\sum_{m=0}^{i-1}\left(\mathrm{E}\left[\tilde{\xi}_m\middle|\mathcal{F}_{m-i}\right]-\mathrm{E}\left[\tilde{\xi}_m\middle|\mathcal{F}_{m-i-1}\right]\right)$$

and according to the definition (2.9)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$U^k\eta = \sum_{i=1}^{\infty}\sum_{m=0}^{i-1}\left(E\left[\tilde{\xi}_{m+k}\middle|\mathcal{F}_{m+k-i}\right] - E\left[\tilde{\xi}_{m+k}\middle|\mathcal{F}_{m+k-i-1}\right]\right).$$

It is also easy to see that the above series are absolutely convergent in L^1 and η is well defined.

We have proved that

$$\tilde{\xi}_k - E(\tilde{\xi}_k) = e_k - U(U^k\eta) + U^k\eta. \tag{3.3}$$

Next we obtain

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(\tilde{\xi}_k - E(\tilde{\xi}_k)\right) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} e_k +$$

$$+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(-U^{k+1}\eta + U^k\eta) = \frac{1}{\sqrt{n}}S_n + \frac{U\eta}{\sqrt{n}} - \frac{U^{n+1}\eta}{\sqrt{n}}.$$

To complete the proof it suffices to show that the last two terms tend to zero in P if $n \to \infty$. We have

$$E\left|(U^{n+1}\eta)\right| \leqslant \sum_{i=1}^{\infty}\sum_{m=1}^{i}\left|E\left[\tilde{\xi}_{m+n}\middle|\mathcal{F}_{m+n-i}\right] - E\left[\tilde{\xi}_{m+n}\middle|\mathcal{F}_{m+n-i-1}\right]\right| \leqslant$$

$$= \sum_{i=1}^{\infty}\sum_{m=1}^{i}\left[E\left|\int_{(m+n-i-1)\Delta}^{(m+n-i)\Delta} g_{m+n}(t)\,d\tilde{W}_t\; 2\left(\int_{-\infty}^{(m+n-i-1)\Delta} g_{m+n}(t)\,d\tilde{W}_t + \right.\right.\right.$$

$$\left.\left.\left. + \int_{\Delta(m+n-i-1)}^{(m+n-i)\Delta} g_{m+n}(t)\,d\tilde{W}_t\right) - \int_{(m+n-i-1)\Delta}^{(m+n-i)\Delta} g_{m+n}^2(t)\,dt\right|\right].$$

Using the Schwarz inequality one can easily show that the above series converges and is uniformly bounded with respect to n. We omit the details. There exists $C < \infty$ depending on a and Δ such that $E|U^{n+1}\eta| \leqslant C$. Therefore $\frac{1}{\sqrt{n}}U^{n+1}\eta \xrightarrow{\text{in P}} 0$.

It is obvious that

$$\frac{1}{\sqrt{n}}U\eta \xrightarrow{\text{in P}} 0.$$

The proof of Lemma 5 is now complete.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f of Theorem 1. We shall prove that $\sqrt{n}^{-1} S_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, E(e_1^2))$. Let us apply the Central Limit Theorem for martingales (see [8]). It suffices to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\left[e_k^2 \middle| \mathcal{F}_{k-1}\right] = E(e_1^2) \quad \text{a.s.} \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} E(e_k^4) = 0. \tag{3.5}$$

Using Lemma 3 we obtain (3.4). In view of the formula (2.6) one can verify that e_k^4 is integrable and $\{e_k^4,\ k=\ldots-1,0,1,\ldots\}$ is also a stationary sequence. Therefore (3.5) follows immediately. The proof of Theorem 1 is now complete.

REFERENCES

[1] A. B a g c h i , V. B o r k a r : Parameter identification in infinite dimensional linear systems, Stochastics 12 (1984) 201-213.

[2] J. D ą b r o w s k i : Parameter identification in linear stochastic differential equations. Stat. and Prob. Letters. 7 (1989) 391-394.

[3] K. I t o , M. N i s i o : On stationary solutions of a stochastic differential equations, J. Math. Kyoto University 4 (1964) 1-75.

[4] G. D a P r a t o : Equations aux derivees partielles stochastiques et applications. Centre de Mathematiques Appliques, 1986, R.I. n148, Unite de Recherche Associée au CNRS-756.

[5] I.I. G i h m a n , A.V. S k o r o h o d : The theory of stochastic processes, vol. I. Berlin 1974.

[6] A. C h o j n o w s k a - M i c h a l i k : On processes of Ornstein-Uhlenbeck Type in Hilbert space, Stochastics 21 (1987) 251-286.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[7] P. Hall, C.C. Heyde: Martingale limit theory and its application. New York 1980.

[8] B.M. Brown: Martingale central limit theorems, Ann. Math. Statistics 42 (1971) 59-66.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received September 27, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Tadeusz Jagodziński

ON GALERKIN APPROXIMATIONS TO PSEUDO-PARABOLIC PROBLEM

1. Introduction

W.H. Ford [2] uses Galerkin approximations to the solution of the pseudo-parabolic problem

$$\begin{cases} Lu = f(x,t,u), & (x,t) \in G\times(0,T], \\ u(x,0) = u_0(x), & x \in G, \\ u(x,t) = 0, & (x,t) \in \partial G\times(0,T], \end{cases}$$

where

$$Lu = m(x,t)u_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u_t}{\partial x_j}\right) - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t,u)\cdot\frac{\partial u}{\partial x_j}\right).$$

G is a bounded domain in R^n with boundary ∂G and f, u_0, m, a_{ij}, b_{ij} are given functions, $T > 0$ given constant.

In this paper we extend this problem to the case of function f depending on the partial derivatives of u with respect to $x_1,\dots,x_n$, i.e.

$$(1)\qquad \begin{cases} Lu = f\left(x,t,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},u\right), & (x,t) \in G\times(0,T], \\ u(x,0) = u_0(x), & x \in G, \\ u(x,t) = 0, & (x,t) \in \partial G\times(0,T]. \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In order to extend the results of [2] we made assumptions about the solution and the terms appearing in problem (1) which differ from those made in [2].

We develope also continuous in time semidiscrete Galerkin approximation and discrete in time Crank-Nicolson-Galerkin approximation to the solution of problem (1).

We show in detail these parts of proofs which are connected with the change of the right side of the equation in problem (1) and the new assumptions.

The reason for terming such pseudo-parabolic problems can be found in [4]. For a discussion of existence and uniqueness results for such problems, see [3] and for similar Galerkin approximations to the solution of the parabolic problem, see [1].

2. Assumptions

Let $(.,.)_0$ be the standard inner product in $L^2(G)$, and $\|\cdot\|_0$ the corresponding norm to $(.,.)_0$. Define by

$$\|\phi\|_1 = \left(\|\phi\|_0^2 + \sum_{i=1}^{n} \left\|\frac{\partial \phi}{\partial x_i}\right\|_0^2\right)^{\frac{1}{2}}$$

the norm in Sobolev space $H^1(G)$ and by

$$\|\phi\|_{1'} = \left(\sum_{i=1}^{n} \left\|\frac{\partial \phi}{\partial x_i}\right\|_0^2\right)^{\frac{1}{2}}$$

another norm in $H_0^1(G)$. If u, f and coefficients in (1) are sufficiently smooth, (1) is equivalent to the generalized problem

$$(2)\quad \begin{cases} (m(.,t)u_t(.,t),v)_0 + a(t;u_t,v) + b(t;u;u,v) = \\ \qquad = (f(.,t,\nabla_x u,u),v)_0, \quad \forall v \in H_0^1(G),\ 0 < t \leqslant T, \\ u(x,0) = u_0(x) \quad \text{in } H_0^1(G), \\ u(x,t) \quad \text{in } H_0^1(G),\ 0 < t \leqslant T, \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where

$$\nabla_x u = (u_{x_1}, \ldots, u_{x_n}),$$

$$a(t;u,v) = \int_G \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j} v_{x_i}\, dx,$$

$$b(t;w;u,v) = \int_G \sum_{i,j=1}^{n} b_{ij}(x,t,w)\, u_{x_j} v_{x_i}\, dx.$$

We make the following assumptions:

(A1) m, a_{ij}, b_{ij}, f are measurable in G and continuous in the remaining variables for each $x \in G$.

(A2) $\partial m/\partial t$ exists and is bounded on $G\times[0,T]$, and there exist constans m_*, $m^* > 0$, such that $m_* \leqslant m,(x,t) \leqslant m^*$ for $(x,t) \in G\times[0,T]$.

(A3) $\partial a_{ij}/\partial t$, $i,j = 1,2,\ldots,n$, exist and are bounded on $G\times[0,T]$. The matrix $(a_{ij}(x,t))$, $i,j = 1,\ldots,n$ is symmetric and there exist constans $\eta_0, \eta_1 > 0$ such that

$$\eta_0 \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,j=1}^{n} a_{i,j}(x,t)\, \xi_i \xi_j \leqslant \eta_1 \sum_{i=1}^{n} \xi_i^2$$

for $(x,t) \in G\times[0,T]$ and $\xi \in R^n$.

(A4) $b_{ij}(x,t,p)$, $i,j = 1,\ldots,n$ are bounded on $G\times[0,T]\times R$ and there exists constant $K_1 > 0$ such that

$$|b_{ij}(x,t,p_1) - b_{ij}(x,t,p_2)| \leqslant K_1 \cdot |p_1 - p_2|$$

for $(x,t) \in G\times[0,T]$ and $p_1, p_2 \in R$.

(A5) There exists a constant $K_2 > 0$ such that

$$\left|f\left(x,t,p_1^{(1)},\ldots,p_1^{(n+1)}\right) - f\left(x,t,p_2^{(1)},\ldots,p_2^{(n+1)}\right)\right| \leqslant$$

$$\leqslant K_2 \sum_{i=1}^{n+1} |p_2^{(i)} - p_1^{(i)}|$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $(x,t) \in G\times[0,T]$ and there exists $g(x) \in L^2(G)$ such that

$$|f(x,t,0,\ldots,0)| \leqslant g(x)$$

for $(x,t) \in G\times[0,T]$ and so for each t, if $u \in H^1(G)$ then $f(.,t,\nabla_x u,u) \in L^2(G)$.

(A6) There exists a unique solution to (2) such that the function $t \longrightarrow u(.,t)$ is a strongly continuously differentiable mapping from $[0,T]$ into $H_0^1(G)$ and $\partial u/\partial x_i$, $i = 1,\ldots,n$, are bounded on $G\times[0,T]$.

By (A3)-(A5), we can state that

$$\text{(i)} \quad \left|\sum_{i,j=1}^{n} \frac{\partial}{\partial t}(a_{ij}(x,t))\,\xi_i\theta_j\right| \leqslant \eta_2\left(\sum_{i=1}^{n}\xi_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\theta_1^2\right)^{\frac{1}{2}}$$

for all $\xi,\theta \in R^n$ where $\eta_2 = n\cdot \max\limits_{1\leqslant i,j\leqslant n}\sup\limits_{G\times[0,T]}\left|\frac{\partial}{\partial t}(a_{ij}(x,t))\right|$,

$$\text{(ii)} \quad \left|\sum_{i,j=1}^{n} b_{ij}(x,t,p)\,\xi_i\theta_j\right| \leqslant \eta_3\left(\sum_{i=1}^{n}\xi_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\theta_i^2\right)^{\frac{1}{2}},$$

for all ξ,θ in R^n, where

$$\eta_3 = n\cdot \max_{1\leqslant i,j\leqslant n}\sup_{G\times[0,T]\times R}\left|b_{ij}(x,t,p)\right|,$$

$$\text{(iii)} \quad \left|f(x,t,\nabla_x\phi(x),\phi(x))\right|^2 \leqslant 2|g(x)|^2 + 2K_2^2(n+1)\left(\sum_{i=1}^{n}\left(\frac{\partial\phi(x)}{\partial x_i}\right)^2 + (\phi(x))^2\right)$$

for $(x,t) \in G\times[0,T]$ and $\phi \in H^1(G)$.

$$\text{(iv)} \quad 2|(f(.,t,\nabla_x V,V),V)_0| \leqslant 2\|g\|_0^2 + (2K_2^2(n+1)+1)\|V\|_1^2$$

for $t \in [0,T]$, $V \in H_0^1(G)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In fact, from (ii) and Schwartz inequality we get

$$2|(f(\cdot,t,\nabla_x V,V),V)_0| \leqslant 2\|f\|_0\|V\|_0 \leqslant \|f\|_0^2 + \|V\|_0^2 \leqslant$$

$$\leqslant 2\|g\|_0^2 + (2K_2^2(n+1))\|V\|_1^2 +$$

$$+ \|V\|_0^2 \leqslant 2\|g\|_0^2 + (2K_2^2\cdot(n+1)+1)\cdot\|V\|_1^2.$$

3. The semidiscrete Galerkin approximation

Let $\widetilde{H}_0$ be the finite dimensional subspace of $H_0^1(G)$ spanned by N linearly independent elements V_p, $p = 1,2,\ldots,N$, bounded on G. We have to find a solution

$$U(x,t) = \sum_{p=1}^{n} a_p(t)\cdot V_p(x),$$

of (2), with $a_p \in C^1[0,T]$, $p = 1,2,\ldots,N$, thus

$$(3)\quad \begin{cases} (m(\cdot,t)U_t(\cdot,t),V)_0 + a(t;U_t,V)+b(t;U;U,V) = \\ \qquad = (f(\cdot,t,\nabla_x U,U),V)_0, \\ V\in\widetilde{H}_0, \quad t\in(0,T], \\ (U(\cdot,0),V)_1 = (u_0,V)_1, \quad V\in\widetilde{H}_0. \end{cases}$$

L e m m a 1. If $U(x,t)$ is strongly continuously differentiable and satisfies (3) for each $t\in[0,\gamma)$, $\gamma<T$, then $\|U(\cdot,t)\|_1$ is bounded in $[0,\gamma]$.

P r o o f . If we apply the identities

$$a(t;U_t,U) = \frac{1}{2}\left(\frac{d}{dt}\,a(t;U,U) - \sum_{i,j=1}^{n}\int_G \frac{\partial a_{ij}}{\partial t}(x,t)\,\frac{\partial U}{\partial x_i}\,\frac{\partial U}{\partial x_j}\,dx\right),$$

$$(m(\cdot,t)U_t,U)_0 = \frac{1}{2}\left[\frac{d}{dt}\,(m(\cdot,t)U,U)_0 - \left(\frac{\partial m}{\partial t}(\cdot,t)U,U\right)_0\right],$$

and the inequalities

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left|\frac{\partial m}{\partial t}(\cdot,t)V,Z)_0\right| \leqslant \sup_{G\times[0,T]}\left|\frac{\partial m}{\partial t}(x,t)\right| \cdot \|V\|_0 \cdot \|Z\|_0,$$

$$\left|\sum_{i,j=1}^{n}\int_G \frac{\partial a_{ij}}{\partial t}(x,t)\,\frac{\partial V}{\partial x_j}\,\frac{\partial Z}{\partial x_i}\,dx\right| \leqslant \eta_2\|V\|_{1'}\cdot\|Z\|_{1'},$$

for $V,Z\in\tilde{H}_0$, which follow from assumptions, and if we let $V = U$ in (3), then we have

$$\frac{d}{dt}(m(\cdot,t)U,U)_0 + \frac{d}{dt}a(t;U,U) \leqslant 2\left|(f(\cdot,t,\nabla_x U,U)_0\right| +$$

$$+ 2|b(t;U;U,U)| + \sup_{G\times[0,T]}\left|\frac{\partial m}{\partial t}(x,t)\right| \cdot \|U\|_0^2 + \eta_2\|U\|_{1'}^2 .$$

By (iv), the Schwartz inequality and the inequality

$$|b(t;W;V,Z)| \leqslant \eta_3\|V\|_{1'}\cdot\|Z\|_{1'}$$

for $V,W,Z\in\tilde{H}_0$, which follows from (ii), we have

$$\frac{d}{dt}(m(\cdot,t)U,U)_0 + \frac{d}{dt}a(t;U,U) \leqslant 2\|g\|_0^2 +$$

$$+ (2K_2^2 n + 2K_2^2 + 1)\cdot\|U\|_1^2 + 2\eta_3\|U\|_{1'}^2 + \sup_{G\times[0,T]}\frac{\partial m}{\partial t}(x,t)\cdot\|U\|_0^2 +$$

$$+ \eta_2\|U\|_{1'}^2 \leqslant 2\|g\|_0^2 + L_1\|U\|_1^2,$$

where $L_1 = \max(2nK_2^2 + 2K_2^2 + 1,\ 2\eta_3 + \eta_2,\ \sup_{G\times[0,T]}\left|\frac{\partial m}{\partial t}(x,t)\right|)$.

If we integrate from $t = 0$ to $t = \tau$, $\tau < \gamma$, and make use of (A2), (A3), getting

$$m_*\|V\|_0^2 \leqslant (m(\cdot,t)V,V)_0 \leqslant m^*\|V\|_0^2 ,$$

$$\eta_0\|V\|_{1'}^2 \leqslant a(t;V,V) \leqslant \eta_1\|V\|_{1'}^2 ,$$

for $V\in\tilde{H}_0$, and if we note that $\|U\|_1^2(0) \leqslant \|u_0\|_1^2$ and apply Gronwall's Lemma (see [2], p.274), then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(m(\cdot,t)U,U)_0\Big|_0^\tau + a(t;U,U)\Big|_0^\tau \leqslant 2\|g\|_0^2\tau + L_1\int_0^\tau \|U\|_1^2 dt$$

and we have

$$\|U\|_1^2(\tau) \leqslant L_3[2\|g\|_0^2 + L_2\|u_0\|_1^2]\cdot \mathrm{Exp}(L_1L_3T),$$

where $L_2 = \max(m_*,\eta_1)$, $L_3 = [\min(m_*,\eta_0)]^{-1}$.

Theorem 1. There exists a unique strongly continuously differentiable solution to (3).

Proof. Let $V\in\tilde{H}_0$. For fixed t in $[0,T]$, $W\in\tilde{H}_0$ we have

$$|(f(\cdot,t,\nabla_x W,W),V)_0| \leqslant \|f\|_0\|V\|_1$$

and $(f(\cdot,t,\nabla_x W,W),V)_0$ is a bounded linear functional.

By assumptions (A2), (i), (ii) we find

$$|(m(\cdot,t)U,V)_0| \leqslant m^*\|U\|_0\|V\|_0 \leqslant m^*\|U\|_1\|V\|_1,$$

$$|a(t;U,V)| \leqslant \eta_2\|U\|_{1'}\|V\|_{1'},$$

$$|(m(\cdot,t)U,V)_0 + a(t;U,V)| \leqslant 2\max(m^*,\eta_2)\|U\|_1\|V\|_1,$$

$$|b(t;W;U,V)| \leqslant \eta_3\|U\|_{1'}\|V\|_{1'} \leqslant \eta_3\|U\|_1\|V\|_1,$$

and $(m(\cdot,t)U,V)_0 + a(t;U,V)$, $b(t;W;U,V)$ are bounded bilinear functionals on the Hilbert space $\tilde{H}_0$. Thus, there exists an element $F(t,\nabla_x W,W)\in\tilde{H}_0$, $W\in H_0$ and bounded linear operators $A(t)$, $B(t,W)$, such that for $U,V\in\tilde{H}_0$ we have

$$(T2)\qquad \begin{cases} (f(\cdot,t,\nabla_x W,W),V)_0 = (F(t,\nabla_x W,W),V)_1, \\ (m(\cdot,t)U,V)_0 + a(t;U,V) = (A(t)U,V)_1, \\ b(t;W;U,V) = (B(t,W)U,V)_1. \end{cases}$$

Hence, solving of (3) is reduced to that of

$$(4)\qquad \begin{cases} A(t)U_t + B(t,U)V = F(t,\nabla_x U,U), \\ (U(0),V)_1 = (u_0,V)_1, \quad V\in\tilde{H}_0, \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, the proof is analogous to that of Theorem 2.3 in [2], (p.277). By assumptions (A2), (A3) we find

$$(A(t)V,V)_1 = (m(\cdot,t)V,V)_0 + a(t;V,V) \geqslant$$

$$\geqslant m_*\|V\|_0^2 + \eta_0\|V\|_1^2, \geqslant \min(m_*,\eta_0)\|V\|_1^2,$$

and $A^{-1}(t)$ exists; so (4) becomes

$$U_t = H(t,U),\ (U(0),V)_1 = (u_0,V)_1,\quad V \in \tilde{H}_0, \tag{5}$$

where $H(t,U) = A^{-1}(t)\left[F(t,\nabla_x U,U) - B(t,U)U\right]$.
From (A5) we get

$$\|F(t,\nabla_x V_1,V_1) - F(t,\nabla_x V_2,V_2)\|_1 < K_2\|V_1-V_2\|_1$$

for $V_1,V_2 \in \tilde{H}_0$, and we can show that $H(t,U)$ satisfies Lipschitz condition uniformly on any bounded subset of $[0,T]$ and by assumptions $H(t,U)$ is a continuous function on $[0,T]\times\tilde{H}_0$.

These results and Lemma 1 imply the thesis of Theorem 1.

4. The discrete Crank-Nicolson-Galerkin approximation

Let $\Delta t = \frac{T}{M}$, M a positive integer, $t_m = m\Delta t$, $t_{m+0,5} = t_m + 0,5\Delta t$, $m = 0,1,\ldots,M$, $U_m = \sum_{p=1}^{N} a_{pm}\cdot V_p$, $m = 0,1,\ldots,M$, where U_m are to be determined by the difference system (cf. (3))

$$\begin{cases} (m(\cdot,t_{m+0,5})(U_{m+1} - U_m)(\Delta t)^{-1},V)_0 + \\ + a(t_{m+0,5};\ (\Delta t)^{-1}(U_{m+1} - U_m),V) + \\ + b(t_{m+0,5};\ 0,5(U_{m+1} + U_m);0,5(U_{m+1} + U_m),V) = \\ = (f(\cdot,t_{m+0,5},\ 0,5\nabla_x(U_{m+1} + U_m),\ 0,5(U_{m+1} + U_m)),V), \\ (U_0,V)_1 = (u_0,V)_1,\quad V \in \tilde{H}_0. \end{cases} \tag{6}$$

T h e o r e m 2. There exists a constant $K_0 > 0$ such that (6) has a solution for $\Delta t < K_0$, and the choice of Δt is independent of $\tilde{H}_0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

p r o o f . Analogously as in [2], (p.278) we shall define the mappings $G_m:\tilde{H}_0 \longrightarrow \tilde{H}_0$, $m = 0,1,\dots,M-1$, where $W = G_m(Y)$ is a solution of the problem

$$(7)\quad \begin{cases} (m(\cdot,t_{m+0,5})(W-U_m)(\Delta t)^{-1},V)_0 + a(t_{m+0,5};(W-U_m)(\Delta t)^{-1},V) + \\ + b(t_{m+0,5};0,5(Y+U_m);0,5(Y+U_m),V) = \\ = (f(\cdot,t_{m+0,5},\nabla_x(Y+U_m),0,5(Y+U_m)),V)_0, \\ (U_0,V)_1 = (u_0,V)_1, \quad V \in \tilde{H}_0. \end{cases}$$

The first relation from (7) implies, by (T1), that

$$(W-U_m)(\Delta t)^{-1} = A^{-1}(t_{m+0,5})[F(t_{m+0,5},\nabla_x 0,5(Y+U_m),0,5(Y+U_m)) - B(t_{m+0,5},0,5(Y+U_m))0,5(Y+U_m)]$$

and so $G_m(Y)$ is uniquely defined and continuous. If G_m has a fixed point Y_m, then $U_{m+1} = Y_m$ is a solution to (6). Let $V = W$ in (7). Nothing that, by (A2)-(A4), (A6), for $V,W,Z \in \tilde{H}_0$ we have

$$|a(t;V,Z)| \leqslant \eta_1 \|V\|_{1'} \|Z\|_{1'},$$

$$\eta_0 \|V\|^2_{1'} \leqslant a(t;V,V) \leqslant \eta_1 \|V\|^2_{1'},$$

$$m_* \|V\|_0 \|Z\|_0 \leqslant (m(\cdot,t)V,Z)_0 \leqslant m^* \|V\|_0 \|Z\|_0,$$

$$|b(t;W;V,Z)| \leqslant \eta_3 \|V\|_{1'} \|Z\|_{1'},$$

and applying the Schwartz inequality, we get

$$\eta_0 \|W\|^2_{1'} + m_* \|W\|^2_0 \leqslant$$

$$\leqslant \Delta t \|f(\cdot,t_{m+0,5},0,5\nabla_x(Y+U_m),0,5(Y+U_m))\|_0 \|W\|_0 + m^* \|U_m\|_0 \|W\|_0 +$$

$$+ \eta_1 \|U_m\|_{1'} \|W\|_{1'} + \eta_3 \Delta t \|W\|_{1'} \|0,5(Y+U_m)\|_{1'}.$$

Using now the inequality $ab \leqslant \varepsilon a^2 + (4\varepsilon)^{-1}b^2$ with either $\varepsilon = \frac{m_*}{4}$ or $\varepsilon = \frac{\eta_0}{4}$, we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\eta_0\|W\|_{1'}^2 + m_*\|W\|_0^2 \leqslant$$

$$\leqslant (\Delta t)^2 m_*^{-1}\|f(\cdot,t_{m+0,5},0,5\nabla_x(Y+U_m),0,5(Y+U_m))\|_0^2 + \frac{m_*}{4}\|W\|_0^2 +$$

$$+ m_*^{-1}m^{*2}\|U_m\|_0^2 + \frac{m_*}{4}\|W\|_0^2 + \eta_0^{-1}\eta_1^2\|U_m\|_{1'}^2 + \frac{\eta_0}{4}\|W\|_{1'}^2 +$$

$$+ \eta_3^2\eta_0^{-1}(\Delta t)^2\cdot\|0,5(Y+U_m)\|_{1'}^2 + \frac{\eta_0}{4}\|W\|_{1'}^2 .$$

So by definition of norms $\|\cdot\|_0$, $\|\cdot\|_1$, $\|\cdot\|_{1'}$, there exists a constant $C>0$ such that

$$C\|W\|_1^2 \leqslant 0,5\ m^*\|W\|_0^2 + 0,5\eta_0\|W\|_{1'}^2 \leqslant$$

$$\leqslant (\Delta t)^2\cdot m_*^{-1}\|f(\cdot,t_{m+0,5},0,5\nabla_x(Y+U_m),0,5(Y+U_m)\|_0^2 +$$

$$+ m_*^{-1}m^{*2}\|U_m\|_0^2 + \eta_0^{-1}\eta_1^2\|U_m\|_{1'}^2 + (\Delta t)^2\eta_0^{-1}\eta_3^2\|0,5(Y+U_m)\|_{1'}^2 .$$

By (iii), there exist constans $C_1, C_2 > 0$, $C_2 > 2K_2^2(n+1)$, independent of m, such that

$$\|f(\cdot,t_{m+0,5},0,5\nabla_x(Y+U_m),0,5(Y+U_m)\|_0^2 \leqslant C_1 + C_2\|Y+U_m\|_1^2 .$$

Then using the inequality $\|Y+U_m\|_1^2 \leqslant 2\|Y\|_1^2 + 2\|U_m\|_1^2$, we can state that there exist constans $C_1, C_2, C_3 > 0$, independent of m, such that

$$\|W\|_1^2 \leqslant C_1(\Delta t)^2\|Y\|_1^2 + C_2(\Delta t)^2 + C_3\|U_m\|_1^2 .$$

Let $\|Y\|_1 \leqslant R_m$. If Δt is so chosen that $C_1(\Delta t)^2 < 1-\varepsilon$, $0<\varepsilon<1$, and that

$$R_m^2 \geqslant (C_2(\Delta t)^2 + C_3\|U_m\|_1^2)\varepsilon^{-1} \geqslant (C_2(\Delta t)^2 + C_3\|U_m\|_1^2)(1-C_1(\Delta t)^2)$$

we find the inequality

$$\|W\|_1^2 \leqslant (1-\varepsilon)R_m^2 + R_m^2\varepsilon = R_m^2$$

which means that G_m maps the ball $\|Y\|_1 \leqslant R_m$ into itself, for $m = 0,1,\ldots,M-1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By the Schauder fixed point theorem ([5], p.189), G_m has a fixed point Y_m such that $G_m(Y_m) = Y_m$, then $U_{m+1} = Y_m$. Notice that $K_0 = \left(\frac{1-\varepsilon}{C_1}\right)^{\frac{1}{2}}$, so the choice of Δt is independent of $\tilde{H}_0$. This ends the proof of Theorem 2.

REFERENCES

[1] J. Douglas, JR., T. Dupont: Galerkin methods for parabolic equations, SIAM J. Numer. Anal. 11 (1974) 155-169.

[2] W.H. Ford: Galerkin approximations to non-linear pseudo-parabolic partial differential equations. Aequationes Math. 14 (1976) 271-291.

[3] R.E. Showalter, T.W. Ting: Pseudoparabolic partial differential equations, SIAM J. Math. Anal. 1 (1970) 1-26.

[4] T.W. Ting: Parabolic and pseudo-parabolic partial differential equations. J. Math. Soc. Japan 21 (1969) 440-453.

[5] A. Friedman: Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, N. Jersey, 1964.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received September 27, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Janina Kotus

THE OSCILLATING TRAJECTORIES AND SADDLES AT INFINITY OF VECTOR FIELDS ON OPEN SURFACES

1. Introduction

In this paper we study limit sets of oscillating trajectories and saddles at infinity for C^r $(r \geqslant 1)$ vector fields on open surfaces with finite genus. These problems have arisen from description of globally structurally stable vector fields on open surfaces (see [1], [2]). We prove that limit sets of oscillating trajectories of Kupka-Smale vector fields contains a saddle at infinity. We also prove that a vector field with hyperbolic critical points has at most a countable many of stable and unstable separatrices of saddles at infinity. For a plane this was already shown in [2], here we generalize it for open surface with finite genus and countable space of ends.

2. Definitions and statement of the results

Let M be an open surface with the genus $g = g(M) < \infty$ and a space of ends E. Any such surface is homeomorphic to N-K, where N is a closed surface of genus g and K is closed, totally disconnected subset of N (see [4]). We note that finite genus is a weaker assumption than finite Euler characteristic, which would require that E be a finite set.

By $H^r(M)$ we denote the space of complete C^r vector fields on M with the C^r Whitney topology $(r \geqslant 1)$. X, Y denote elements of $H^r(M)$, ϕ_X denotes the flow induced by X. For $x \in M$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$O_X(x)$ ($O_X^+(x)$, $O_X^-(x)$) is the trajectory of x (the positive semitrajectory, the negative semitrajectory) under ϕ_X. By $O_X[x,y]$ we denote the closed X-trajectory segment from x to y.

We distinguish three kinds of asymptotic behavior for each semi-trajectory $O_X^{\pm}(x)$:

(a) $O_X^{\pm}(x)$ is bounded if it is contained in some compact set $K \subset M$,

(b) $O_X^{\pm}(x)$ escapes to infinity if for each compact set $K \subset M$ there is a point $y \in O_X^{\pm}(x)$ for which $O_X^{\pm}(y) \cap K = \emptyset$,

(c) $O_X^{\pm}(x)$ oscillates if it is neither bounded nor escapes to infinity.

These kinds of behavior for $O_X^+(x)$ (resp. $O_X^-(x)$) can be also described in terms of the ω-limit (resp. α-limit) set of $x \in M$ under ϕ_X:

$$\omega(O_X(x)) = \{ y \in M : \exists t_n \to +\infty \text{ such that } \phi_X(x,t_n) \to y \},$$

$$\alpha(O_X(x)) = \{ y \in M : \exists t_n \to -\infty \text{ such that } \phi_X(x,t_n) \to y \}.$$

It is easy to see that:

(a) $O_X^+(x)$ is bounded iff $\omega(O_X(x))$ is compact (and nonempty),

(b) $O_X^+(x)$ escapes to infinity iff $\omega(O_X(x)) = \emptyset$,

(c) $O_X^+(x)$ oscillates iff $\omega(O_X(x))$ is a non-compact and non-empty subset of M.

We extend the definition of ω-limit (resp. α-limit) set of $x \in M$ to ω^*-limit (α^*-limit) set

$$\omega^*(O_X(x)) = \{ y \in M \cup E : \exists t_n \to +\infty \text{ such that } \phi_X(x,t_n) \to y \},$$

$$\alpha^*(O_X(x)) = \{ y \in M \cup E : \exists t_n \to -\infty \text{ such that } \phi_X(x,t_n) \to y \}.$$

Thus:

(a) $O_X^+(x)$ escapes to infinity iff there is $P \in E$ such that $\omega^*(O_X(x)) = P$,

(b) $O_X^+(x)$ oscillates iff $\omega(O_X(x)) \neq \emptyset$ and there is $P \in E$ such that $P \in \omega^*(O_X(x))$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $Per(X)$, $\Omega(X)$ denote respectively the set of periodic points, the set of non-wandering points of X i.e.

$$Per(X) = \{x \in M : \phi_X(x,t) = x \text{ for some } t > 0\},$$

$$\Omega(X) = \{x \in M : \exists x_n \to x,\ t_n \to +\infty \text{ such that } \phi_X(x,t_n) \to x\}.$$

We define the first positive (resp. negative) prolongation limit set of $x \in M$ by

$$J_X^{\pm}(x) = \{y \in M : \exists x_n \to x,\ t_n \to \pm\infty \text{ such that } \phi_X(x_n,t_n) \to y\}$$

In general $Per(X) \subset \Omega(X)$, $\omega(O_X^+(x)) \subset J_X^+(x)$, $\alpha(O_X^-(x) \subset J_X^-(x)$ and $\Omega(X) = \{x \in M : x \in J_X^+(x)\}$.

We say that two unbounded semi-trajectories $O_X^+(x)$ and $O_X^-(y)$ form a saddle at infinity if $y \in J_X^+(x)$ and $O_X^+(x)$, $O_X^-(y)$ escape to infinity (i.e. there are points $P,Q \in E$ such that $\omega^*(O_X(x)) = P$, $\alpha^*(O_X(x)) = Q$. In this case, call $O_X^+(x)$ (resp. $O_X^-(y)$) the stable (resp. unstable) separatix of the saddle at infinity. By $W_X^{\pm}$ we denote the union of all stable (unstable) separatices of saddles at infinity.

Let $H_0^r(M)$ be the set of vector fields $X \in H^r(M)$ with hyperbolic critical points, $H_{K-S}^r(M)$ be the set of Kupka-Smale vector fields i.e. $X \in H_{K-S}^r(M)$ if $X \in H^r(M)$ and satisfies:

(a) for each $x \in Per(X)$ the trajectory $O_X(x)$ is hyperbolic,

(b) there are no saddles connection between fixed saddles.

In [3] it was proved that $H_0^r(M)$ is open and dense in $H^r(M)$ while $H_{K-S}^r(M)$ is residual in $H^r(M)$.

T h e o r e m 1. Let M be an open surface with finite genus and countable space of ends E. For each $X \in H_0^r(M)$ the sets $W_X^{\pm}$ contains at most countably many different trajectories.

T h e o r e m 2. Let M be an open surface with finite genus, $X \in H_{K-S}^r(M)$. If $O_X^+(x)$ is an oscillating semi-trajectory then ω-limit set of x contains a saddle at infinity.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Proofs of the results

Let M be an open surface with finite genus and countable space of ends E. For $X \in H_o^r(M)$ and $P \in E$ we define sets

$$W^+_{X,P} = \{ O^+_X(a) \in W^+_X : \omega^*(O_X(a)) = P \},$$

$$W^-_{X,P} = \{ O^-_X(a) \in W^-_X : \alpha^*(O_X(a)) = P \}.$$

Then $W^\pm_X = \bigcup_{P \in E} W^\pm_{X,P}$.

In the sequel we assume that S is a transverse section of $X \in H_o^r(M)$, $p_1, p_2 \in S$, $\omega^*(O_X(p_1)) = \omega^*(O_X(p_2)) = P$ for some $P \in E$ and:

(i) $L = O^+_X(p_1) \cup O^+_X(p_2) \cup [p_1, p_2]$ ($[p_1, p_2] \subset S$) separates M into components D^+, D^-,

(ii) if $x \in D^\pm \cup (p_1, p_2)$ then $O^\pm_X(x) \subset D^\pm$,

(iii) D^+ is homeomorphic to an open set $G \subset R^2$.

L e m m a 1. Let $O^+_X(a) \in W^+_{X,P}$ for $a \in (p_1, p_2)$. Then $B_{a,Q} = \{ O^-_X(b) \in W^-_{X,Q} : b \in J^+_X(a) \}$ is at most countable for any $Q \in E$.

P r o o f . It is enough to prove that a set of accumulation points of $B^d_{a,Q} \subset D^+$ is empty. Suppose that there is a sequence of different trajectories $O^-_X(b_n) \in B_{a,Q}$ and we choose an accumulation point b such that $b_n \longrightarrow b \in M$. By (ii) $O^+_X(b_n)$, $O^+_X(b) \subset \bar{D}^+$. We may assume that $X(b) \neq 0$. Otherwise, if $b \in \mathrm{Per}(X)$ then b is a fixed saddle as $X \in H_o^r(M)$ and its stable and unstable separatrices are contained in the set $B^d_{a,Q}$. Let $S' \subset \bar{D}^+$ be a transverse section of X at b. Then $O^+_X(b_n) \cap S' \neq \emptyset$ for large n. We also may assume that $b_n \in S'$ and $O^-_X(b_n) \cap S' = \emptyset$ as $\alpha^*(O_X(b_n)) = Q$. Choose three such points denoted by b_1, b_2, b_3. Let $L_1 = O^-_X(b_1) \cup O^-_X(b_3) \cup [b_1, b_3]$, where $b_2 \in (b_1, b_3) \subset S'$. By (iii) L_1 separates D^+ into two components G^+ and G^- which are respectively forward and backward invariant. Since $a \in J^-_X(b_2)$ there are points $y_{2n} \in (b_1, b_3)$ and num-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

bers $t_{2n} \longrightarrow -\infty$ such that $y_{2r} \longrightarrow b_2$, $x_{2n} = \phi_X(y_{2n}, t_{2n}) \in S$ and $x_{2n} \longrightarrow$ a what contradicts property that $O_X^-(x_{2n}) \subset G^-$. See Fig.1. Thus $B^d_{a,Q} = \emptyset$.

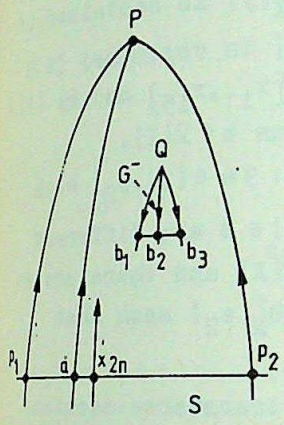


Fig.1

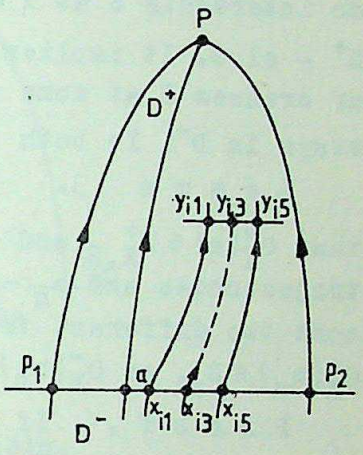


Fig.2

C o r o l l a r y 1. Let $O_X^+(a) \in W^+_{X,P}$ for $a \in (p_1, p_2)$. Then $B_a = \left\{ O_X^-(b) \in W_X^- : b \in J_X^+(a) \right\}$ is countable.

R e m a r k 1. For $b \in D^+$ such that $O_X^-(b) \in W_X^-$ a set $A_b = \left\{ O_X^+(a) \in W^+_{X,P} : a \in (p_1, p_2) \wedge a \in J_X^-(b) \right\}$ has at most two trajectories.

We omit proof of this remark.

L e m m a 2. Let B_a be as in Corollary 1. If $O_X^-(b_n) \in B_a$ are different trajectories and $b_n \longrightarrow b \in M$ then $b \in \Omega(X)$.

P r o o f . Since $b_n \in J_X^+(a)$ then $O_X(b) \subset D^+$ and $O_X(b) \subset D^+$. Suppose that $b \notin \Omega(X)$. Let $S' \subset D'$ be a transverse section of X at b. Then $O_X^+(b) \cap S' = \emptyset$ for $x \in S'$ and $O_X^+(b_n) \cap S' \neq \emptyset$ for large n. We choose five such points denoted by $b_1, \ldots, b_5$. By assumption $b_i \in J_X^+(a)$, so there are points $x_{iK} \in (p_1, p_2)$ and numbers $t_{iK} \longrightarrow +\infty$ such that $x_{iK} \longrightarrow a$, $y_{iK} = \phi_X(x_{iK}, t_{iK}) \in S'$ and $y_{iK} \longrightarrow b_i$, $i = 1, \ldots, 5$, $k \longrightarrow \infty$. We may assume that y_{iK} is the first common point of $O_X^+(x_{iK})$ and S'. Let $\varepsilon = \frac{1}{2} \min_{1 \leq i \neq j \leq 5} \left\{ \varrho(b_i, b_j) \right\}$, where ϱ is a Riemann metric on M.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then there are y_{iK} satisfying $\varrho(y_{iK},b_i) < \varepsilon$ and $y_{i1} < b_2 < y_{i3} < b_4 < y_{i5}$. See Fig.2 A set $L_1 = O_X[x_{i1},y_{i1}] \cup [x_{i1},x_{i5}] \cup \cup [y_{i1},y_{i5}]$ (where $[x_{i1},x_{i5}] \subset S$, $[y_{i1},y_{i5}] \subset S'$) separates D^+ into two components such that for $y \in [y_{i1},y_{i5}]$ the semitrajectory $O_Y^-(y)$ either is contained in the region G bounded by L_1 or intersects S at $x \in [x_{i1},x_{i5}]$, while $O_X^+(y)$ is contained in $D^+ - \operatorname{cl} G$. It implies that either $O_X^-(y_{3K})$ is contained in G or crosses S at some point belonging to $[x_{i1},x_{i5}]$ and by (ii) stays in D^-. In both cases $b_3 \notin J_X^+(a)$. Thus $b \in \Omega(X)$.

L e m m a 3. Let $B = \{ O_X^-(b) \in W_X^- : \exists\, a \in (p_1,p_2)$ such that $O_X^+(a) \in W_{X,P}^+$ and $b \in J_X^+(a)\}$. If $O_X^-(b_n) \in B$ are different trajectories and $b_n \to b \in M$ then be $b \in \Omega(X)$ and there are at most two different trajectories $O_X^+(a_1)$, $O_X^+(a_2)$ such that $O_X^-(b_n) \in B_{a1}$ or $O_X^-(b_n) \in B_{a2}$ for $n \in N$.

P r o o f . If $b \notin \Omega(X)$ there is a transverse section S' of X at b. By Lemma 2 and Remark 1 there are $a_n \in (y_1,y_2)$, $b_n \in S'$ such that $O_X^+(a_n) \in W_{X,P}^+$, $O_X^-(b_n) \in W_X^-$, $b_n \in J_X^+(a_n)$ and $a_n \neq a_m$, $b_n \neq b_m$ for $n,m \in N$. We will show that a sequence (a_n) contains at least three points a_1, a_2, a_3 such that $O_X^+(a_i) \cap \cap S' \neq \emptyset$, $i = 1,2,3$. We choose nine points belonging to (b_n). Let $\varepsilon = \frac{1}{2} \min_{1 \leqslant i \neq j \leqslant 9} \{\varrho(b_i,b_j)\}$. Since $b_i \in J_X^+(a_i)$ then there are points $y_{iK} \in S'$, $x_{iK} \in (p_1,p_2)$, numbers $t_{iK} \to -\infty$ for $k \to \infty$ such that $x_{iK} = \phi_X(y_{iK},t_{iK})$, $\varrho(y_{iK},b_i) < \varepsilon$, $b_1 < x_{i2} < b_3 < \ldots < \ldots < x_{i8} < b_9$ and y_{iK} is the first common point of $O_X^+(x_{iK})$ and S'. Like in Lemma 2 $L_1 = O_X[x_{i2},y_{i2}] \cup O_X[x_{i8},y_{i8}] \cup \cup [x_{i2},x_{i8}] \cup [y_{i2},y_{i8}]$ separates D^+. See Fig.3. Let G be a domain bounded by L_1. Then $O_X[x_{iK},y_{iK}]$, $i=3,5,7$ separates G, so if $\omega^*(O_X(a_i)) = P$ then $O_X^+(a_i)$ meets S' at some a_i'. Let G_2 be a forward invariant domain bounded by $L_2 = O_X^+(a_3') \cup O_X^+(a_7') \cup [a_3',a_7']$. Thus $O_X(b_5)$ is contained either in a domain G_3 bounded by $O_X^+(a_3') \cup O_X^+(a_5') \cup [a_3',a_5']$ or in a domain G_4 bounded

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

by $O_X^+(a'_5) \cup O_X^+(a'_7) \cup [a'_5, a'_7]$ what contradicts assumption that $b_5 \in S'$

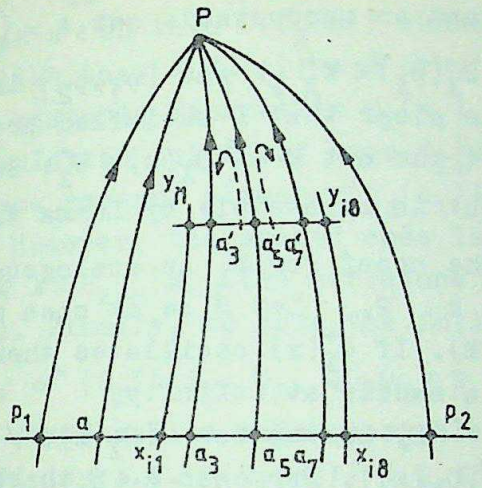


Fig. 3

L e m m a 4. The set B is countable.

P r o o f . Suppose contrary, then there is an uncountable family of different trajectories $O_X^-(b_i)$, $i \in I$, which has an accumulation point $b \in M$. By Lemma 3 there are at most two different trajectories $O_X^+(a_1)$, $O_X^+(a_2) \in W_{X,P}^+$ such that either $O_X^-(b_i) \in B_{a_1}$ or $O_X^-(b_i) \in B_{a_2}$. It implies that B_{a_1} or B_{a_2} contains uncountably many different trajectories. On the other hand B_{a_i} is countable by Corollary 1. This proves the lemma.

T h e o r e m 1. Let M be an open surface with finite genus and countable space of ends E. If $X \in H_o^r(M)$ then $W_X^{\pm}$ has at most countably many different trajectories.

P r o o f . By assumption M is homeomorphic to N-K, where N is a closed surface, K is a countable subset of N. Richards [4] proved that homeomorphism maybe replaced by diffeomorphism of class C^∞. So, for any $P \in E$ there exists a sequence of neighborhoods $\{B_n\}$ of p in M diffeomorphic to a ball in $\mathbb{R}^2$ without countably many points such that its boundary FrB_n is a compact one-dimensional smooth submanifold of M and $clB_{n+1} \subset B_n$. As $X \in H_o^r(M)$, we may assume that X is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

transverse to $\mathrm{Fr}B_n$ except for a few points. If $W^+_{X,P}$ is uncountable for some $P \in E$ then there are $L = O^+_X(p_1) \cup O^+_X(p_2) \cup [p_1,p_2] \subset B_{n_o}$ and an uncountable set $A = \{a \in (p_1,p_2) : O_X(a) \in W^+_{X,P}\}$, where $O_X(p_i) \in W^+_{X,P}$, and $[p_1,p_2]$ is a transverse section of X. It is clear that L satisfied properties (i)-(iii). Thus by Remark 1 the set $B = \{O_X(b) \in W^-_X : \exists a \in A \text{ and } b \in J^+_X(a)\}$ is countable what is impossible by Lemma 4. Hence $W^+_X = \bigcup_{P \in E} W^+_{X,P}$ is countable. The proof for W^-_X is analogous.

T h e o r e m 2. Let M be an open surface with finite genus, $X \in H^r_{K-S}(M)$. If $O^+_X(x)$ oscillates then ω-limit set of $O_X(x)$ contains a saddle at infinity.

P r o o f . By assumption $\omega(O_X(x)) \neq \emptyset$ and there is $R \in$ such that $R \in \omega^*(O_X(x))$. For each $R \in E$ there is a sequence of neighborhoods $\{B_n\}$ of R in M such that $B_{n+1} \subset B_n$, B_n is diffeomorphic to a ball in R^2 without a set of dimension zero. We may choose B_n such that ∂B_n is compact one-dimensional smooth submanifold of M and X is transverse to ∂B_n except for a few points, since $X \in H^r_{K-S}(M)$. Thus for $n \geq m$ $O^+_X(x)$ crosses ∂B_n infinitely many times. Moreover, there are two sequences of transverse sections $\{S_{1n}\}$, $\{S_{2n}\}$, $S_{1n}, S_{2n} \subset \partial B_n$ and points $x_{nk} = \phi_X(x,t_{nk})$, $y_{1n} = \phi_X(x,s_{nk})$, $n,k \in N$, such that $x_{nk} \in S_{1n}$, $y_{nk} \in S_{2n}$, $O_X[x_{nk},y_{nk}] \subset B_n$ and $|t_{nk} - s_{nk}| \longrightarrow +\infty$ for $k \longrightarrow +\infty$. Let P be a Poincare's map i.e. for $z \in D \subset S_{1m}$ $P(z)$ is the first intersection point of $O^+_X(z)$ with S_{2m}. Of course D is non-empty as $x_{mk} \in D$ and D is the union of intervals $(a_k,a_{k+1}) \subset S_{1m}$. We prove that either $a_k \in \partial S_{1m}$ or $O^+_X(a_k) \in W^+_X$. Suppose that $a = a_k \notin \partial S_{1m}$. Then $a \in S_{1m}$, $O^+_X(a) \subset B_{1m}$ and there is a point $b \in J^+_X(a)$ such that $b \in \partial P(D) \subset S_{2m}$. We have to show that if $\omega(O_X(a)) \neq \emptyset$, $\alpha(O_X(b)) \neq \emptyset$ then $\omega(O_X(a))$ and $\alpha(O_X(b))$ are fixed saddles. Let $z_o \in \omega(O_X(a))$ and $z_o \notin \mathrm{Per}(X)$, S' be a transverse section of X at z_o, a', a'' be two consecutive intersection points of $O^+_X(a)$ with S' and $L_1 = O_X[a',a''] \cup [a',a']$ where $[a\ ,a\ ] \subset S'$. Since L_1 separates B_m into two components

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

then for $z \in [a', a'']$ $O_X^+(z)$ remainds bounded and consequently $O_X^+(x)$ does not corss S_{2m}. Thus either $\omega(O_X(a)) = \emptyset$ or $\omega(O_X(a)) \in Per(X)$. As $a \in \partial D$ then $\omega(O_X(a))$ cannot contains sinks or sources. It implies that $\omega(O_X(a)) = \emptyset$ or $\omega(O_X(a))$ is a fixed saddle which unstable separatrix $O_X(a_1)$ is contained in the set of accumulation points of arcs $O_X[x_{mk}, y_{mk}]$ for $k \to +\infty$. By the same reasons either $\omega(O_X(a_1)) = \emptyset$ or $\omega(O_X(a_1))$ is a fixed saddle. However, the second case is impossible by the assumption that $X \in H^r_{K-S}(M)$. The structure of $\alpha(O_X(b))$ is similar to $\omega(O_X(a))$. Finally, we have the following cases: (i) $\omega^*(O_X(a)) = Q_1$, $\alpha^*(O_X(b)) = Q_2$, $Q_1, Q_2 \in E$ and $b \in J_X^+(a)$; (ii) $\omega(O_X(a))$ is a fixed saddle which unstable separatrix $O_X(a_1)$ tends to infinity i.e. $\omega(O_X(a_1)) = Q_1 \in E$, $\alpha^*(O_X(b)) = Q_2 \in E$ and $b \in J_X^+(a_1)$; (iii) $\omega(O_X(a))$ and $\alpha(O_X(b))$ are fixed saddles which respectively unstable separatrix $O_X(a_1)$ and stable separatrix $O_X(b_1)$ tend to infinity and $b_1 \in J_X^+(a_1)$. The last case (iv) is symmetric to (ii). In all cases $O_X^+(a) \in W_X^+$, $O_X^-(b) \in W_X^-$. Now $P_n : D_n \subset S_{1m} \longrightarrow S_{1n}$ be a sequence of Poincare's maps i.e. $P_n(z)$ is the first intersection point of $O_X^+(z)$ with S_{1n} such that $P(z) \in O_X^+(P_n(z))$ for $n > m$. Then $D_n \subset D$ and $D_n \neq \emptyset$ since $O_X^+(x) \cap D_n \neq \emptyset$. D_n is a union of intervals $(a_k, a_{k+1}) \subset S_{1m}$ such that $O_X^+(a_k) \subset B_m$, $O_X^+(a_k) \in W_X^+$, $D_{n+1} \subset D_n$, diam $D_n \longrightarrow 0$ and $D_o = \bigcap_{n>m}^{\infty} D_n \neq \emptyset$. So there is $a \in \partial D_o$ and a sequence $a_n \in \partial D_n$ such that $a = \lim a_n$ and $O_X^+(a) \subset B_m$ since otherwise $O_X^+(a_n)$ crosses ∂B_m for large n. Suppose that $\omega(O_X(a)) \neq \emptyset$. Let $z_o \in \omega(O_X(a))$, $X(z_o) \neq \emptyset$, S' be a transversal section at z_o, a', a'' be two consecutive intersection points of $O_X^+(a)$ with S' and $L_1 = O_X[a', a''] \cup [a', a'']$. Since L_1 separates B_m into two components then for $z \in [a', a'']$ $O_X^+(z)$ remainds bounded. As $a \in \omega(O_X(x))$ then $O_X^+(x)$ intersects $[a', a'']$ and $O_X^+(x)$ cannot intersect S_{2m}. Then either $\omega(O_X(a)) = \emptyset$ or

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\omega(O_X(a))$ is a fixed saddle which unstable separatrix $O_X(a_1)$ is contained in the set of accumulation points of arcs $O_X[x_{mk}, y_{mk}]$ for $k \to +\infty$, $O_X(a_1) \subset B_m$ and $\omega(O_X(a_1)) = Q_1 \in E$. Analogously one can prove that there is a point $b \in \omega(O_X(x)) \cap S_{2m}$ such that $O_X^-(b) \subset B_m$, $\alpha(O_X(b)) = \emptyset$ or $\alpha(O_X(b))$ is a fixed saddle which stable separatrix $O_X(b_1)$ is contained in B_m, $\alpha(O_X(b_1)) = Q_2 \in E$ and $b_1 \in J_X^+(a)$. It implies that $O_X^+(a)$ (or $(O_X^+(a_1))$ and $O_X^-(b)$ (or $O_X^-(b_1)$) define a saddle at infinity. It happens that $Q_1 \neq R \neq Q_2$ as in the following example where R is an accumulation point of other points $Q_n \in \omega^*(O_X(x))$.

E x a m p l e . Let $\vec{w} = [w_1, \alpha w_2]$ be a constant vector field on $\mathbb{R}^2$, α be an irrational number, $K = \bigcup_{n=1}^{\infty} (\{(-\frac{1}{n}, \frac{\alpha}{n})\} \cup \{(0,0)\} \cup \{(\frac{1}{n}, \frac{\alpha}{n})\})$. There is a C^∞ smooth function $f: \mathbb{R}^2 \to \mathbb{R}_+$ such that $f(x,y) = 0$ iff $(x,y) \in K$. Then $X = f \cdot \vec{w}$ defines a Kupka-Smale vector field on an open surface $M = \mathbb{R}^2 - K/_{Z \times Z}$, where M is diffeomorphic with $T^2 - K$, while a space of ends E is diffeomorphic with K. For any point $(x,0)$, $x \in (-\frac{1}{2}, \frac{1}{2})$ $O_X^+(x)$ and $O_X^-(x)$ oscillate, $\omega(O_X(x)) = \alpha(O_X(x)) = T^2 - K$ and $R = (0,0)$ is the accumulation point of $Q_n^+ = \{(\pm\frac{1}{n}, \frac{\alpha}{n})\} \in \omega^*(O_X(x)) = \alpha^*(O_X(x))$.

REFERENCES

[1] C. C a m a c h o , M. K r y c h , R. M a n é , Z. N i t e c k i : An extension of Peixito's structural stability theorem to open surfaces with finite genus, Lecture Notes in Math. 1007 (1983), 60-87.

[2] J. K o t u s , M. K r y c h , Z. N i t e c k i : Global structural stability of flows on open surfaces, Mem. Amer. Math. Soc. 37 (261) (1982).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] M.M. P e i x i t o : On an approximation theory of Kupka and Smale, J. Diff. Equat. 3 (1967), 214-227.

[4] I. R i c h a r d s : On the classification of noncompact surfaces, Trans. Amer. Math. Soc. 106 (1963), 259-269.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received September 27, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

OF A NO

1. In

In [

value pr

resonance

optimal

where co

different

Since for

no or mor

here belo

sense of

tic probl

nally we

After

tion 2 we

necessary

of the ex

compactne

quences o

We remark

can be de

elliptic)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Manfred Goebel, Marian Majchrowski

DISTRIBUTED AND BOUNDARY CONTROL OF A NONLINEAR ELLIPTIC MIXED BOUNDARY VALUE PROBLEM

1. Introduction

In [2] for a certain nonlinear elliptic mixed boundary value problem were proved existence theorems both in the resonance and nonresonance case. This paper is devoted to an optimal control problem for the same boundary value problem, where control parameters occur on the right hand sides of the differential equation and the boundary condition as well. Since for fixed control parameters the state equation may have no or more than one solution, the control problem considered here belongs to the class of singular control problems in the sense of J.L. Lions [8]. References related to nonlinear elliptic problems can be found in the monographs [8], [9]; additionally we mention the papers [1], [5], [11].

After giving the control problem to be considered in section 2 we prove an existence theorem in section 3 and derive necessary optimality conditions in section 4. While the proof of the existence theorem is based on paper [2] and well-known compactness arguments, the optimality conditions are consequences of [4] (cf. [3] for short summary and some extensions). We remark that, using [3], [4], necessary optimality conditions can be derived for much more complicated elliptic (and non-elliptic) control problems.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Problem statement

There is given a bounded domain $\Omega \in R^n$, $n \geq 2$, with boundary $\partial\Omega = S_0 \cup S_1 \cup S_2$, where S_1 is a Lipschitz surface, S_2 is of class C^1, both S_1 and S_2 are disjoint and open in $\partial\Omega$ with strictly positive measure and $\mathrm{mes} S_0 = 0$.

We shall deal with the following optimal control problem (P): Find $\inf J(y)$ with

$$(1) \qquad J(y) = \int_\Omega \varphi(x)(y(x)-p(x))^2 dx + \int_{S_2} \psi(x)(y(x)-q(x))^2 dS_2,$$

subject to

$$(2) \quad \begin{cases} (Ly)(x) = f(x,y(x)) + v(x), & x \in \Omega, \\ y(x) = 0, \ x \in S_1, \ (ly)(x) = g(x,y(x)) + w(x), & x \in S_2, \end{cases}$$

$$(3) \quad \begin{cases} u = \{v,w\} \in U_{ad} = V_{ad} \times W_{ad} \subset L^\infty(\Omega) \times L^\infty(S_2), \\ V_{ad} = \{ v \in L^\infty(\Omega) : p_0(x) \leq v(x) \leq p_1(x), \ \text{a.e. } x \in \Omega \}, \\ W_{ad} = \{ w \in L^\infty(S_2) : q_0(x) \leq w(x) \leq q_1(x), \ \text{a.e. } x \in S_2 \}. \end{cases}$$

Here the functions $\varphi \in L^\infty(\Omega)$, $\psi \in L^\infty(S_2)$, with $\varphi(x) \geq 0$, $\psi(x) \geq 0$, $\|\varphi\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(S_2)} > 0$, and $p \in L^2(\Omega)$, $p_0, p_1 \in L^\infty(\Omega)$, $q \in L^2(S_2)$, $q_0, q_1 \in L^\infty(S_2)$ are given. L denotes a linear uniformly elliptic differential operator given by

$$Ly = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial y}{\partial x_i} \right) + a_0 y,$$

where $a_{ij}, a_0 \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$,

$$(4) \quad \begin{cases} \alpha_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \, \xi_i \xi_j \leq \alpha_1 |\xi|^2, \\ \text{a.e. } x \in \Omega, \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in R^n, \text{ with } \alpha_0, \alpha_1 \in R, \ \alpha_0 > 0, \end{cases}$$

and 1 is the respective linear boundary operator such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\ell y = \sum_{i,j=1}^{n} a_{ij} \vartheta_j \frac{\partial y}{\partial x_i} + \delta y,$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ is the unite outer normal to S_2 and

(5) $\quad \delta \in L^\infty(S_2), \ \delta(x) \geqslant \delta_0 > 0, \quad$ a.e. $x \in S_2, \quad \delta_0 \in R.$

The right hand sides of the differential equation and the boundary condition in (2) are given Carathéodory functions satisfying the growth conditions

(6) $$\begin{cases} |f(x,y)| \leqslant \mu_f(x) + k_f|y|, & \text{a.e. } x \in \Omega, \ \forall y \in R, \\ |g(x,y)| \leqslant \mu_g(x) + k_g|y|, & \text{a.e. } x \in S_2, \ \forall y \in R, \end{cases}$$

where $\mu_f \in L^2(\Omega)$, $\mu_g \in L^2(S_2)$ and $k_f, k_g \in \langle 0, +\infty)$. Further assumptions will be specified below.

Of course, under the above assumptions, for fixed $u = \{v,w\} \in L^\infty(\Omega) \times L^\infty(S_2)$ a solution of state equation (2) has to be understood in the sense of Sobolev. So we introduce the Sobolev space

$$V^1(\Omega) = \{ y \in H^1(\Omega) : y(x) = 0 \text{ on } S_1 \},$$

$$\|y\|^2_{V^1(\Omega)} = \|y\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial y}{\partial x_i} \right\|^2_{L^2(\Omega)},$$

$$(y,z)_{V^1(\Omega)} = (y,z)_{L^2(\Omega)} + \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}, \frac{\partial z}{\partial x_i} \right)_{L^2(\Omega)}.$$

Then $y \in V^1(\Omega)$ is said to be a solution to (2), if it solves the nonlinear operator equation

(7) $$Ay + By = f^* u$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

with the linear bounded Fredholm operator $A \in \mathcal{L}(V^1(\Omega), V^{-1}(\Omega))$ defined by

$$(8) \qquad \langle Ay, z \rangle = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} \, dx + \int_{\Omega} a_0 yz dx + \int_{S_2} \sigma yz dS_2, \qquad y, z \in V^1(\Omega),$$

the nonlinear continuous bounded operator $B: V^1(\Omega) \longrightarrow V^{-1}(\Omega)$ defined by

$$(9) \qquad \langle By, z \rangle = - \int_{\Omega} f(x,y) z dx - \int_{S_2} g(x,y) z dS_2, \qquad y, z \in V^1(\Omega),$$

and $f^* u \in V^{-1}(\Omega)$ defined by

$$(10) \qquad \begin{cases} \langle f^* u, z \rangle = \int_{\Omega} vz dx + \int_{S_2} wz dS_2, \\ u = \{v, w\} \in L^{\infty}(\Omega) \times L^{\infty}(S_2), \quad z \in V^1(\Omega). \end{cases}$$

Here $V^{-1}(\Omega)$ denotes the dual space of $V^1(\Omega)$ and $\langle \cdot, \cdot \rangle$ the generalized inner product between both spaces. Now introducing the set of admissible pairs by setting

$$P_{ad} = \left\{ \{u, y\} \in (L^{\infty}(\Omega) \times L^{\infty}(S_2)) \times V^1(\Omega) : u \in U_{ad}, \; Ay + By = f^* u \right\}$$

the control problem under consideration is given precisely by

$$(P) \qquad \inf \left\{ J(y) : \{u, y\} \in P_{ad} \right\}.$$

3. Existence of optimal controls

In this section we shall prove a theorem stating the existence of at least one optimal solution to the control problem (P) provided that in the cost functional J the weight $\varphi = \varphi(x)$ is greater than a positive constant φ_0 for almost

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

every $x \in \Omega$. Other cases, for example, if either φ vanishes on a measurable subset of Ω or φ is identical zero, would be of interest, too. To prepare the wanted existence theorem we need an estimate for the solution set of the state equation (2).

L e m m a 1. There are positive constants k_1, k_2 such that

$$(\alpha_0-\varepsilon)\sum_{i=1}^{n}\left\|\frac{\partial y}{\partial x_i}\right\|^2_{L^2(\Omega)} \leqslant \varepsilon^{-1}k_2^2 + k_1\|y\|_{L^2(\Omega)} +$$

$$+ (\|a_0\|_{L^\infty(\Omega)} + k_f + \varepsilon)\|y\|^2_{L^2(\Omega)} + (k_g - \sigma_0)\|y\|^2_{L^2(S_2)}$$

holds true for any $\{u,y\} \in P_{ad}$ and any $\varepsilon > 0$.

P r o o f . Let $\{u,y\} \in P_{ad}$ be arbitrary. Then

$$\langle Ay,y\rangle = -\langle By,y\rangle + \langle f^*u,y\rangle.$$

Using assumptions (4)-(6) we find

$$\langle Ay,y\rangle \geqslant \alpha_0 \int_\Omega \sum_{i=1}^{n}\left(\frac{\partial y}{\partial x_i}\right)^2 dy + \int_\Omega a_0 y^2 dy + \sigma_0 \int_{S_2} y^2 dS_2 \geqslant$$

$$\geqslant \alpha_0 \sum_{i=1}^{n}\left\|\frac{\partial y}{\partial x_i}\right\|^2_{L^2(\Omega)} - \|a_0\|_{L^\infty(\Omega)}\|y\|^2_{L^2(\Omega)} + \sigma_0\|y\|^2_{L^2(S_2)}$$

and on the other hand

$$-\langle By,y\rangle + \langle f^*u,y\rangle \leqslant \int_\Omega (|\mu_f| + |v|)|y|dx + k_f\int_\Omega y^2dx +$$

$$+ \int_{S_2}(|\mu_g| + |w|)|y|dS_2 + k_g\int_{S_2} y^2dS_2 \leqslant$$

$$\leqslant \left(\|\mu_f\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)\|y\|_{L^2(\Omega)} + k_f\|y\|^2_{L^2(\Omega)} +$$

$$+ \left(\|\mu_g\|_{L^2(S_2)} + \|w\|_{L^2(S_2)}\right)\|y\|_{L^2(S_2)} + k_g\|y\|_{L^2(S_2)}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since $V^1(\Omega)$ is continuously imbedded into $L^2(S_2)$, there is a positive constant k such that

$$\|y\|_{L^2(S_2)} \leqslant k\|y\|_{V^1(\Omega)}, \quad \forall y \in V^1(\Omega).$$

By definition, both V_{ad} and W_{ad} are bounded in $L^2(\Omega)$ and $L^2(S_2)$, respectively. Therefore, there are two positive constants k_1, k_2 such that

$$\|\mu_f\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leqslant k_1, \quad \forall\, v \in V_{ad},$$

$$k\left(\|\mu_g\|_{L^2(S_2)} + \|w\|_{L^2(S_2)}\right) \leqslant 2k_2, \quad \forall\, w \in W_{ad}.$$

Combining all these inequalities we come to the estimate

$$\alpha_0 \sum_{i=1}^{n} \left\| \frac{\partial y}{\partial x_i} \right\|^2_{L^2(\Omega)} \leqslant k_1\|y\|_{L^2(\Omega)} + (\|a_0\|_{L^\infty(\Omega)} +$$

$$+ k_f)\|y\|^2_{L^2(\Omega)} + 2k_2\|y\|_{V^1(\Omega)} + (k_g - \sigma_0)\|y\|^2_{L^2(S_2)}$$

and because of

$$2k_2\|y\|_{V^1(\Omega)} \leqslant \varepsilon^{-1}k_2^2 + \varepsilon\|y\|^2_{V^1(\Omega)}, \quad \forall\, y \in V^1(\Omega), \quad \forall\, \varepsilon > 0,$$

the Lemma is proved.

T h e o r e m 1. Suppose that

(i) $P_{ad} \neq \emptyset$,

(ii) $k_g \leqslant \sigma_0$,

(iii) $\varphi(x) = \varphi_0 > 0$, a.e. $x \in \Omega$, $\varphi_0 \in R$.

Then there exists at least one solution

$$\{u_0, y_0\} = \{v_0, w_0, y_0\} \in L^\infty(\Omega) \times L^\infty(S_2) \times V^1(\Omega)$$

of the optimal control problem (P).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. In virtue of (i), there is a minimizing sequence $\{\{u_\nu, y_\nu\}\} \subset P_{ad}$, $u_\nu = \{v_\nu, w_\nu\}$, which particularly means

$$\lim_{\nu \to \infty} J(y_\nu) = \inf\{J(y) : \{u,y\} \in P_{ad}\} < \infty.$$

Taking advantage of Lemma 1 with $0 < \varepsilon < \alpha_0$, assumption (ii) and the inequality (cf. (iii))

$$J(y) \geq \varphi_0 \|y-p\|^2_{L^2(\Omega)} \geq (\varphi_0/2)(\|y\|^2_{L^2(\Omega)} - 2\|p\|^2_{L^2(\Omega)}), \quad \forall y \in V^1(\Omega),$$

we see that $\{y_\nu\}$ is bounded in $V^1(\Omega)$. Hence, since the imbeddings of $V^1(\Omega)$ into $L^2(\Omega)$ and $L^2(S_2)$ are compact (cf. [7]) and since both V_{ad} and W_{ad} are sequentially weakly* compact in $L^\infty(\Omega)$ and $L^\infty(S_2)$, respectively, we can assume without loss of generality

$$(11)\quad \begin{cases} y_\nu \longrightarrow y_0 \text{ weakly in } V^1(\Omega) \text{ and strongly in } L^2(\Omega), \\ y_\nu \longrightarrow y_0 \text{ strongly in } L^2(S_2), \\ v_\nu \longrightarrow v_0 \text{ weakly* in } L^\infty(\Omega), \\ w_\nu \longrightarrow w_0 \text{ weakly* in } L^\infty(S_2), \end{cases}$$

$$(12)\quad u_0 = \{v_0, w_0\} \in U_{ad} = V_{ad} \times W_{ad}.$$

Consider now $Ay_\nu + By_\nu = f^* u_\nu$, $\nu = 1,2,\ldots$, that means

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial y_\nu}{\partial x_i} \frac{\partial z}{\partial x_j}\, dx + \int_\Omega a_0 y_\nu z\, dx +$$

$$+ \int_{S_2} \sigma y_\nu z\, dS_2 = \int_\Omega f(x, y_\nu) z\, dx + \int_{S_2} g(x, y_\nu) z\, dS_2 +$$

$$+ \int_\Omega v_\nu z\, dx + \int_{S_2} w_\nu z\, dS_2, \quad \forall z \in V^1(\Omega), \ \nu = 1,2,\ldots$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The properties (11) allows to conclude $Ay_0 + By_0 = f^* u_0$ which, together with (12), proves that $\{u_0, y_0\} \in P_{ad}$. Because (11) implies also $\lim_{\nu \to \infty} J(y_\nu) = J(y_0)$ so the proof of Theorem 1 is complete.

On the basis of [2] we give now two collections of conditions being sufficient for assumption (i) of Theorem 1. Let dim N(A) denote the dimension of the kernel of the Fredholm operator $A \in \mathcal{L}(V^1(\Omega), V^{-1}(\Omega))$ defined in (8) and let us introduce the sets

$$\Omega^+(z) = \{x \in \Omega : z(x) > 0, \text{ a.e. } x \in \Omega\},$$
$$\Omega^-(z) = \{x \in \Omega : z(x) < 0, \text{ a.e. } x \in \Omega\},$$
$$S_2^+(z) = \{x \in S_2 : z(x) > 0, \text{ a.e. } x \in S_2\},$$
$$S_2^-(z) = \{x \in S_2 : z(x) < 0, \text{ a.e. } x \in S_2\}$$

for any nontrivial element $z \in V^1(\Omega)$.

L e m m a 2. If dim N(A) = 0 (nonresonance case) and if the constants k_f, k_g of assumption (6) are sufficiently small, then $P_{ad} \neq \emptyset$.

L e m m a 3. If dim N(A) > 0 (resonance case), then $P_{ad} \neq \emptyset$ provided that the following assumptions (i)-(iv) are satisfied

(i) $a_0(x) \leqslant 0$, a.e. $x \in \Omega$;

(ii) it holds (6) with $k_f = k_g = 0$;

(iii) there exist the limits

$$\lim_{y \to -\infty} f(x,y) = f_0(x), \quad \lim_{y \to \infty} f(x,y) = f_1(x),$$

$$\lim_{y \to -\infty} g(x,y) = g_0(x), \quad \lim_{y \to \infty} g(x,y) = g_1(x),$$

where $f_0, f_1 \in L^2(\Omega)$ and $g_0, g_1 \in L^2(S_2)$;

(iv) there exists an admissible control $u = \{v,w\} \in U_{ad}$ such that the inequality

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\int_{\Omega^+(z)} (f_1 + v)|z|dx - \int_{\Omega^-(z)} (f_0 + v)|z|dx +$$

$$+ \int_{S_2^+(z)} (g_1 + w)|z|dS_2 - \int_{S_2^-(z)} (g_0 + w)|z|dS_2 > 0$$

holds true for any $z \in N(A)$ with $\|z\|_{V^1(\Omega)} = 1$.

Note that $N(A)$ is a set of solutions of the linear homogeneous elliptic mixed boundary value problem (cf. (2), (8))

$$(Ly)(x) = 0, \quad x \in \Omega,$$

$$y(x) = 0, \ x \in S_1, \ (ly)(x) = 0, \ x \in S_2.$$

Lemmas 2 and 3 are an immediate consequence of Theorem 3.1 and Theorem 4.2 in [2], respectively.

4. Optimality conditions

The optimal control problem (P), formulated at the end of section 2, can be viewed as a specified control problem in the state space $X = V^1(\Omega)$ with the control space $U = L^\infty(\Omega) \times L^\infty(S_2)$. Therefore, for deriving necessary optimality conditions we may use [3] (section 2) which presents a summary of the respective general theory developed in [4].

Throughout the whole section let $\{u_0, y_0\} = \{v_0, w_0, y_0\} \in L^\infty(\Omega) \times L^\infty(S_2) \times V^1(\Omega)$ be an optimal solution to (P). We have to verify the hypotheses H1-H5 of paper [3].

Obviously, the cost functional $J: V^1(\Omega) \longrightarrow R$ is Fréchet-differentiable at any point $y \in V^1(\Omega)$. Denoting the Fréchet derivative at y_0 by J'_0, $J'_0 \in V^{-1}(\Omega)$, we have

$$\langle J'_0, y \rangle = 2 \int_\Omega \varphi(y_0 - p)y dx + 2 \int_{S_2} \psi(y_0 - q)y dS_2, \quad y \in V^1(\Omega).$$

Since we need Fréchet differentiability of the operator $B: V^1(\Omega) \longrightarrow V^{-1}(\Omega)$ we have to replace assumption (6). To this end we recall that the compact imbeddings (cf. [7])

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$H^1(\Omega) \subset L^r(\Omega), \quad H^1(\Omega) \subset L^s(S_2),$$

in case of $r,s \in R$, are such that

$$r \in (2,\infty), \ s \in (2,\infty), \text{if } n = 2,$$

$$r \in (2,2n/(n-2)), \ s \in (2,2(n-1)/(n-2)), \text{if } n > 2,$$

and assume the following conditions:

(13) $f = f(x,y)$ and $g = g(x,y)$ have partial derivatives with respect to y being Carathéodory functions on $\Omega \times R$ and $S_2 \times R$, respectively;

(14) there are two functions $\mu_f \in L^{r/(r-1)}(\Omega)$, $\chi_f \in L^{r/(r-2)}(\Omega)$ and two constants $k_f, c_f \geq 0$ such that

$$|f(x,y)| \leq \mu_f(x) + k_f|y|^{r-1}, \text{ a.e. } x \in \Omega, \ \forall y \in R,$$

$$|f_y(x,y)| \leq \chi_f(x) + c_f|y|^{r-2}, \text{ a.e. } x \in \Omega, \ \forall y \in R;$$

(15) there are two functions $\mu_g \in L^{s/(s-1)}(S_2)$, $\chi_g \in L^{s/(s-2)}(S_2)$ and two constants $k_g, c_g \geq 0$ such that

$$|g(x,y)| \leq \mu_g(x) + k_g|y|^{s-1}, \text{ a.e. } x \in S_2, \ \forall y \in R,$$

$$|g_y(x,y)| \leq \chi_g(x) + c_g|y|^{s-2}, \text{ a.e. } x \in S_2, \ \forall y \in R.$$

Then it is readily checked that the operator $B: V^1(\Omega) \to V^{-1}(\Omega)$ defined in (9) is continuously Fréchet-differentiable at any point $y \in V^1(\Omega)$ (cf. [10] part X, §1 or [6] part 5, §20). In particular, at y_0 its Fréchet derivative $B_0' \in \mathcal{L}(V^1(\Omega), V^{-1}(\Omega))$ is given by the formula

$$\langle B_0'y, z \rangle = -\int_\Omega f_y(x,y_0)yz\,dx - \int_{S_2} g_y(x,y_0)yz\,dS_2, \quad y,z \in V^1(\Omega).$$

In view of the compact imbeddings mentioned above the operator B_0' is compact and, consequently, $A + B_0' \in \mathcal{L}(V^1(\Omega), V^{-1}(\Omega))$ is a Fredholm operator. Since, furthermore, the map $u \to f^*u$ as a map from $L^\infty(\Omega) \times L^\infty(S_2)$ into $V^{-1}(\Omega)$ is linear and bounded

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(cf. (10)) and hence Fréchet-differentiable, we may apply [3] section 2, [4].

In accordance with this theory, we introduce the adjoint state $z_0 \in V^1(\Omega)$ as a solution to the linear operator equation

$$(A + B'_{\varphi})z = -J'_0. \tag{16}$$

Note that this equation is always solvable, because $\{u_0, y_0\}$ is optimal. Its solutions $z \in V^1(\Omega)$ are solutions in the sense of Sobolev of the linear mixed boundary value problem

$$\begin{cases} (Lz)(x) = f_y(x, y_0(x))z(x) + 2\varphi(x)(p(x) - y_0(x)), & x \in \Omega, \\ z(x) = 0, \quad x \in S_1, \\ (lz)(x) = g_y(x, y_0(x))z(x) + 2\psi(x)(q(x) - y_0(x)), & x \in S_2. \end{cases} \tag{17}$$

Clearly, the homogeneous operator-equation (16) corresponds to the linear homogeneous mixed boundary value problem related to (17), that means to

$$\begin{cases} (Lz)(x) = f_y(x, y_0(x))z(x), & x \in \Omega, \\ z(x) = 0, \ x \in S_1, \ (lz)(x) = g_y(x, y_0(x))z(x), & x \in S_2. \end{cases} \tag{18}$$

We have to distinguish two cases

I: (18) has only the trivial solution,

II: (18) has $\varkappa$ linear independent solutions $z_1, \ldots, z_\varkappa \in V^1(\Omega)$.

Thus, applying [3] section 2, [4] to our control problem (P), we find the following necessary optimality conditions.

T h e o r e m 2. Suppose that $\{u_0, y_0\} = \{v_0, w_0, y_0\} \in L^\infty(\Omega) \times L^\infty(S_2) \times V^1(\Omega)$ is an optimal solution to (P) and it holds case I. Then, under the above assumptions, the variational inequalities

$$\begin{cases} \int_\Omega (v_0(x) - v(x))z_0(x)dx \geqslant 0, \quad \forall v \in V_{ad}, \\ \int_{S_2} (w_0(x) - w(x))z_0(x)dS_2 \geqslant 0, \quad \forall w \in W_{ad} \end{cases} \tag{19}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

are satisfied. Here $z_0 \in V^1(\Omega)$ denotes the unique solution to (17) (adjoint state).

T h e o r e m 3. Suppose that $\{u_0, y_0\} = \{v_0, w_0, y_0\} \in L^\infty(\Omega) \times L^\infty(S_2) \times V^1(\Omega)$ is an optimal solution to (P) and it holds case II. Suppose the above and the following additional assumptions are satisfied:

(i) $p_1(x) - p_0(x) \geqslant \text{const.} > 0$, a.e. $x \in \Omega$,

$q_1(x) - q_0(x) \geqslant \text{const.} > 0$, a.e. $x \in S_2$;

(ii) there is an $u = \{v, w\} \in U_0 \cap \operatorname{int} U_{ad}$, where

$$U_0 = \Big\{ u \in L^\infty(\Omega) \times L^\infty(S_2) : \int_\Omega (v_0 - v) z_j \, dx + \int_{S_2} (w_0 - w) z_j \, dS_2 = 0, \quad j = 1, \ldots, \varkappa \Big\}.$$

Then it holds the orthogonality condition

$$\int \varphi (y_0 - p) z_j \, dx + \int_{S_2} \psi (y_0 - q) z_j \, dS_2 = 0, \quad j = 1, \ldots, \varkappa,$$

and there are $\varkappa+1$ real numbers $\lambda_0, \lambda_1, \ldots, \lambda_\varkappa$, $\lambda_0 \geqslant 0$, $|\lambda_0| + \ldots + |\lambda_\varkappa| > 0$, and a solution $z_0 \in V^1(\Omega)$ to (17) (adjoint state) such that

$$\int_\Omega (v_0(x) - v(x)) z(x) \, dx \geqslant 0, \quad \forall v \in V_{ad},$$

$$\int_{S_2} (w_0(x) - w(x)) z(x) \, dS_2 \geqslant 0, \quad \forall w \in W_{ad},$$

where

$$z = \lambda_0 z_0 + \lambda_1 z_1 + \ldots + \lambda_\varkappa z_\varkappa.$$

In both cases the optimality conditions are variational inequalities which, by a Lemma due to Krasnoselski (cf. [6] part 5, § 17.1), are equivalent to local optimality conditions. For example, in case I condition (19) is equivalent to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(v_0(x) - v)z_0(x) \geq 0, \quad \forall v \in [p_0(x), p_1(x)], \text{ a.e. } x \in \Omega,$$

$$(w_0(x) - w)z_0(x) \geq 0, \quad \forall w \in [q_0(x), q_1(x)], \text{ a.e. } x \in S_2.$$

From this, for the optimal control $u_0 = \{v_0, w_0\}$, we find

$$v_0(x) = \begin{cases} p_0(x), & x \in \Omega^-(z_0) \\ p_1(x), & x \in \Omega^+(z_0) \end{cases}, \qquad w_0(x) = \begin{cases} q_0(x), & x \in S_2^-(z_0) \\ q_1(x), & x \in S_2^+(z_0) \end{cases},$$

where the sets $\Omega^+(z_0)$, $\Omega^-(z_0)$, $S_2^+(z_0)$ and $S_2^-(z_0)$ are defined in section 3.

REFERENCES

[1] J.F. Bonnans, E. Casas: Optimal control of semilinear multistate systems with state constraints, Rapports de Recherche, No. 722, INRIA 1987.

[2] J. Chmaj, M. Majchrowski: Existence of the weak solution of a nonlinear elliptic mixed boundary value problem, Demonstratio Math. 17 (1984), 1091-1109.

[3] M. Goebel: On control problems for a quasilinear second order ordinary differential equation, Math. Nachr., in print.

[4] M. Geobel, L. v. Wolfersdorf: Optimale Steuerprobleme bei Noetherschen Operatorgleichungen III, Math. Nachr. 82 (1978), 77-85.

[5] V. Komornik, D. Tiba: Contrôles de systèmes fortement non linéaire, C.R. Acad. Sci. Paris, Série I, 12 (1985), 393-396.

[6] М.А. Красносельский и др.: Интегральные операторы в пространствах суммируемых функций, Наука, Москва 1966.

[7] A. Kufner, O. John, S. Fučik: Function spaces, Academia, Prague 1977.

[8] J.L. Lions: Contrôle de systèmes distribués singuliers, Dunod, Paris 1983.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[9] У. Раитум: Задачи оптимального управления для эллиптических уравнений, Научн. Зинатне, Рига 1989.

[10] П.П. Забрейко и др.: Интегральные уравнения, Наука, Москва 1968.

[11] D. Žubrinić: On the existence of optimal controls for nonresonant elliptic equations, Glasnik Matematički, Ser. III, 21 (1986) 387-391.

DEPARTMENT OF MATHEMATICS, MINING ACADEMY FREIBERG,
DDR - 9200 FREIBERG, G.D.R.;
INSTITUTE OF MATHEMATICS, WARSAW TECHNICAL UNIVERSITY,
00-661 WARSZAWA

Received October 10, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 1 1990

Bogusław Kaczmarski

ON THE CATEGORY AND BOREL TYPE OF THE SET OF POINTS OF ONE-SIDED NON-DIFFERENTIABILITY

In the present paper we shall consider finite real functions of a real variable. We shall apply the following notations:

E_f^* - the set of points at which the function f possesses no finite or infinite one-sided derivative;

$E_f^+(+\infty) = \{x : f'_+(x) = +\infty\}$, $E_f^+(-\infty) = \{x : f'_+(x) = -\infty\}$

the sets $E_f^-(+\infty)$ and $E_f^-(-\infty)$ are defined analogously;

$$E_f(\pm\infty) \quad E_f^+(+\infty) \cup E_f^+(-\infty) \cup E_f^-(+\infty) \cup E_f^-(-\infty);$$

Λ_f - the set of points at which on one side there is an infinite derivative, and on the other - the Dini derivatives are equal to $+\infty$ and $-\infty$;

Δ_f^* - the set of points at which exists finite f'_+ or f'_- but f' not exists;

$m(A)$ - the Lebesgue measure of a set A;

R - the set of all real numbers.

The problem considered in the paper is the question of the Borel characterization of the set E_f^* where f is a continuous function. It was proved that, for a continuous function, $E_f^* \in G_{\delta\sigma}$ (see [8]). In papers [8] and [2] Zahorski and Filipczak showed that if a set E is the union of a set of type G_δ and a set of type $G_{\delta\sigma}$ of measure zero, then there exists a continuous function f for which $E = E_f$. In paper

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[4] we showe the existence of a continuous function $f : R \to R$ for which the set E_f^* cannot be represented in the form of the union of a set of type G_δ and a set of type $G_{\delta\sigma}$ of measure zero.

In our paper we consider the problem: Exists for any set $E \in G_{\delta\delta}$ the continuous function $f : R \to R$ such that $E = E_f^*$.

T h e o r e m 1. Let $f : R \to R$ be a non-degenerate continuous function. If E_f^* is a first category set, then for each closed non-degenerate interval I $m(E_f^* \cap I) < m(I)$.

P r o o f . Suppose that there exists a closed interval $I_0 = [a,b]$ such that $m(E_f^* \cap I_0) = m(I_0)$.

Let

$$h(x) = \begin{cases} f(x) & \text{for } x \in I_0 \\ f(a) & \text{for } x < a \\ f(b) & \text{for } x > b. \end{cases}$$

Evidently, h is a continuous function. By our assumption $E_f^* \cap I_0$ is a first category set in R and in I_0. Since $E_h^* \subset E_f^* \cap I_0$, therefore E_h^* is a first category set in I_0.

Let $P = (E_h(\pm\infty) \setminus \{x : h'(x) = \pm\infty\}) \cup \Delta_h^*$. We know see the proof of Theorem 4 from paper [3] that

$$P = \Lambda_h \cup M \cup \Delta_h^* \cup \left[E_h^+(+\infty) \cap \{x : \bar{h}(x) = +\infty \text{ and } \underline{h}(x) \neq \pm\infty\}\right] \cup$$

$$\cup \left[E_h^+(-\infty) \cap \{x : \bar{h}(x) \neq \pm\infty \text{ and } \underline{h}(x) = -\infty\}\right] \cup \left[E_h^-(+\infty) \cap\right.$$

$$\left. \cap \{x : \bar{h}^+(x) = +\infty \text{ and } \underline{h}^+(x) \neq \pm\infty\}\right] \cup \left[E_h^-(-\infty) \cap \{x : \bar{h}^+(x) \neq \pm\infty\right.$$

$$\left. \text{and } \underline{h}^+(x) = -\infty\}\right]$$

where the set M is at most countable. Evidently $m(P) = 0$.

Since the function h is continuous, therefore in virtue of paper [8] the set Λ_h and the sets occurring in square brakets are a first category sets. We have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\Delta_h^* \subset \{x : h'_+(x) \neq h'_-(x)$ and $h'_+(x), h'_-(x)$ exist$\} \cup$

$\cup \{x : h'_+(x)$ - exists, finite and $h'_-(x)$ does not exist, either finite or infinite$\} \cup \{x : h'_-(x)$ - exists, finite and $h'_+(x)$ - does not exist, either finite or infinite$\}$.

The first set on the right-hand side of the above equality is at most countable. The other two sets are a first category sets (see [5]). This implies that, Δ_h^* is a first category set.

Let $Q_h = \{x \in I_o : h'(x)$ does not exists, either finite or infinite$\}$.

We have, $Q_h = P \cup E_h^*$. Since P and E_h^* are a first category sets, therefore Q_h is a first category set.

By Theorem 4 from paper [1] we have

$$Q_h = G \cup H,$$

where $G \in G_\delta$, $H \in G_{\delta\delta}$ and $m(H) = 0$. We see that

$$m(G) = m(Q_h) = m(E_h^*) = m(E_h^* \cap I_o) = m(I_o).$$

Hence it appears that $G \in G_\delta$ and G is dense in I_o. Since G is a subset of type G_δ of the metric space I_o, therefore G is a residual set in I_o. Since I_o is a complete space, therefore in virtue of the Baire theorem G is of the second category. The set G, being a subset of Q_h is also of the first category. We have obtained a contradiction which ends the proof.

C o r o l l a r y . There exist the set $E \in G_{\delta\delta}(F_\sigma)$, such that does not exist a continuous function $f : R \longrightarrow R$ for which $E = E_f^*$.

P r o o f . We see that $R = A \cup B$, where $A \in G_\delta$, $m(A) = 0$, $B = R \setminus A$ and B is a first category set (see [6]). Of course $B \in F_\sigma \subset G_{\delta\delta}$. For each interval I, $m(B \cap I) = m(I)$. In virtue Theorem 1 does not exist continuous function $f : R \longrightarrow R$ such that $E_f^* = B$. Let $B = E$.

C o r o l l a r y 2. Let $f : R \longrightarrow R$ be continuous function. If $\{x : f'(x) \neq \pm\infty\}$ is a residual set then for each closed non-degenerate interval I, $m(E_f \cap I) < m(I)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

P r o o f. This follows from the fact that $E_f^* \subset R \setminus \{x : f'(x) \neq \pm\infty\}$ and Theorem 1.

C o r o l l a r y 3. Let $f : R \to R$ be continuous function. If $\bar{f}^r \in B_1$ then for each closed non-degenerate interval I, $m(E_f^* \cap I) < m(I)$. ($f \in B_1$ if and only if f is in the first Baire class).

P r o o f. If $\bar{f}^+ \in B_1$ then $\bar{f}^+$ is continuous on a residual set. From the Dini theorem (see [7]) it follows that f is differentiable on a residual set. From Corollary 2 it follows that $m(E_f^* \cap I) < m(I)$.

REFERENCES

[1] А. Б р у д н о : Непрерывность и дифференцируемость, Mat. Sb. (N.S.) 55, 13 (1943) 119-134.

[2] F.M. F i l i p c z a k : Sur la structure de l'ensemble des points ou une fonction continue n'admet pas de dérivée symetrique, Dissertationes Math. 130 (1976) 1-54.

[3] B. K a c z m a r s k i : On the set of points at which a function has no one-sided derivative, Demonstratio Math. 18 (1985), 1127-1141.

[4] B. K a c z m a r s k i : One the measure and Borel type of the set of points of one-sided non-differentiability, Demonstratio Math. 22 (1989) (in print).

[5] C. N e u g e b a u e r : A theorem on derivatives, Acta Sci. Math. (Szeged) 23 (1962).

[6] J. O x t o b y : Measure and category, Springer-Verlag, New York 1971.

[7] S. S a k s : Zarys teorii całki. Warszawa 1930.

[8] Z. Z a h o r s k i : O zbiorze punktów nieróżniczkowalności funkcji dowolnej, Dodatek do Rocznika PTM, t. 21, 23-26.

INSTITUTE OF TEACHER EDUCATION, 00-561 WARSZAWA

Received January 20, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Marek Rochowski

A SCHEME OF GENERATING CANONICAL FRAMES IN NORMAL BUNDLES OF IMMERSED MANIFOLDS IN EUCLIDEAN SPACES

1. Introduction

In this paper we define canonical frames in the normal bundle NM^n of a differentiable manifold M^n immersed in the Euclidean space E^{n+N}. The geometric idea of this construction was explained in [3]. More generally we repeat this concept as follows. Around a curve $x(I) \subset E^n$, $n \geq 3$, where E^n denotes the n-dimensional Euclidean space and

$$x : I \longrightarrow E^n, \quad I = [0,1] \tag{1.1}$$

is an immersion, we take a tubular neighborhood consisting of a one-parameter family of (n-1)-dimensional disks of radius $\varepsilon > 0$. There exists such an $\varepsilon_o > 0$ that for every ε, $0 < \varepsilon \leq \varepsilon_o$, the boundary $T^{n-1}(\varepsilon)$ of the tubular neighborhood is a regular surface. The Gauss curvature

$$K : T^{n-1}(\varepsilon) \longrightarrow R, \tag{1.2}$$

where R denotes the real numbers, of $T^{n-1}(\varepsilon)$ has the following property. Let us suppose that the first curvature $k_1(s)$ of (1.1) is greater than zero for every $s \in [0,1]$. We consider the restriction of (1.2)

$$K / S^{n-2}(s, \varepsilon), \tag{1.3}$$

where

$$S^{n-2}(s, \varepsilon) \subset T^{n-1}(\varepsilon) \tag{1.4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

denotes the spherical "fiber" with radius ε and center $x(s)$. Then there exist exactly two antipodal points $y_1(s)$, $y_2(s)$ of the sphere (1.4) such that the function (1.3) attains its maximal value at $y_1(s)$ and its minimal value at $y_2(s)$ and the direction in E^n defined by the pair $(y_1(s), y_2(s))$ is that defined by the principal normal $\tilde{e}_2(s)$ of (1.1). In the following such a direction is called canonical.

We replace $x(I)$ by a surface $x(M^n) \subset E^{n+N}$, the tubular neighborhood by the normal bundle $NM^n = NM^n(x)$ of $x(M^n)$ and the Gauss curvature K by an elementary, symmetric function (called in the following also a scalar function) of the characteristic polynomial of the second quadratic form of $x(M^n)$. The stationary values of such a function restricted to the unit vectors of a fiber $N_p M^n(x)$, $p \in M^n$, of $NM^n(x)$ define directions in E^{n+N} which are called canonical.

Let

$$e_2 : I \rightarrowtail S^2, \quad I = [0,1], \quad e_2(0) = e_1(1), \tag{1.5}$$

denotes an immersion such that the direction of the principal normal $\tilde{e}_2$ of e_2 is identical with that of e_2. Then the second curvature of e_2 is zero and therefore $e_2(I) = S^1$, where $S^1 \subset S^2$ is the 1-dimensional unit sphere.

As an application of the method explained in Section 7 we define canonical immersions of manifolds in spheres as a generalization of (1.5) and prove that the image of a closed and connected manifold M^n, $\dim M^n = n$, $n \geqslant 2$, in the $(n+N-1)$-dimensional, unit sphere S^{n+N-1} by means of a canonical immersion is a unit sphere $S^n \subset S^{n+N-1}$ and M^n is diffeomorphic with S^n.

An outline of the content. In Section 2 we describe the scheme of generating canonical frames in an arbitrary vector bundle. In Section 3 basic notations are introduced. In Section 4 scalar functions in the normal bundle are defined and the construction of canonical frames is carried out. In Section 5 necessary conditions which define canoni-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

cal vectors are rewritten as exterior form equations. In Section 6 we define sufficient conditions to get the uniqueness of the construction. In Section 7 we consider canonical cross sections of immersions of manifolds into spheres.

Cartan's method of moving frames and special related problems are considered by Ph. Griffiths [1] and G.R. Jenssen [2].

2. A scheme of generating canonical frames in vector bundles

Let

$$\mathcal{B} = (B, \pi, M, V)$$

denotes a vector bundle with total space B, base space M, the projection π and a standard fiber V. B and M are differentiable manifolds and V is a finite dimensional vector space. We denote

$$\dim M = n, \quad \dim V = N, \quad \dim B = n+N.$$

A metric g on the bundle $\mathcal{B}$ is a function which assigns to every $p \in M$ a positive definite scalar product g_p in the fiber V_p over $p \in M$, such that for every open set $U \subset M$ and every differentiable, local cross sections $s_1, s_2 : U \to B$, the function $g(s_1, s_2) : U \to R$ defined by $g(s_1, s_2)(p) = g_p(s_1(p), s_2(p))$ is differentiable. Differentiability means differentiability of class C^∞. We suppose throughout the paper that considered manifolds and functions are differentiable. By

(2.1)
$$\mathcal{N}_1 = (N_1, \pi_2, M, O(N)),$$

where O(N) denotes the orthogonal group acting on E^N from the right, we denote the principal bundle of orthonormal frames such that $\mathcal{B}$ is associated with $\mathcal{N}_1$. Let $L : B \to R$ denote a differentiable function, called a scalar function on $\mathcal{B}$. With the aid of L we define locall cross sections in $\mathcal{N}_1$, called canonical frames (with respect to L) as follows. Let $U \subset M$ denotes an open set such that there exists a vector bundle chart $H : \pi^{-1}(U) \to U \times V$. This implies that $\mathcal{B}$ restricted to U

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

admits N linearly independent cross sections $(p,\bar{e}_r(p))$, $p \in U$, $\bar{e}_r(p) \in V_p = \pi^{-1}(p)$. These cross sections we denote in the following simply by $\bar{e}_r$. We denote

$$S_p^{N-1} = \left\{ e \in V_p : g_p(e,e) = 1 \right\}. \tag{2.2}$$

The function L restricted to the sphere (2.2) admits a maximal value at a point $\tilde{e}_{n+1}(p) \in S_p^{N-1}$ called the first canonical vector. Thus, $\tilde{e}_{n+1}(p)$ is a stationary point of the restriction L/S_p^{N-1}. If $\tilde{e}_{n+1}(p)$ is a non degenerate stationary point, i.e., if the Hessian of the function L/S_p^{N-1} is different from zero at $\tilde{e}_{n+1}(p)$, then $\tilde{e}_{n+1}(p)$ is isolated in the sphere S_p^{N-1}.

We have the following

Proposition 1. If $\tilde{e}_{n+1}(p) \in S_p^{N-1} \subset V_p$ is a non degenerate stationary point of the restriction L/S_p^{N-1} such that this function attains its maximal (or minimal) value at $\tilde{e}_{n+1}(p)$, then there exists a neighborhood $Q \subset M$ of $p \in M$ such that $\tilde{e}_{n+1}(q)$, $q \in Q$, is a non degenerate stationary point of L/S_q^{N-1} at which this function attains its maximal (or minimal) value, and $\tilde{e}_{n+1}(q)$, $q \in Q$, is a differentiable cross section in the bundle $\mathcal{B}$ restricted to $Q \subset M$.

Proof. Let $H^{-1}: U \times V \longrightarrow B$ denotes a coordinate mapping, where U is an open set. We choose U such that (U,h) is a chart of M such that $h: U \longrightarrow K^n$ is a diffeomorphism, where K^n denote the n-dimensional, unit and open disk in E^n referred to the coordinates $(u_1,\ldots,u_n)$. By $H_1^{-1}: U \longrightarrow B$ we denote the mapping defined by

$$H_1^{-1}(q,e) = H^{-1}(q,e), \quad q \in U, \quad e \in S^{N-1}. \tag{2.3}$$

By $\left\{ (V,a),(W,b) \right\}$ we denote an atlas on S^{N-1} such that $U \cup W = S^{N-1}$ and $a(e) = (v_1,\ldots,v_{N-1}) \in K^{N-1}$, $b(e) = (w_1,\ldots,w_{N-1}) \in K^{N-1}$ are local coordinates of $e \in S^{N-1}$. The function $L: B \longrightarrow B$ written in local coordinates has the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.4) $f(u;v) = L(H_1^{-1}(h^{-1}(u),a^{-1}(v)))$ in the charts (U,H), (V,a)

and similarly

(2.5) $f(u;w) = L(H_1^{-1}(h^{-1}(u),b^{-1}(w)))$ in the charts (U,h), (W,b),

where $u = (u_1,\dots,u_n)$, $v = (v_1,\dots,v_{N-1})$, $w = (w_1,\dots,w_{N-1})$. If $(u^o;v^o) = (u_1^o,\dots,u_n^o;v_1^o,\dots,v_{N-1}^o)$ are coordinates of a non degenerate stationary point $\tilde{e}_{n+1}(p)$ of the function L/S_p^{N-1}, then

(2.6) $\frac{\partial f}{\partial v_\alpha}(u^o;v^o) = 0$, $\det\left(\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta}(u^o;v^o)\right) \neq 0$, $1\leq\alpha,\beta\leq N-1$.

By the implicit function theorem there exists a neighborhood $Q\subset U$ of $p\in U$, where $h(p) = (u_1^o,\dots,u_n^o)$, and uniquely defined differentiable functions $v_\alpha(u_1,\dots,u_n)$ such that $v_\alpha(u_1^o,\dots,u_n^o) = v_\alpha^o$, $1\leq\alpha\leq N-1$, and

(2.7) $$\frac{\partial f}{\partial v}(u_1,\dots,u_n;v_1(u),\dots,v_{N-1}(u)) = 0,$$

(2.8) $$\det\frac{\partial^2 f}{\partial v_\alpha \partial v_\beta}(u_1,\dots,u_n;v_1(u),\dots,v_{N-1}(u))) \neq 0,$$

where $u = (u_1,\dots,u_n)\in h(Q)\subset K^n$. The second differential of (2.4) with respect to the variables v_α, $1\leq\alpha\leq N-1$, is negative (or positive) definite at $(u_1^o,\dots,u_n^o;v_1^o,\dots,v_{N-1}^o)$. Hence from (2.8) we get that this differential is also negative (or positive) definite at every point $(u_1,\dots,u_n;\ v_1(u),\dots,v_{N-1}(u))$ for $u = (u_1,\dots,u_n)\in h(Q)\subset K^n$. This means that

(2.9) $$\tilde{e}_{n+1}(q) = H_1^{-1}(j^{-1}(u),a^{-1}(v(u)))\in S_q^{N-1},$$

where $h(q) = u$, $q\in Q$, $v(u) = (v_1(u),\dots,v_{N-1}(u))$, is a differentiable cross section of the bundle $\mathcal{B}$ restricted to $Q\subset U\subset M$ and the function L/S_q^{N-1} attains its maximal (or minimal) value at $\tilde{e}_{n+1}(q)$. In the case (2.5) the proof is the same.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

As an immediate consequence of Proposition 1 we get the

Corollary 1. If for every $q \in U$, $U = h^{-1}(K^n)$, the Hessian of L/S_q^{N-1} is different from zero at a stationary point $\tilde{e}_{n+1}(q)$ such that L/S_q^{N-1} attains its maximal (or minimal) value at $\tilde{e}_{n+1}(q)$, then there exists a uniquely defined cross section of the bundle $\mathcal{B}$ restricted to $U \subset M$ and represented by the formula (2.9) for $q \in U$.

If

$$\tilde{e}_{n+1}(p), \ldots, \tilde{e}_{n+\varrho-1}(p), \quad 2 \leq \varrho < N, \quad p \in U = h^{-1}(K^n), \tag{2.10}$$

are canonical mutually orthogonal vectors, then $\tilde{e}_{n+\varrho}(p) \in S_p^{N-1}$ is defined as a non degenerate stationary point of the function $L/S_p^{N-\varrho}$, where $S_p^{N-\varrho} \subset S_p^{N-1}$ is contained in the linear subspace of V_p which is orthogonal to the vectors (2.10) such that $L/S_p^{N-\varrho}$ attains its maximal value at $\tilde{e}_{n+\varrho}(p)$. The last canonical vector $\tilde{e}_{n+N}(p)$ is uniquely determined by the foregoing canonical vectors and the orientation of V_p.

In the following we shall identify the function L restricted to $\pi^{-1}(U)$, $U \subset M$, and LH^{-1}, hence we set

$$L(e(p)) = L(p,e). \tag{2.11}$$

The vector $e \in S^{N-1}$ at the right of (2.11) is therefore a "projection" of $e(p) \in S_p^{N-1}$ on S^{N-1} defined by $e(p) \longrightarrow (p,e) \longrightarrow e$. In the following we shall identify the "projection" $\tilde{e}_{n+1} \in S^{N-1}$ with the cross section (2.9) for $q \in U$. Hence we set

$$\tilde{e}_{n+1}(q) = \tilde{e}_{n+1} \quad \text{for} \quad q \in U. \tag{2.12}$$

From Corollary 1 with the use of the identification (2.12) we get

Proposition 2. If for every $p \in U$, $U = h^{-1}(K^n)$, the Hessian of $L/S_p^{N-\varrho}$, where $S_p^{N-\varrho}$ is the $(N-\varrho)$-dimensional, unit sphere contained in the subspace of V_p orthogonal to the vectors (2.10), $1 \leq \varrho < N$, is different from zero at a stationary point $\tilde{e}_{n+\varrho}$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$L/S_p^{N-\varrho}$ attains its maximal value at $\tilde{e}_{n+\varrho}$, then the function which sends $p \in U$ to $\tilde{e}_{n+\varrho}$, $1 \leq \varrho < N$, is a cross section over U in $\mathcal{B}$ and

$$o(p)\tilde{e}_{n+1}\cdots\tilde{e}_{n+N} \in N_1 \tag{2.13}$$

is a uniquely defined orthonormal frame of V_p attached to the origin $o(p)$ of V_p. The function which sends $p \in U$ to (2.13) is called a canonical cross section over U in $\mathcal{N}_1$ (see (2.1)) and $o: M \to B$ is the zero section in $\mathcal{B}$.

Finally we have

Proposition 3. If the function L/V_p is linear for every $p \in U$, then exactly one canonical vector $\tilde{e}_{n+1}$ can be defined by means of L.

Proof. Since the nullspace of L/V_p is an (N-1)-dimensional subspace $W_p \subset V_p$, which, as follows from the definition of $\tilde{e}_{n+1}$, is orthogonal to $\tilde{e}_{n+1}$ it follows that the function L/S_p^{N-2}, $S_p^{N-2} \subset W_p$, is identically zero.

Generally, if L is identically constant on a subspace of V_p orthogonal to the already defined vectors (2.10), then the definition of $\tilde{e}_{n+\varrho}$ with the aid of L is not possible. In such a case besides L we need further scalar functions $L_\varrho: B_\varrho \to R$, where B_ϱ denotes the total space of the subbundle $\mathcal{B}_\varrho$ of the restriction $\mathcal{B}/U$, whose fiber over $p \in U$ is the subspace of V_p orthogonal to the vectors (2.10). In applications the scalar function L is a homogeneous polynomial of N variables and of degree less than or equal to dim M = n. Since the estimation of stationary points of a polynomial can be described as an algebraic procedure, it follows that the process of defining canonical cross sections (2.13) is rather an algebraic and not differential geometric procedure. The function L depend on the second derivatives of an immersion of M in another manifold and the polynomial L_ϱ depend on derivatives of this immersion of order greater than two.

We define

$$L_r(p) = L(p, \tilde{e}_r), \quad p \in U \subset M. \tag{2.14}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Preliminaries

Let

$$x: M^n \longrightarrow E^{n+N} \tag{3.1}$$

denotes an immersion of a differentiable manifold M^n of dimension n in the Euclidean space E^{n+N}. We use the following notations. TM^n denote the tangent bundle of M^n, T_pM^n denote the tangent space to M^n at $p \in M^n$, $NM^n(x)$ denote the normal bundle of M^n determined by (3.1). The fiber $N_pM^n(x)$ over $p \in M^n$ is the N-dimensional subspace of E^{n+N} of such vectors $e \in E^{n+N}$ that

$$e.dx(p) = 0. \tag{3.2}$$

In the following we frequently set

$$\|e\| = 1. \tag{3.3}$$

By $\mathcal{T}(x)$ we denote the principal tangent bundle of orthonormal frames $x(p)e_1 \ldots e_n \in T(x)$, where $e_i \in x_*(T_pM^n)$, $1 \leq i \leq n$, denote vectors tangent to the surface $x(M^n)$ at the point $x(p)$. We denote

$$\mathcal{T}(x) = (T(x), \pi_1, M^n, O(n)), \tag{3.4}$$

By $\mathcal{N}_1(x)$ we denote the principal normal bundle of orthonormal frames

$$x(p)e_{n+1} \ldots e_{n+N} \in N_1(x) \quad e_r \in N_pM^n(x), \quad n+1 \leq r \leq n+N. \tag{3.5}$$

We denote

$$\mathcal{N}_1(x) = (N_1(x), \pi_2, M^n, O(N)). \tag{3.6}$$

By $\mathcal{F}_1(x) = \mathcal{T}(x) \oplus \mathcal{N}_1(x)$ we denote the Whitney sum of bundles, i.e.

$$\mathcal{F}_1(x) = (F_1(x), \pi, M^n, O(x)xO(N)). \tag{3.7}$$

The elements of the total space $F_1(x)$ are orthonormal frames

$$x(p)e_1 \ldots e_n e_{n+1} \ldots e_{n+N} \in F_1(x). \tag{3.8}$$

The orthogonal groups $O(n)$, $O(N)$ and $O(n+N)$ act from the left on the respective orthonormal frames. Every frame (3.8) defines an element of the Euclidean group $E(n+N)$ which acts on E^{n+N} from the right, i.e.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.9) $$x \longrightarrow xA + a, \quad A \in O(n+N), \quad a \in T(n+N), \quad x \in E^{n+N},$$

where $T(n+N)$ denotes the translation group, isomorphic with E^{n+N}. The group operations in $E(n+N)$ have the form

(3.10) $$(A,a)\cdot(B,b) = (AB, aB+b), \quad (A,a)^{-1} = (A^T, -aA^T),$$

where $(A,a),(B,b) \in E(n+N)$ and A^T denotes the matrix transposed to A. The equations of structure of $E(n+N)$ have the form

(3.11) $$d\omega' = \omega' \wedge \Omega', \quad d\Omega' = \Omega' \wedge \Omega',$$

where $\omega' = dA^T = (\omega'_A)_{1 \leq A \leq n+N}$, $\Omega' = dA \cdot A^T = (\omega_{AB})_{1 \leq A, B \leq n+N}$, and on the right sides of the equations (3.11) we have matrix multiplication. In the following we use the convention

$$1 \leq i,j,k,l \leq n, \quad n+1 \leq r,s,t,u \leq n+N, \quad 1 \leq \rho, \sigma \leq N,$$

and repeated indices denote summation. The total space $F_1(x)$ of (3.7) can be identified with the set

(3.12) $$\left\{ (A,a) \in E(n+N) / e_i = (a_{i1}, \ldots, a_{i,n+N}) \in x_*(T_p M^n),\ a = x(p),\ p \in M^n \right\},$$

where $A = (a_{AB})_{1 \leq A,B \leq n+N} \in O(n+N)$, $a = x(p) \in E^{n+N}$ and x_* denotes the differential of (3.1). The vectors $e_r = (a_{r1}, \ldots, a_{r,n+N})$ are orthogonal to $x(M^n)$ at $x(p)$. By means of the identification (3.12) we define the inclusion map

(3.13) $$F_1(x) \overset{\iota}{\subset} E(n+N).$$

We define linear forms

(3.14) $$\omega = \iota^* \omega' = (\omega_A), \quad \Omega = \iota^* \Omega' = (\omega_{AB}), \quad 1 \leq A, B \leq n+N,$$

on $F_1(x)$, where ι^* denotes the pull back of the components ω', Ω' of the moving frame of the Euclidean group $E(n+N)$, which corresponds to the inclusion map (3.13). On $F_1(x)$ we have

(3.15) $$\iota^*(da) \cdot e_r = dx(p) \cdot e_r = \omega_r = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence

$$dx = \omega_i e_i. \tag{3.16}$$

Furthermore, we have

$$de_i = \omega_{ik} e_k + \omega_{ir} e_r \tag{3.17}$$

and the Weingarten equations have the form

$$de_r = \omega_{ri} e_i + \omega_{rs} e_s. \tag{3.18}$$

The equations of structure of the bundle $\mathcal{F}_1(x)$ have the form

$$d\omega_i = \omega_k \wedge \omega_{ki}, \tag{3.19}$$

$$d\omega_{ik} = \omega_{ij} \wedge \omega_{jk} + \omega_{ir} \wedge \omega_{rk}, \tag{3.20}$$

$$d\omega_{ir} = \omega_{ij} \wedge \omega_{jr} + \omega_{is} \wedge \omega_{sr}, \tag{3.21}$$

$$d\omega_{rs} = \omega_{rj} \wedge \omega_{js} + \omega_{rt} \wedge \omega_{ts}. \tag{3.22}$$

The formulas (3.20) are called the Gauss equations and (3.21) the Codazzi-Mainardi equations. From (3.11), (3.14) and (3.15) we get

$$\omega_{ri} \wedge \omega_i = 0 \tag{3.23}$$

and from Cartan's Lemma it follows

$$\omega_{ir} = A_{rij} \omega_j, \qquad A_{rij} = A_{rji}. \tag{3.24}$$

We define the curvature forms by

$$\Theta_{ik} = \omega_{ir} \wedge \omega_{rk} = -A_{rij} A_{rkl} \omega_j \wedge \omega_l = R_{ijkl} \omega_j \wedge \omega_l, \tag{3.25}$$

where R_{ijkl} are the components of the Riemann curvature tensor. From (3.16) it follows that the induced metric on M^n defined by (3.1) has the form

$$ds^2 = \sum_{i=1}^{n} \omega_i^2 \tag{3.26}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and the volume element of M^n has the form

$$dV_n = \omega_1 \wedge \ldots \wedge \omega_n. \tag{3.27}$$

From (3.16), (3.18) and (3.24) we get

$$d^2x \cdot e_r = -dx \cdot e_r = \omega_i \omega_{ri} = -A_{rij}\omega_i\omega_j. \tag{3.28}$$

From (3.28) we have that A_{rij} defined on $F_1(x)$ are coefficients of the second quadratic form of (3.1) in the direction defined by e_r. The coefficients of the characteristic polynomial

$$\det(\lambda\delta_{ij} - A_{rij}) \tag{3.29}$$

are scalar functions defined on $NM^n(x)$. With the use of the identifications (2.11) and (2.12) we denote by $L_k(p,e_r)$ the coefficient by λ^{n-k} in (3.29).

We have

Proposition 4. For every $p \in M^n$ there exist at most $\frac{n(n+1)}{2}$ canonical vectors which can be defined by means of L_k/S_p^{N-1}, $2 \leq k \leq n$.

Proof. We denote

$$f_{ij} = (A_{n+1,ij}, \ldots, A_{n+N,ij}). \tag{3.30}$$

We prove that f_{ij} is a vector of $N_pM^n(x)$ in the base (3.5). Let

$$e_r = a_{rs}\bar{e}_s, \quad (a_{rs}) \in O(N), \tag{3.31}$$

denotes a change of bases in $N_pM^n(x)$. We have

$$\omega_{ir} = de_i \cdot e_r = a_{rs}\bar{\omega}_{is} = a_{rs}de_i \cdot \bar{e}_s. \tag{3.32}$$

From (3.24) and (3.32) we get

$$A_{rij}\omega_j = a_{rs}\bar{A}_{sij}\omega_j. \tag{3.33}$$

Since ω_j are linearly independent we get

$$A_{rij} = a_{rs}\bar{A}_{sij}. \tag{3.34}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From (3.31) and (3.34) it follows that A_{rij} undergo a change as a vector of $N_pM^n(x)$. Since $A_{rij} = A_{rji}$, we have $f_{ij} = f_{ji}$. Therefore the vectors f_{ij} with coordinates (3.30) define a subspace $N_p^1M^n(s) \subset N_pM^n(x)$ and

$$\dim N_p^1M^n(x) \leq \frac{n(n+1)}{2}. \tag{3.35}$$

The subspace $N_p^1M^n(x)$ remains invariant by change of bases in T^pM^n. Indeed, we have

$$A_{rij} = a_{ik}a_{jl}\bar{A}_{rkl}, \tag{3.36}$$

where

$$e_i = a_{ij}\bar{e}_j, \qquad (a_{ij}) \in O(n), \tag{3.37}$$

is a change of bases in T_pM^n. From (3.30) and (3.36) we get $f_{ij} = a_{ik}a_{jl}\bar{f}_{kl}$, i.e. f_{ij} depend linearly on $\bar{f}_{ij}$.

For $p \in M^n$ fixed L_1 is a linear function and therefore from Proposition 3 it follows that by means of L_1 at most one canonical vector can be defined.

4. Canonical frames in $NM^n(x)$

The coefficient by λ^{n-k} in (3.29) has the form

$$L_k(p,e_r) = (-1)^k \sum_{i_1<\ldots<i_k} \det(A_{rpq})_{p,q\in\{i_1,\ldots,i_k\}}, \tag{4.1}$$

where $1 \leq i_l \leq n$, $1 \leq l \leq k$, $1 \leq k \leq n$. In particular we get

$$L_1(p,e_r) = -\sum_{i=1}^{n} A_{rii}, \tag{4.2}$$

$$L_2(p,e_r) = \sum_{i<j} \det(A_{rpq})_{p,q\in\{i,j\}}, \tag{4.3}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$L_n(p,e_r) = L(p,e_r) = -1)^n \det(A_{rij}). \tag{4.4}$$

The function (4.4) is the Killing-Lipschitz curvature. The functions (4.2), and (4.3) are considered more detailed below in a) and b) respectively. The general case is considered in c). Let

$$x(p)\bar{e}_{n+1}\cdots\bar{e}_{n+N} \in N_1(x), \quad p \in U, \tag{4.5}$$

denotes a cross section from an open set $U \subset M^n$ in $\mathcal{N}_1(x)$ (see (3.6)).

a) From (3.34) and (4.2) we get

$$L_1(p,e_{n+1}) = -a_{n+1,s} \sum_{i=1}^{n} \bar{A}_{sii}. \tag{4.6}$$

We apply the method of Lagrange to determine stationary points of the function (4.6) restricted to S_p^{N-1}. We have

$$\sum_{s=n+1}^{n+N} a_{n+1,s}^2 = 1. \tag{4.7}$$

We get

$$\sum_{i=1}^{n} \bar{A}_{sii} - 2l_{1,n+1}a_{n+1,s} = 0, \tag{4.8}$$

where $l_{1,n+1}$ is a constant for $p \in U$ fixed. From (4.6), (4.7) and the formula

$$\sum_{r=n+1}^{n+N} A_{rij}^2 = \sum_{r=n+1}^{n+N} (a_{rs}\bar{A}_{sij})^2 = a_{rs}a_{rt}\bar{A}_{sij}\bar{A}_{tij} = \sum_{r=n+1}^{n+N} \bar{A}_{rij}^2 \tag{4.9}$$

we get

$$4l_{1,n+1}^2 = \sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} A_{sii} \right)^2 = \sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} A_{sii} \right)^2. \tag{4.10}$$

The function $l_{1,n+1} = l_{1,n+1}(p)$, $p \in U$, is an invariant of $NM^n(x)$. This is the maximal (or minimal) value of $L_1(p,\tilde{e}_{n+1})$ restricted to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

S_p^{N-1} (see (2.14)). We have $I_{1,n+1}(p) = L_1(p,\tilde{e}_{n+1})$. In the following an invariant $I_{kr}(p) = L_k(p,\tilde{e}_r)$ of $NM^n(x)$ is called simply an invariant of (3.1) or of the surface $x(M^n)$. Denoting by $\tilde{A}_{n+1,ij}$ the coefficients of the second quadratic form (3.28) evaluated in the direction of $\tilde{e}_{n+1}$ we get

$$2I_{1,n+1} = -\sum_{i=1}^{n} \tilde{A}_{n+1,ii}. \tag{4.11}$$

If $I_{1,n+1}(p) \neq 0$ for every $p \in U \subset M^n$, then from (3.31) and (4.8) we get

$$\tilde{e}_{n+1} = -\frac{1}{2I_{1,n+1}} \sum_{i=1}^{n} \bar{A}_{sii}\bar{e}_s. \tag{4.12}$$

From (4.12) it follows that $\tilde{e}_{n+1}$ is a differentiable cross section from U to $NM^n(x)$. This is the mean curvature vector and $I_{1,n+1}$ defined by (4.11) is the mean curvature of (3.1) in the direction of $\tilde{e}_{n+1}$. If we choose the cross section (4.5) such that $\bar{e}_{n+1} = \tilde{e}_{n+1}$ and $\bar{e}_s$, $n+2 \leqslant s \leqslant n+N$, remains arbitrary, then from (4.10) and (4.11) we get

$$\sum_{i=1}^{n} \bar{A}_{sii} = 0, \quad n+2 \leqslant s \leqslant n+N. \tag{4.13}$$

Proposition 5. If $I_{1,n+1}(p) \neq 0$ for every $p \in M^n$, then there exists a cross section $\tilde{e}_{n+1}: M^n \longrightarrow NM^n(x)$ and for every open set $U \subset M^n$ which admit a cross section (4.5) $2I_{1,n+1}$ has the form (4.11) and $\tilde{e}_{n+1}$ the form (4.12).

Proof. It suffices to prove that the definition (4.12) of $\tilde{e}_{n+1}$ does not depend on the cross section (4.5). Let $x(p)\hat{e}_{n+1}\ldots\hat{e}_{n+N} \in N_1(x)$ denotes another cross section defined on an open set V such that $p \in U \cap V$. There exists a matrix $(a_{rs}) \in O(N)$ such that

$$\bar{e}_r = a_{rs}\hat{e}_s, \qquad \bar{A}_{rii} = a_{rs}\hat{A}_{sii}$$

for $p \in U \cap V$. We have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\bar{A}_{rii}\bar{e}_r = a_{rs}\hat{A}_{sii}a_{rt}\hat{e}_t = \sigma_{st}\hat{A}_{sii}\hat{e}_t = \hat{A}_{tii}\hat{e}_t.$$

b) The function L_2 defined by (4.3) satisfies $L_2(p,e_{n+1}) = L_2(p,-e_{n+1})$ and therefore by means of L_2 we define canonical directions only. Let us suppose inductively that by means of L_2 we have defined canonical vectors (2.10) and invariants $I_{2,n+\sigma}$, $1\leq\sigma\leq\rho-1$, of $r(M^n)$ at $p\in U\subset M^n$. The vectors (2.10) are defined up to orientation. We define the canonical vector $\tilde{e}_{n+\rho}$ as follows. In (3.34) we change the range of indices such that $n+\rho\leq r,s\leq n+N$ and substitute

$$A_{n+\rho,ij} = a_{n+\rho,s}\bar{A}_{sij}, \quad n+\rho\leq s\leq n+N, \tag{4.14}$$

in (4.3). The seeked unit vector $\tilde{e}_{n+\rho}$ is orthogonal to the vectors (2.10) and therefore can be written in the form

$$\tilde{e}_{n+\rho} = a_{n+\rho,s}\bar{e}_s, \quad n+\rho\leq s\leq n+N, \tag{4.15}$$

where $\bar{e}_s$ together with $\tilde{e}_{n+\sigma}$, $1\leq\sigma\leq\rho-1$, define a base (4.5) of $M_p^n(x)$. The coordinates $(a_{n+\rho,n+\rho},\ldots,a_{n+\rho,n+N})$ of a stationary point $\tilde{e}_{n+\rho}$ of $L_2/S_p^{N-\rho}$ satisfy the condition

$$\sum_{r=n+\rho}^{n+N} a_{n+\rho,r}^2 = 1 \tag{4.16}$$

and the system of equations

$$a_{n+\rho,s}\sum_{i<j}(\bar{A}_{rii}\bar{A}_{sjj}+\bar{A}_{rjj}\bar{A}_{sii}-2\bar{A}_{rij}\bar{A}_{sij}) - 2I_{2,n+\rho}a_{n+\rho,r} = 0, \tag{4.17}$$

where $1\leq\rho<N$, $n+\rho\leq r,s\leq n+N$. If we choose in (4.5) $\bar{e}_{n+\sigma}$ as canonical vectors $\tilde{e}_{n+\sigma}$, $1\leq\sigma\leq\rho$, and $\bar{e}_s$, $n+\rho+1\leq s\leq n+N$, remains arbitrary, then $\tilde{e}_{n+\rho}$ has coordinates $a_{n+\rho,n+\rho} = 1$, $a_{n+\rho,r} = 0$, where $n+1\leq r\leq n+N$, $r\neq n+\rho$, and the system of equations (4.17) takes the form

$$\sum_{i<j}(\tilde{A}_{n+\rho,ii}\tilde{A}_{n+\rho,jj} - \tilde{A}_{n+\rho,ij}^2) = I_{2,n+\rho}, \tag{4.18}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4.19)\qquad \sum_{i<j} (\tilde{A}_{n+\rho,ii}\bar{A}_{rjj} + \tilde{A}_{n+\rho,jj}\bar{A}_{rii} - 2\tilde{A}_{n+\rho,ij}\bar{A}_{rij}) = 0,$$

where $1\leq\rho\leq N-1$, $n+\rho+1\leq r\leq n+N$. Because of our inductive method the identities (4.19) are valid also for the range of indices $1\leq\sigma\leq\rho$, $n+\sigma+1\leq r\leq n+N$, and therefore they are valid also for the range of indices

$$(4.20)\qquad 1\leq\sigma\leq\rho\,,\quad n+\rho+1\leq r\leq n+N.$$

Hence the coefficients of the quadratic form $\bar{A}_{rij}$, $n+\rho+1\leq r\leq n+N$, cannot be arbitrary numbers, since they satisfy ρ equations of the form (4.19), where in (4.19) the index ρ is replaced by σ and the range of indices is defined by (4.20). The same remark concerning conditions imposed on $\bar{A}_{rij}$, $n+\rho+1\leq r\leq n+N$, by the foregoing steps of the construction is valid also in the general case c) considered below. Direct calculations by means of the formula (4.9) yield to the result that besides l_{2r}, $n+1\leq r\leq n+N$, also

$$(4.21)\qquad \sum_{r=n+1}^{n+N}\ \sum_{i<j} (A_{rii}A_{rjj} - A_{rij}^2)$$

is an invariant of $x(M^n)$. For $n = 2$ (4.21) is the Gauss curvature of $x(M^2)\subset E^3$. Since $L_2/S_p^{N-\rho}$, $p\in U$, is a quadratic form of the variables $a_{n+\rho,r}$, $n+\rho\leq r\leq n+N$, which satisfy (4.16), it follows that $l_{2,n+\rho}$, $1\leq\rho\leq N$, defined by (4.18) are eigenvalues of this quadratic form.

c) In general case (4.1) we have $L_k(p,-e_{n+1}) = (-1)^k L_k(p,e_{n+1})$. Hence for k even we determine directions only. We substitute (4.14) into (4.1). We get

$$(4.22)\quad L_k(p,e_{n+\rho}) = (-1)^k \sum_{i_1<\ldots<i_k} \det(a_{n+\rho,s_1}\bar{A}_{s_1 i_1 q},\ldots,a_{n+\rho,s_k}\bar{A}_{s_k i_k q})$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $1 \leq \rho < N$, $1 \leq i_1 \leq n$, $n+\rho \leq s_1 \leq n+N$, $1 \leq l \leq k$, $q \in \{i_1, \ldots, i_k\}$. The symbol $a_{n+\rho,s_1} \bar{A}_{s_1 i_1 q}$ denotes the l-th row of the determinant at the right of (4.22). The function $L_k / S_p^{N-\rho}$ is defined by the formula (4.22), where the variables $a_{n+\rho,s}$, $n+\rho \leq s \leq n+N$, in (4.22) satisfy the condition (4.16). The coordinates $(a_{n+\rho,n+\rho}, \ldots, a_{n+\rho,n+N})$ of a stationary point $\tilde{e}_{n+\rho}$ of $L_k / S_p^{N-\rho}$ satisfies besides (4.16) the system of equations

$$(4.23)\quad (-1)^k \sum_{l=1}^{k} \sum_{i_1 < \ldots < i_k} \det(A_{n+\rho,i_1 q}, \ldots, \bar{A}_{r i_l q}, \ldots, A_{n+\rho,i_k q}) = 2l_{k,n+\rho} a_{n+\rho,r},$$

where $1 \leq \rho < N$, $n+\rho \leq r \leq n+N$, and $A_{n+\rho,i_m q}$, $m \neq l$, $1 \leq m \leq k$, is an abbreviation for $a_{n+\rho,s_m} \bar{A}_{s_m i_m q}$, $n+\rho \leq s_m \leq n+N$, in (4.23). If we choose (4.5) such that $\bar{e}_{n+\sigma}$ are canonical vectors $\tilde{e}_{n+\sigma}$, $1 \leq \sigma \leq \rho$, then in such a base $\tilde{e}_{n+\rho}$ has coordinates $a_{n+\rho,n+\rho} = 1$, $a_{n+\rho,r} = 0$, $n+1 \leq r \leq n+N$, $r \neq n+\rho$, and the system of equations (4.23) takes the form

$$(4.24)\quad (-1)^k k \sum_{i_1 < \ldots < i_k} \det(\tilde{A}_{n+\rho,pq})_{p,q \in \{i_1 \ldots i_k\}} = 2l_{k,n+\rho},$$

$$(4.25)\quad \sum_{l=1}^{k} \sum_{i_1 < \ldots < i_k} \det(\tilde{A}_{n+\rho,i_1 q}, \ldots, \bar{A}_{r i_l q}, \ldots, \tilde{A}_{n+\rho,i_k q}) = 0,$$

where $1 \leq \rho < N$, $n+\rho+1 \leq r \leq n+N$. For $k = n$ the equations (4.24) and (4.25) have the form

$$(4.26)\quad (-1)^n n \det(\tilde{A}_{n+\rho,ik}) - 2l_{n,n+\rho} = 0,$$

$$(4.27)\quad \sum_{i=1}^{n} \det(\tilde{A}_{n+\rho,1k}, \ldots, \bar{A}_{rik}, \ldots, \tilde{A}_{n+\rho,nk}) = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

5. Canonical frames defined by systems of exterior equations

From (3.24) and (3.27) we get the identity

$$\sum_{i=1}^{n} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \omega_{ir} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n = \sum_{i=1}^{n} A_{rii} dV_n. \tag{5.1}$$

From (3.24) we get

$$\tilde{\omega}_{i,n+1} = de_i \cdot \tilde{e}_{n+1} = \tilde{A}_{n+1,ij} \omega_j, \tag{5.2}$$

where $\tilde{e}_{n+1}$ in (5.2) is defined by (4.12). Denoting by $I_{1,n+1}$ the maximal value of the function (4.6) restricted to the sphere (4.7) we get from (4.11) and $L_1(p,-e_{n+1}) = -L_1(p,e_{n+1})$ the inequality

$$-\sum_{i=1}^{n} \tilde{A}_{n+1,ii} \geqslant 0. \tag{5.3}$$

From (4.10), (4.11) and (5.3) we get

$$-\sum_{i=1}^{n} \tilde{A}_{n+1,ii} = \left(\sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} A_{sii} \right)^2 \right)^{\frac{1}{2}}. \tag{5.4}$$

From (5.1), (5.2) and (5.4) we get

$$-\sum_{i=1}^{n} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \tilde{\omega}_{i,n+1} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n = \tag{5.5}$$

$$= \left(\sum_{s=n+1}^{n+N} \left(\sum_{i=1}^{n} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \omega_{is} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n \right)^2 \right)^{\frac{1}{2}}.$$

The equations (4.13) can be written in the equivalent form

$$\sum_{i=1}^{n} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \bar{\omega}_{is} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n = 0, \quad n+2 \leqslant s \leqslant n+N, \tag{5.6}$$

where

$$\bar{\omega}_{is} = de_i \cdot \bar{e}_s = \bar{A}_{sij} \omega_j. \tag{5.7}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the case b) from (3.24) and (3.27) we get the identity

$$(5.8)\quad \sum_{i<j} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \omega_{ir} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_{j-1} \wedge \omega_{jr} \wedge \omega_{j+1} \wedge \ldots \wedge \omega_n =$$

$$= \sum_{i<j} (A_{rii} A_{rjj} - A_{rij}^2) dV_n.$$

The equations (4.19) can be written in the equivalent form

$$(5.9)\quad \sum_{i<j} (\omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \bar{\omega}_{ir} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_{j-1} \wedge \tilde{\omega}_{j,n+\rho} \wedge \omega_{j+1} \wedge \ldots \wedge \omega_n +$$

$$+ \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \tilde{\omega}_{i,n+\rho} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_{j-1} \wedge \bar{\omega}_{jr} \wedge \omega_{j+1} \wedge \ldots \wedge \omega_n) = 0,$$

where $1 \leq \rho < N$, $n+\rho+1 \leq r \leq n+N$, and

$$(5.10)\quad \tilde{\omega}_{i,n+\rho} = de_i \cdot \tilde{e}_{n+\rho}.$$

Gurukul Kangri Vishwavidyalaya HARDWAR

The invariant (4.21) is defined by means of the invariant form

$$(5.11)\quad \sum_{r=n+1}^{n+N} \sum_{i<j} \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge \omega_{ir} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_{j-1} \wedge \omega_{jr} \wedge$$

$$\wedge \omega_{j+1} \wedge \ldots \wedge \omega_n.$$

In the general case c) we get from (3.24) and (3.27) the identity

$$(5.12)\quad \sum_{i_1<\ldots<i_k} \omega_1 \wedge \ldots \wedge \omega_{i_1-1} \wedge \omega_{i_1 r} \wedge \omega_{i_1+1} \wedge \ldots \wedge \omega_{i_k-1} \wedge \omega_{i_k r} \wedge$$

$$\wedge \omega_{i_k+1} \wedge \ldots \wedge \omega_n = \sum_{i_1<\ldots<i_k} \det(A_{rpq})_{p,q \in \{i_1,\ldots,i_k\}} dV_n.$$

The equations (4.25) can be written in the equivalent form

$$(5.13)\quad \sum_{l=1}^{k} \sum_{i_1<\ldots<i_k} \omega_1 \wedge \ldots \wedge \omega_{i_1-1} \wedge \tilde{\omega}_{i_1,n+\rho} \wedge \omega_{i_1+1} \wedge \ldots \wedge \omega_{i_l-1} \wedge$$

$$\wedge \bar{\omega}_{i_l r} \wedge \omega_{i_l+1} \wedge \ldots \wedge \omega_{i_k-1} \wedge \tilde{\omega}_{i_k,n+\rho} \wedge \omega_{i_k+1} \wedge \ldots \wedge \omega_n = 0,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $1 \leq \rho < N$, $n+\rho+1 \leq r \leq n+N$. In particular (4.27) can be written in the form

$$\sum_{i=1}^{n} \tilde{\omega}_{1,n+\rho} \wedge \ldots \wedge \bar{\omega}_{ir} \wedge \ldots \wedge \tilde{\omega}_{n,n+\rho} = 0. \tag{5.14}$$

6. Local canonical cross sections in $NM^n(x)$

In Section 4 we have defined canonical frames

$$x(p)\tilde{e}_{n+1} \ldots \tilde{e}_{n+N} \in N_1(x) \tag{6.1}$$

at a point $p \in M^n$ by means of the functions (4.1). From Proposition 1 and Corollary 1 it follows that if the Hessian

$$\det(h^{\rho}_{kru})_{n+\rho+1 \leq r, u \leq n+N}, \quad 1 \leq \rho < N, \quad 2 \leq k \leq n, \tag{6.2}$$

of $L_k/S_p^{N-\rho}$ is different from zero at a stationary point $\tilde{e}_{n+\rho}$, then there exists a neighborhood $U \subset M^n$, $p \in U$, diffeomorphic with the unit disk K^n, such that for every $q \in U$ the Hessian (6.2) is different from zero and

$$\tilde{e}_{n+\rho} : U \longrightarrow NM^n(x) \tag{6.3}$$

is a differentiable cross section. Let

$$x(p)\tilde{e}_{n+1} \ldots \tilde{e}_{n+\rho} \bar{e}_{n+\rho+1} \ldots \bar{e}_{n+N} \in N_1(x), \quad p \in U. \tag{6.4}$$

We suppose that $\tilde{e}_{n+\sigma}$, $1 \leq \sigma < \rho$, are cross sections defined by means of the functions L_j, $1 \leq j \leq n$, on $U \subset M^n$ and $\bar{e}_r$, $n+\rho+1 \leq r \leq n+N$, are arbitrary cross sections of $NM^n(x)$ defined on $U \subset M^n$. From (4.1) and (6.4) it follows that the function $L_k/S_p^{N-\rho}$ can be written in the form

$$L_k(p, e_{n+\rho}) = (-1)^k \sum_{i_1 < \ldots < i_k} \det(A_{n+\rho,pq})_{p,q \in \{i_1, \ldots, i_k\}}, \tag{6.5}$$

where

$$e_{n+\rho} = a_{n+\rho,n+\rho}\tilde{e}_{n+\rho} + \sum_{r=n+\rho+1}^{n+N} a_{n+\rho,r}\bar{e}_r \tag{6.6}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and $(a_{n+\rho,n+\rho},\ldots,a_{n+\rho,n+N})$ satisfies (4.16). The second derivative of (6.5) with respect to $a_{n+\rho,r}$, $a_{n+\rho,u}$, $n+\rho+1\leq r,u\leq n+N$, at $a_{n+\rho,r}=0$, where

$$A_{n+\rho,pq}=a_{n+\rho,n+\rho}\tilde{A}_{n+\rho,pq}+\sum_{r=n+\rho+1}^{n+N}a_{n+\rho,r}\bar{A}_{rpq} \tag{6.7}$$

and

$$a_{n+\rho,n+\rho}=\left(1-\sum_{r=n+\rho+1}^{n+N}a_{n+\rho,r}^2\right)^{\frac{1}{2}}, \tag{6.8}$$

has the form

$$h^{\rho}_{kru}=(-1)^{k+1}\delta_{ru}\sum_{i_1<\ldots<i_k}\det(\tilde{A}_{n+\rho,pq})_{p,q\in\{i_1,\ldots,i_k\}}+ \tag{6.9}$$

$$+(-1)^k\sum_{i_1<\ldots<i_k}\sum_{l<m}(\det(\tilde{A}_{n+\rho,i_1q},\ldots,\bar{A}_{ri_lq},\ldots,\bar{A}_{ui_mq},\ldots,\tilde{A}_{n+\rho,i_kq})$$

$$+\det(\tilde{A}_{n+\rho,i_1q},\ldots,\bar{A}_{ui_lq},\ldots,\bar{A}_{ri_mq},\ldots,\tilde{A}_{n+\rho,i_kq})),$$

where $2\leq k\leq n$, $1\leq l<m\leq k$, $1\leq\rho<N$, $n+\rho+1\leq r,u\leq n+N$, and $q\in\{i_1,\ldots,i_k\}$ denote the number of a column in the determinants of (6.9).

From (6.9) we get

$$h^{\rho}_{2ru}=-2\delta_{ru}\sum_{i<j}(\det(\tilde{A}_{n+\rho,pq})_{p,q\in\{i,j\}}+ \tag{6.10}$$

$$+\bar{A}_{rii}\bar{A}_{ujj}+\bar{A}_{rjj}\bar{A}_{uii}-2\bar{A}_{rij}\bar{A}_{uij}).$$

If in (6.1) $\tilde{e}_{n+\rho}$, $\bar{e}_r$ are eigenvectors $\tilde{e}_{n+\rho}$, $\tilde{e}_r$, $n+\rho+1\leq r\leq n+N$, of the quadratic form $L_2/S_p^{N-\rho}$ (not necessarily uniquely defined), then from (4.19) it follows

$$\bar{A}_{rii}\bar{A}_{ujj}+\bar{A}_{rjj}\bar{A}_{uii}-2\bar{A}_{rij}\bar{A}_{uij}=\tilde{A}_{rii}\tilde{A}_{ujj}+\tilde{A}_{rjj}\tilde{A}_{uii}-2\tilde{A}_{rij}\tilde{A}_{uij}=0 \tag{6.11}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and (6.10) get by (6.11) a diagonal matrix

$$h^{\rho}_{2rr} = -2(l_{2,n+\rho} - l_{2r}), \quad n+\rho+1 \leq r \leq n+N, \tag{6.12}$$

$$h^{\rho}_{2ru} = 0 \text{ for } r \neq u, \quad n+\rho+1 \leq r,u \leq n+N, \tag{6.13}$$

where l_{2r} are defined by (4.18).

We have

Proposition 6. Let

$$x(p)\tilde{e}_{n+1}\dots\tilde{e}_{n+\rho}\dots\tilde{e}_{n+N} \in N_1(x), \tag{6.14}$$

where $\tilde{e}_r$, $n+1 \leq r \leq n+N$, in (6.14) are eigenvectors of L_2/S^{N-1}_p. If the eigenvalue $l_{2,n+\rho}(p)$ is different from $l_{2r}(p)$ for every r, $r \neq n+\rho$, $n+1 \leq r \leq n+N$, then there exists a neighborhood $U \subset M^n$, such that $l_{2,n+\rho}(q) \neq l_{2r}(q)$ for every $q \in U$ and a 1-dimensional, trivial vector bundle with fiber $\varepsilon_{n+\rho}(q)$, $q \in U$, such that $\varepsilon_{n+\rho}(q)$ is determined by a uniquely defined eigenvector $\tilde{e}_{n+\rho}$.

Proof. From (6.12) it follows that the definition

$$h^{\sigma}_{2,n+\rho,n+\rho} = -h^{\rho}_{2,n+\sigma,n+\sigma}, \tag{6.15}$$

where $1 \leq \rho < N$, $\rho+1 \leq \sigma < N$, is correct. From (6.15) we get

$$h^{\rho}_{2rr} = 2(l_{2n+\rho} - l_{2r}), \tag{6.16}$$

where $n+1 \leq r \leq n+\rho-1$, and similarly as (6.13) we get

$$h^{\rho}_{2ru} = 0 \text{ for } r \neq u, \ n+1 \leq r,u \leq n+N, \ r,u \neq n+\rho. \tag{6.17}$$

By assumption we have

$$h^{\rho}_{2rr} \neq 0, \quad r \neq n+\rho, \ n+1 \leq r \leq n+N, \tag{6.18}$$

at $p \in M^n$. From (6.12), (6.16), (6.17) and (6.18) it follows that the Hessian

$$\det(h^{\rho}_{2ru}) \neq 0, \quad r,u \neq n+\rho, \ n+1 \leq r,u \leq n+N, \tag{6.19}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

at $p \in M^n$. Hence there exists a neighborhood $U \subset M^n$, $p \in U$, such that (6.19) is satisfied for every $q \in U$ and therefore from Proposition 1 and Corollary 1 it follows that there exists a cross section

$$\tilde{e}_{n+\rho} : U \longrightarrow NM^n(x) \tag{6.20}$$

which for every $q \in U$ defines the canonical direction $\varepsilon_{n+\rho}(q)$.

The extension of (6.12), (6.13) by (6.16), (6.17) is necessary, since we do not assume that $\tilde{e}_{n+\sigma}$, $1 \leq \sigma \leq \rho$, in (6.14) is a non degenerate stationary point of $L_2/S_p^{N-\sigma}$ and therefore that it is uniquely defined as the vector $\tilde{e}_{n+\sigma}$ in (6.4) is.

7. Immersion of manifolds in spheres

Let

$$\bar{e}_{n+1} : M^n \longrightarrow E^{n+N} \tag{7.1}$$

denotes an immersion such that

$$\bar{e}_{n+1}(M^n) \subset S^{n+N-1}, \tag{7.2}$$

where $S^{n+N-1} \subset E^{n+N}$ denotes the unit sphere.

We have

Proposition 7. If $\tilde{e}_{n+1}$ denotes the first canonical vector defined by means of L_k/S_p^{N-1}, then

$$L_k(p, \tilde{e}_{n+1}) \geq \binom{n}{k} \tag{7.3}$$

for every $p \in M^n$.

Proof. Let us take

$$\bar{e}_{n+1}(p) e_1 \cdots e_n \bar{e}_{n+1} e_{n+2} \cdots e_{n+N} \in F_1(\bar{e}_{n+1}), \tag{7.4}$$

where $F_1(x)$ denotes the total space of (3.7). From (3.15) it follows $d\bar{e}_{n+1} \cdot e_r = \omega_{n+1,r} = 0$ and therefore

$$d\bar{e}_{n+1} = \bar{\omega}_{n+1,i} e_i. \tag{7.5}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From (3.24) and (7.5) it follows

$$\bar{\omega}_{i,n+1} = \bar{A}_{n+1,ij}\bar{\omega}_{n+1,j} \tag{7.6}$$

and therefore

$$\bar{A}_{n+1,ij} = -\delta_{ij}. \tag{7.7}$$

From (7.4) it follows $\bar{e}_{n+1} \in S_p^{N-1} \subset N_p M^n(\bar{e}_{n+1})$. Hence setting (7.7) in (4.1) we get $L_k(p,\bar{e}_{n+1}) = \binom{n}{k}$. Since $L_k(p,\tilde{e}_{n+1})$ is the maximal value of $L_k(p,e_{n+1})$, $e_{n+1} \in S_p^{N-1}$, we have

$$L_k(p,\tilde{e}_{n+1}) \geqslant L_k(p,\bar{e}_{n+1}) = \binom{n}{k}. \tag{7.8}$$

We have

Proposition 8. Let $\tilde{e}_{n+1}$ denotes the canonical vector of (7.1) defined by means of L_1/S_p^{N-1}. There exists a cross section

$$\tilde{e}_{n+1}: M^n \longrightarrow NM^n(\bar{e}_{n+1}). \tag{7.9}$$

and for every $p \in M^n$ the vector $\tilde{e}_{n+1}$ is transversal to S^{n+N-1}. For every $e_{n+2} \in S_p^{N-1} \subset N_p M^n(\bar{e}_{n+1})$ such that

$$e_{n+2} \cdot \tilde{e}_{n+1} = 0 \tag{7.10}$$

the function (4.3) takes the form

$$L_2(p,e_{n+2}) = -\frac{1}{2}\sum_{i,j=1}^{n} A^2_{n+2,ij}. \tag{7.11}$$

Proof. Fron (7.3) we have

$$L_1(p,\tilde{e}_{n+1}) \geqslant n. \tag{7.12}$$

From (4.2), (4.11) and (7.12) we get

$$2I_{1,n+1} = -\sum_{i=1}^{n} \tilde{A}_{n+1,ii} \geqslant n \tag{7.13}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for every $p \in M^n$. Hence, from Proposition 5 we get (7.9). From (7.4) we have

$$\tilde{e}_{n+1} = a_{n+1,n+1}\bar{e}_{n+1} + \sum_{r=n+2}^{n+N} a_{n+1,r}e_r. \tag{7.14}$$

From (4.12) and (7.14) we get

$$a_{n+1,n+1} = -\frac{1}{2I_{1,n+1}}\left(\sum_{i=1}^{n}\bar{A}_{n+1,ii}\right). \tag{7.15}$$

From (4.2) and (7.13) it follows that (7.15) can be written in the form

$$a_{n+1,n+1} = \frac{L_1(p,\bar{e}_{n+1})}{L_1(p,\tilde{e}_{n+1})}. \tag{7.16}$$

From (7.8) and (7.16) we get $0 < a_{n+1,n+1} \leq 1$, hence

$$0 < \tilde{e}_{n+1}\cdot\bar{e}_{n+1} \leq 1 \tag{7.17}$$

for every $p \in M^n$. Denoting by $\alpha(p)$, $p \in M^n$, the angle between $\tilde{e}_{n+1}$ and $\bar{e}_{n+1}$, (7.17) can be written in the form

$$0 < \cos\alpha(p) \leq 1 \quad \text{or equivalently} \quad 0 \leq \alpha(p) < \frac{\pi}{2}. \tag{7.18}$$

From (7.18) it follows that $\tilde{e}_{n+1}$ is transversal to S^{n+N-1} at every point $\bar{e}_{n+1}(p) \in S^{n+N-1}$, $p \in M^n$.

For $r = n+2$ we get from (3.34)

$$A_{n+2,ij} = a_{n+2,s}\bar{A}_{sij}. \tag{7.19}$$

From (4.2), (4.13) and (7.19) we get

$$L_1(p,e_{n+2}) = -\sum_{i=1}^{n} A_{n+2,ii} = -\sum_{s=n+2}^{n+N} a_{n+2,s}\sum_{i=1}^{n}\bar{A}_{sii} = 0, \tag{7.20}$$

where e_{n+2} in (7.20) satisfies (7.10). From (7.20) it follows that the function $L_2(p,e_{n+2})$ defined by (4.3) take the form (7.11).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By $\tilde{e}_{k,n+\rho}$, $1 \leq k \leq n$, $1 \leq \rho < N$, we denote the ρ-th canonical vector defined by means of L_k, i.e. $L_k(p, \tilde{e}_{k,n+\rho})$ is the maximal value of $L_k / S_p^{N-\rho}$.

An immersion (7.1) such that (7.2) is satisfied is called canonical, if there exists a k, $1 \leq k \leq n$, such that for every $p \in M^n$

$$\bar{e}_{n+1} = \tilde{e}_{k,n+1} \quad \text{or} \quad \bar{e}_{n+1} = -\tilde{e}_{k,n+1} \tag{7.21}$$

and

$$L_{2j}(p, \tilde{e}_{2j,n+2}) = 0 \quad \text{for every} \quad j, \ 1 < 2j \leq n. \tag{7.22}$$

Finally we prove the following

Theorem. If (7.1) is a canonical immersion of a closed and connected manifold M^n, $n \geq 2$, then

$$\bar{e}_{n+1}(M^n) = S^n \subset S^{n+N-1}, \tag{7.23}$$

where S^n denotes the unit sphere and M^n is diffeomorphic with S^n.

Proof. From (7.7) and (7.21) it follows

$$\bar{A}_{n+1,ij} = \pm \tilde{A}_{n+1,ij} = -\delta_{ij}, \tag{7.24}$$

where $\tilde{A}_{n+1,ij}$ are coefficients of the second quadratic form evaluated in the direction of $\tilde{e}_{k,n+1}$. From (7.24) it follows that the equations (4.25) for $\rho = 1$ take the form

$$\sum_{l=1}^{k} \sum_{i_1 < \ldots < i_k} \det(-\delta_{i_1 q}, \ldots, \bar{A}_{r i_l q}, \ldots, -\delta_{i_k q}) = 0, \tag{7.25}$$

where $n+2 \leq r \leq n+N$, $1 \leq k \leq n$, $1 \leq i_m \leq n$, $1 \leq m \leq k$, $q \in \{i_1, \ldots, i_k\}$, and $\bar{A}_{rij}$ are coefficients of the second quadratic form evaluated in the direction of $\bar{e}_r$. The vector $\bar{e}_r$ is an element of a base in $N_p M^n(\bar{e}_{n+1})$ of the form

$$\bar{e}_{n+1}(p) \tilde{e}_{k,n+1} \bar{e}_{n+2} \cdots \bar{e}_{n+N} \in N_1(\bar{e}_{n+1}). \tag{7.26}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We have

(7.27) $$\det(-\delta_{i_1 q},\ldots,\bar{A}_{ri_1 q},\ldots,-\delta_{i_k q}) = (-1)^{k-1}A_{ri_1 i_1}.$$

Hence

(7.28) $$\sum_{l=1}^{k}\det(-\delta_{i_1 q},\ldots,\bar{A}_{ri_l q},\ldots,-\delta_{i_k q}) =$$

$$= (-1)^{k-1}(\bar{A}_{ri_1 i_1} + \ldots + \bar{A}_{ri_k i_k}) = 0,$$

where $n+2\leq r\leq n+N$. From (7.25) and (7.28) we get

(7.29) $$\sum_{i_1<\ldots<i_k}(\bar{A}_{ri_1 i_1} + \ldots + \bar{A}_{ri_k i_k}) = 0, \quad n+2\leq r\leq n+N.$$

The equations (7.29) are exactly the equations (4.13). Hence we have proved that for every k, $1\leq k\leq n$, if $\tilde{e}_{k,n+1}$ satisfies (7.21), then the equations (4.13) are satisfied.

Let $\tilde{e}_{j,n+2}\in N_p M^n(\bar{e}_{n+1})$, $1\leq j\leq n$, denotes a second canonical vector. We choose a base

(7.30) $$\bar{e}_{n+1}(p)\tilde{e}_{k,n+1}\tilde{e}_{j,n+2}\bar{e}_{n+3}\ldots\bar{e}_{n+N}\in N_1(\bar{e}_{n+1}).$$

The equations (4.25) now take the form

(7.31) $$\sum_{l=1}^{j}\sum_{i_1<\ldots<i_j}\det(\tilde{A}_{n+2,i_1 q},\ldots,\bar{A}_{ri_l q},\ldots,\tilde{A}_{n+2,i_j q}) = 0,$$

where $n+3\leq r\leq n+N$ and $\tilde{A}_{n+2,ij}$ are coefficients of the second quadratic form evaluated in the direction of $\tilde{e}_{j,n+2}$. Let

(7.32) $$\bar{e}_{n+1}(p)\bar{e}_1\ldots\bar{e}_n\in T(\bar{e}_{n+1}),$$

where T(x) is the total space of (3.4), denotes a base in $T_p M^n$ such that $\bar{e}_i$ define principal axes of the matrix $(\tilde{A}_{n+2,ij})$. Then $(\tilde{A}_{n+2,ij})$ is a diagonal matrix and (7.31) takes the form

(7.33) $$\sum_{l=1}^{i}\sum_{i_1<\ldots<i_j}\det(\tilde{A}_{n+2,i_1 i_1},\ldots,\bar{A}_{ri_l q},\ldots,\tilde{A}_{n+2,i_j i_j}) = 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We have

$$\det(\tilde{A}_{n+2,i_1i_1},\ldots,\bar{A}_{ri_1q},\ldots,\tilde{A}_{n+2,i_ji_j}) = \tag{7.34}$$

$$= \tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_1i_1}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

From (7.33) and (7.34) it follows that (7.31) takes the form

$$\sum_{l=1}^{i}\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_1i_1}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0, \tag{7.35}$$

where $n+3\leqslant r\leqslant n+N$. We have

$$\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_1i_1}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} =$$

$$= \frac{1}{j!}\sum_{i_1,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_1i_1}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

Hence, we get

$$\sum_{l=1}^{j}\sum_{i_1<\ldots<i_j}\tilde{A}_{n+2,i_1i_1}\cdot\ldots\cdot\bar{A}_{ri_1i_1}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} =$$

$$= \frac{1}{(j-1)!}\sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j}.$$

Therefore the equations (7.33) can be written in the form

$$\sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0, \tag{7.36}$$

Therefore the equations (7.33) can be written in the form

$$\sum_{i=1}^{n}\bar{A}_{rii}\sum_{i_2,\ldots,i_j=1}^{n}\tilde{A}_{n+2,i_2i_2}\cdot\ldots\cdot\tilde{A}_{n+2,i_ji_j} = 0, \tag{7.36}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $n+3 \leq r \leq n+N$. From (4.13) it follows that (7.36) are identities. This means that for every choice of

$$e_{n+2} = a_{n+2,s}\bar{e}_s, \quad \sum_{s=n+2}^{n+N} a^2_{n+2,s} = 1, \tag{7.37}$$

where $\bar{e}_s$ are the vectors of (7.24), the function $L_j(p,e_{n+2})$ has the same maximal value. Therefore

$$L_j(p,e_{n+2}) = B_j = \text{const} \tag{7.38}$$

for every j, $1 \leq j \leq n$. From $L_{2m+1}(p,-e_{n+2}) = -L_{2m+1}(p,e_{n+2})$ and (7.38) we get

$$B_{2m+1} = 0 \quad \text{for every } m,\ 1 \leq 2m+1 \leq n. \tag{7.39}$$

From (7.22) and (7.38) we get

$$B_{2m} = 0 \quad \text{for every } m,\ 1 < 2m \leq n. \tag{7.40}$$

From (4.1), (7.38), (7.39) and (7.40) we get

$$\sum_{i_1<\ldots<i_j} \det(A_{n+2,pq})_{p,q\in\{i_1,\ldots,i_k\}} = 0, \tag{7.41}$$

where $A_{n+2,ij}$ are coefficients of the second quadratic form evaluated in the direction of the vector (7.37). Let

$$\bar{e}_{n+1}(p)e_1\ldots e_n \in T(\bar{e}_{n+1}) \tag{7.42}$$

denotes a base in T_pM^n such that e_i define principal axes of $(A_{n+2,ij})$. Then (7.41) has the form

$$\sum_{i_1<\ldots<i_j} A_{n+2,i_1i_1},\ldots,A_{n+2,i_ji_j} = 0 \tag{7.43}$$

for every j, $1 \leq j \leq n$. From (7.43) it follows that the eigenvalues $A_{n+2,ii} = 0$, $1 \leq i \leq n$. Hence in an arbitrary base of the form (7.42) of T_pM^n we get $A_{n+2,ij} = 0$, $1 \leq i,j \leq n$. This proves (7.23).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since M^n is by (7.23) locally diffeomorphic with S^n and compact, it follows that M^n is a finite covering of S^n. Such a connected covering is for $n \geqslant 2$ diffeomorphic with S^n.

REFERENCES

[1] Ph. Griffiths: On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, Duke Math. J. 41 (1974), 775-814.

[2] G.R. Jenssen: Higher order contact of submanifolds of homogeneous spaces, Lecture Notes in Mathematics 610 (1977).

[3] M. Rochowski: The Frenet frame of an immersion, J. Diff. Geom., 10, (1975), 181-200.

Author's address: Fleiner Strasse 7C, 7000 Stuttgart 40, BRD/RFN

Received September 9, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

T.D. Narang*)

METRIC PROJECTIONS IN LINEAR METRIC SPACES

Various properties of the metric projection are known in normed linear spaces (see e.g. [7]). G. Pantelidis [5] discussed some of these properties in linear metric spaces. In this paper we also discuss some necessary and sufficient conditions for the metric projections to be continuous and Lipschitzian in linear metric spaces.

Let G be a subspace of a linear metric space (E,d) and $x \in E$. An element $g_o \in G$ is said to be a best approximation to x in G if

$$d(x,g_o) = d(x,G).$$

The set of all such $g_o \in G$ is denoted by $L_G(x)$ i.e.

$$L_G(x) = \left\{ g_o \in G : d(x,g_o) = d(x,G) \right\}.$$

G is said to proximinal if $L_G(x)$ is non-empty for each $x \in E$ and it is said to be Chebyshev if $L_G(x)$ consists of exactly one element for each $x \in E$. The mapping π_G, which takes each element x of E to the set $L_G(x)$ is called the metric projection of E onto G. For Chebyshev sets G, π_G is single-valued.

The canonical mapping $W_G : E \longrightarrow E/G$ defined by $W_G(x) = x + G$ is clearly related to best approximation, since

$$L_G(x) = \left\{ g_o \in G : d(x,g_o) = \mathrm{dist}(W_G(x),0) \right\}.$$

*)The author is thankful to U.G.C., India for financial support.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We shall denote by $P_G^{-1}(o)$ the set

$$P_G^{-1}(o) = \left\{ x \in E : 0 \in L_G(x) \right\}.$$

This set is called kernel of the mapping π_G.

The first two theorems give necessary and sufficient conditions for the continuity of the metric projections in linear metric spaces. In normed linear spaces, the first theorem was proved by Holmes [2] and the second by Cheney and Wulbert [1] and Holmes [2].

Theorem 1. For a Chebyshev subspace G of a linear metric space (E,d), the metric projection π_G is continuous if and only if the restriction $W = W_G\big|_{P_G^{-1}(o)}$ of the canonical mapping $W_G : E \to E/G$ to the set $P_G^{-1}(o)$ is homeomorphism of $P_G^{-1}(o)$ onto E/G.

Proof. As G is a Chebyshev subspace, W is one-to-one (see [4], Theorem 2.3). Further, W is always continuous (see [6]) and a mapping onto E/G, since for any $x + G \in E/G$, we have $x - \pi_G(x) \in P_G^{-1}(o)$ and $W_G[x - \pi_G(x)] = x + G$. Thus the condition that $W = W_G\big|_{P_G^{-1}(o)}$ be a homeomorphism onto E/G is equivalent to the continuity of W^{-1}.

Assume that W^{-1} is continuous. We shall show that π_G is continuous. Let x_n, $x \in E$ and $x_n \to x$ i.e. $d(x_n, x) \to 0$ as $n \to \infty$. Then as above, $x_n - \pi_G(x_n) = W^{-1}(x_n + G)$ and $x - \pi_G(x) = W^{-1}(x+G)$, and so

$$d\left[\pi_G(x_n), \pi_G(x)\right] \leq d\left[\pi_G(x_n) - \pi_G(x),\ x_n - x\right] + d(x_n, x) =$$

$$= d\left[W^{-1}(x+G),\ W^{-1}(x_n+G)\right] + d(x_n, x) \to 0 \quad \text{as} \quad n \to \infty$$

and so π_G is continuous.

Now assume that π_G is continuous. We shall show that W^{-1} is continuous. Let $x_n + G$, $x + G \in E/G$, $\lim(x_n + G) = x + G$ and $\varepsilon > 0$ be given. Since π_G is continuous at the point $W^{-1}(x+G) \in P_G^{-1}(o)$, there exist a $\delta > 0$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$d[z,W^{-1}(x+G)]<\delta \Longrightarrow d[\pi_G(z),0] = d[\pi_G(z),\pi_G(W^{-1}(x+G)] < \frac{\varepsilon}{2}$.

Consider the open ball

$$V = \left\{ z \in E:\ d[z,W^{-1}(x+G)] < \min(\delta, \tfrac{\varepsilon}{2}) \right\}.$$

Since the canonical mapping W_G is open (see [6]), the set $W_G(v)$ is open and obviously, $x+G \in W_G(V)$. Hence $x_n+G \in W_G(V)$ for all $n>N=N(\varepsilon)$ and so there exist elements $z_n \in V$ such that

$$z_n+G = W_G(z_n) = x_n+G, \quad n>N,$$

i.e. $x_n-z_n \in G$ and hence by the quasi-additivity of π_G (see [5]), we have

$$\pi_G(x_n) = \pi_G(x_n-z_n+z_n) = x_n-z_n+\pi_G(z_n).$$

Consequently, since $z_n \in V$ $(n>N)$, we obtain

$$\begin{aligned} d\left[W^{-1}(x_n+G),W^{-1}(x+G)\right] &= d\left[x_n-\pi_G(x_n),\ W^{-1}(x+G)\right] \leqslant \\ &\leqslant d\left[x_n-\pi_G(x_n),z_n\right] + d\left[z_n,W^{-1}(x+G)\right] = \\ &= d\left[\pi_G(z_n),0\right] + d\left[z_n,W^{-1}(x+G)\right] < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad n>N \end{aligned}$$

and thus W^{-1} is continuous.

Remark. The above proof is similar to one given in [7] (Theorem 4.2) for normed linear spaces.

Theorem 2. For a Chebyshev subspace G of a linear metric space (E,d), the following statements are equivalent:

(i) The metric projection π_G is continuous.

(ii) π_G is continuous at each point of $P_G^{-1}(o)$.

(iii) The direct sum decomposition $E = G \oplus P_G^{-1}(o)$ is topological (i.e. $\lim x_n = x$ if and only if $\lim \pi_G(x_n) = \pi_G(x)$ and $\lim\left[x_n-\pi_G(x_n)\right] = x-\pi_G(x)$).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) The functional $\Phi_G(x) = d(\pi_G(x).o)$, $x \in E$ is continuous.

Proof. We shall show that (i) $\Longleftrightarrow$ (ii), (i) $\Longleftrightarrow$ (iii) and (i) $\Longleftrightarrow$ (iv). (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i) Suppose (i) does not hold i.e. $x_n \longrightarrow x$ but $\pi_G(x_n) \longrightarrow \pi_G(x)$, $x_n, x \in E$. Then $x_n - \pi_G(x) \longrightarrow x - \pi_G(x) \in P_G^{-1}(o)$ but $\pi_G(x_n - \pi_G(x)) = \pi_G(x_n) - \pi_G(x) \longrightarrow 0$ contradicting (ii).

(i) $\Longleftrightarrow$ (iii) is obvious and so is (i) $\Longrightarrow$ (iv). Now we show that (iv) $\Longrightarrow$ (i).

Let $x_n \longrightarrow x \in P_G^{-1}(o)$, then $d\left[\pi_G(x_n), \pi_G(x)\right] = d\left[\pi_G(x_n), 0\right] \longrightarrow d\left[\pi_G(x), 0\right] = 0$ i.e. (iv) $\Longrightarrow$ (ii) but (ii) $\Longrightarrow$ (i). This completes the proof of Theorem 2.

Remark. The proof of Theorem 2 is a minor modification of the one given in [7] for normed linear spaces (Proposition 4.1).

The following theorem deals with the Lipschitzian metric projections in linear metric spaces. In normed linear spaces this theorem was proved by Holmes [2].

Theorem 3. For a Chebyshev subspace G of a linear metric space (E,d), the metric projection π_G is Lipschitzian if and only if the restriction $W = W_G\big|_{P_G^{-1}(o)}$ of the canonical mapping $W_G : E \longrightarrow E/G$ is a Lipschitzian homeomorphism of $P_G^{-1}(o)$ onto E/G.

Proof. For any $x+G \in E/G$, we have $x - \pi_G(x) \in P_G^{-1}(o)$ and $W\left[x - \pi_G(x)\right] = x+G$. Since

$$d\left[W_G(x), W_G(y)\right] = d(x+G, y+G) = d(x-y, G) \leq d(x-y, o) = d(x,y)$$

for all $x, y \in E$, the condition amounts to W^{-1} being Lipschitzian.

Let W^{-1} be Lipschitzian, consider

$$d\left[\pi_G(x), \pi_G(y)\right] = d\left[\pi_G(x) - x + x, \pi_G(y) - y + y\right] =$$

$$= d\left[-W^{-1}(x+G) + x, -W^{-1}(y+G) + y\right] \leq d\left[W^{-1}(y+G), W^{-1}(x+G)\right] + d(x,y) \leq$$

$$\leq d(y+G, x+G) + d(x,y) \leq 2d(x,y)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

i.e. π_G is Lipschitzian. Suppose now that π_G is Lipschitzian. Consider

$$d\left[W^{-1}(x+G),\ W^{-1}(y+G)\right] = d\left[x-\pi_G(x),\ y-\pi_G(y)\right] =$$

$$= d\left[x-\pi_G(x+g), y-\pi_G(y-g)\right] \qquad \text{for all} \quad g \in G$$

$$= d\left[x-\pi_G(x)+g, y-\pi_G(y-g)\right] \qquad \text{for all} \quad g \in G$$

$$\leq d(x+g,y)+d\left[\pi_G(x),\pi_G(y-g)\right] \qquad \text{for all} \quad g \in G$$

$$\leq d(x,y-g)+d(x,y-g) \qquad \text{for all} \quad g \in G.$$

Therefore, $d\left[W^{-1}(x+G),W^{-1}(y+G)\right] \leq 2d(x,y+G) = 2d(x+G,y+G)$ i.e. W^{-1} is Lipschitzian.

The following theorem deals with the continuity of metric projections in quotient spaces of linear metric spaces. This result in normed linear spaces was proved by Cheney and Wulbert [1].

Theorem 4. Let P be a Chebyshev subspace of a linear metric space (E,d) with continuous π_P and let Q be a subspace of E which contains P. Then the following conditions are equivalent:

(i) Q is a Chebyshev subspace of E with continuous π_Q.

(ii) Q/P is a Chebyshev subspace of E/P with continuous $\pi_{Q/P}$.

Proof. (i) $\Longrightarrow$ (ii). Since Q is proximinal in E, Q/P is proximinal in E/P (Theorem 2 [3]). Since P is proximinal in E and Q is semi-Chebyshev in E, Q/P is semi-Chebyshev in E/P (Theorem 2 [3]). Hence Q/P is a Chebyshev subspace in E/P. Now we show that $\pi_{Q/P}$ is continuous.

Let $f_n+P \longrightarrow f+P$. Put $g_n = f_n - \pi_Q f_n$ and $g = f - \pi_Q f$. Then $g_n+P \longrightarrow g+P$ and $g \in \pi_Q^{-1}(o)$. It follows that dist $(g_n-g,P) \longrightarrow 0$ i.e. $d\left[g_n-g,\pi_P(g_n-g)\right] \longrightarrow 0$ and so $g_n-\pi_P(g_n-g) \longrightarrow g$. By the continuity of π_P (hypothesis), the quasi-additivity of π_P (see [5]) and idempotency of π_P (see [5]), it follows that

$$\pi_P(g_n)-\pi_P(g_n-g) \longrightarrow \pi_P(g) = 0 \quad \text{as} \quad g = f-\pi_Q f.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Thus $g_n - \pi_P(g_n - g) \longrightarrow g$. By the continuity of π_Q,

$$\pi_Q g_n - \pi_P(g_n - g) \longrightarrow \pi_Q(g) = 0.$$

Thus dist $(\pi_Q g_n, P) \longrightarrow 0$, dist $(\pi_Q f_n - \pi_Q f, P) \longrightarrow 0$, and $\pi_Q f_n + P \longrightarrow$ $\longrightarrow \pi_Q f + P$. By Theorem 2 [3], this implies that $\pi_{Q/P}(f_n + P) \longrightarrow$ $\longrightarrow \pi_{Q/P}(f + P)$.

(ii) $\Longrightarrow$ (i) Since Q/P is Chebyshev in E/P and P is Chebyshev in E, Q is Chebyshev in E (Theorem 2[3]). Now we show that π_Q is continuous. Since $E = Q \oplus P_Q^{-1}(o)$ (Theorem 2.3 [4]), it is sufficient to prove the continuity of π_Q at the points of $P_Q^{-1}(o)$. Let $f_n \longrightarrow f \in P_Q^{-1}(o)$. Then $f_n + P \longrightarrow f + P$, and by the continuity of $\pi_{Q/P}(f_n + P) \longrightarrow \pi_{Q/P}(f + P)$. By Theorem 2 [3], $\pi_Q f_n + P \longrightarrow \pi_Q f + P = P$. Hence $f - f_n + \pi_Q f_n + P \longrightarrow P$. It follows that

$$\operatorname{dist}(f - f_n + \pi_Q f_n, P) \longrightarrow 0,$$

$$d\left[f - f_n + \pi_Q f_n,\ \pi_P(f - f_n + \pi_Q f_n)\right] \longrightarrow 0$$

$$f_n - \pi_Q f_n + \pi_P(f - f_n + \pi_Q f_n) \longrightarrow f \longrightarrow (a)$$

By the continuity of π_P, we have

$$\pi_P(f_n - \pi_Q f_n) + \pi_P(f - f_n + \pi_Q f_n) \longrightarrow \pi_P f = 0.$$

Since $\pi_P(f_n - \pi_Q f_n) = 0$, we have $\pi_P(f - f_n + \pi_Q f_n) \longrightarrow 0$. It follows from (a) that $f_n - \pi_Q f_n \longrightarrow f$, whence $\pi_Q f_n \longrightarrow 0 = \pi_Q(f)$.

Remark. The converse of Theorem 4 is not true even in normed linear spaces (see [1]).

REFERENCES

[1] E.W. Cheney, D.E. Wulbert: The existence and unicity of best approximations, Math. Scand. 24 (1969), 113-140.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] R.B. Holmes: On the continuity of best approximation operators, Proc. Symp. Infinite Dimensional Topology, Ann. Math. Studies 69, Princeton Univ. Press, Princeton (1972), 137-157.

[3] T.D. Narang: Existence and unicity of best approximation and different types of continuity of proximity maps, Bull. Calcutta Math. Soc. 69 (1977), 39-43.

[4] T.D. Narang: Best approximation in metric linear spaces, Math. Today, 5 (1987), 21-28.

[5] G. Pantelidis: Approximationstheorie für metrische linear Räume, Math. Ann., 184 (1969), 30-48.

[6] W. Rudin: Functional analysis. Tata McGraw Hill, 1974.

[7] I. Singer: Best approximation in normed linear spaces, Constructive aspects of functional analysis, Part II, Edizioni Cremonese, Roma (1973).

DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, AMRITSAR - 143005 (INDIA)

Received March 2nd, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

M. Kirane, A. Youkana

A REACTION-DIFFUSION SYSTEM MODELLING THE POST IRRIDIATION OXYDATION OF AN ISOTACTIC POLYPROPYLENE

1. Introduction

The spontaneous oxydation of isotactic polypropylene is a reaction of great interest not only in practice but also in modelling theory.

So in [2] a kinetic submodel have been studied, describing the reaction of post-irridiaton between oxygen and the most stable macro-radical alkyl, produced by irridiation in vacuum, of an isotactic polypropylene.

Here, we are concerned with the reaction-diffusion system originated from [2] which is the following weakly coupled system of nonlinear partial differential equations

$$\partial_t u_i - \mu_i \partial_{xx} u_i = F_i(U), \quad i \in I := \{1,2,3,4\}; \; x \in \Omega := (0,1), \; t \geq 0, \tag{1.i}$$

in the concentration functions u_1, u_2, u_3, u_4 with

$U = (u_1, u_2, u_3, u_4)$,

$F_1(U) = -k_1 u_1 u_2 + k_2 u_3^2$,

$F_2(U) = -k_1 u_1 u_2 - 2k_6 u_2^2 - k_5 u_3 u_2 + k_3 u_3 + k_8 u_4$,

$F_3(U) = k_1 u_1 u_2 - k_5 u_2 u_3 - (k_2 + k_3) u_3 - 2k_3 u_3^2$,

$F_4(U) = 2k_3 u_3^2 - (k_7 + k_8) u_4$.

The diffusitivities μ_i, $i \in I$, and the coefficients k_j, $j=1,\ldots,8$, are positive constants, they are factors of proportionality and measure the proportion of reactional collisions.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The concentrations u_i, $i \in I$, satisfy the boundary conditions

$$(1.5)\qquad \begin{cases} u_i(t,0) = u_i(t,1) = 0, & t>0 \quad \text{for} \quad i=2,3,4, \\ \partial_x u_2(t,0) = \partial_x u_2(t,1) = 0, & t>0, \end{cases}$$

and the initial data

$$(1.6)\qquad u_i(0,x) = \phi_i(x), \qquad x \in \bar{\Omega}, \qquad i \in I,$$

where $\phi_i(0) = \phi_i(1) = 0$, $i = 1,2,3$, and $\phi_2'(0) = \phi_2'(1) = 0$.

Given are the initial concentrations $\phi_i(x)$, $i \in I$, all of which are assumed to be nonnegative and continuous on $[0,1]$.

Our aim will be to establish the existence of global bounded solutions to problem (1.1)-(1.6) and analyze their large time behaviour by methods of functional analysis, specifically the theory of semigroups of linear operators in Banach spaces. We shall prove that for problem (1.1)-(1.6) there exists a unique nonnegative solution which goes to 0, uniformly in every closed interval contained in Ω, for each $i \in I$. Moreover, we shall show that there exist a positive T large enough, positive constants $K_i > 0$, $i = 1,3,4$, and α, C_0, C_1 with $C_0 < C_1$ such that for any $t \geq T$, we have

$$\|u_i\|_\infty \leq K_1 \exp(-\alpha t) \quad \text{for} \quad i=1,3,4,$$

$$C_0/t \leq \|u_2\|_\infty \leq C_1/t.$$

It is worth noting that the problem (1.1)-(1.6) can be analyzed neither in the framework of the theory of monotonic or quasimonotonic systems nor by the techniques of comparison [1], [4], [8], [12].

2. Notations and preliminary results

In order to study the problem (1.1)-(1.6), we introduce the Banach space $X := C([0,1],R^4)$ (of the functions $u = (u_1,u_2,u_3,u_4)^T$ (T for transpose) which are continuous in $[0,1]$), endowed with the norm

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|U\| = \sum_{j=1}^{4} \sup|u_j(x)| \quad \text{for} \quad x \in [0,1].$$

Let A be defined as the linear operator $D\partial_{xx} = \mathrm{diag}(\mu_1,\mu_2,\mu_3,\mu_4)\partial_{xx}$ with domain

$$D(A) := \Big\{U \in X \,\Big|\, D\partial_{xx}U \in X \quad \text{and} \quad u_i(t,0) = u_i(t,1) = 0,\ i=1,3,4,$$

$$\partial_x u_2(t,0) = \partial_x u_2(t,1) = 0\Big\}$$

and let F denote the nonlinear operator

$$F(U) = (F_1(U),F_2(U),F_3(U),F_4(U))^T \in X \text{ for } U \in X,$$

so (1.1)-(1.6) can be written in the abstract form

$$d/dt(U(t)) = AU(t) + F(U(T)), \tag{2.1}$$

$$U(0) = \phi \in X_+, \tag{2.2}$$

where X_+ denotes the set of nonnegative functions from X.

By Theorem 1.5 of [10] we can show that a unique local classical solution of problem (2.1)-(2.2) exists; by this we mean that

$$U \in (C^{2,1}(\Omega x (0,T),R) \cap C^{1,0}(\bar{\Omega} x (0,T),R))^4$$

and satisfies system (1.1)-(1.6). This can be proved using general results concerning the semigroup theory because each operator $\mu_i\partial_{xx}$ generates an analytic semigroup on $C(\bar{\Omega})$.

Moreover, we have the following alternative:

(Alt.) Either $T = \infty$,

or $T < \infty$ and $\lim_{t \to \infty} \|U\| = \infty$.

An important fact is that $U(t,x)$ remains nonnegative, if $\phi_i(x) \geq 0$, $i \in I$.

Lemma 2.1. Let U be a local solution to (1.1)-(1.6). If $\phi_i(x) \geq 0$, $i \in I$, then $u_i \geq 0$, $i \in I$, on $[0,T) \times \Omega$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Observe first that u_4 satisfies the inequality

$$\partial_t u_4 - \mu_4 \partial_{xx} u_4 + (k_7 + k_8)u_4 = 2k_3 u_3^2 \geq 0$$

as $\phi_4 \geq 0$, by the maximum principle $u_4 \geq 0$.

Now we restrict ourselves to $t \in [0, T_0]$ for a fixed $T_0 < T$. By the local result, we can choose a constant $C = C(T_0)$ such that $p(t,x) = C + k_1 u_1(t,x) \geq 0$, then we can write

$$\partial_t(\exp(-Ct)u_1) - \mu_1 \partial_{xx}(\exp(-Ct)u_1) + p(t,x)(\exp(-Ct)u_1) = k_3 \exp(-Ct)u_3^2 \geq 0$$

so, again by the maximum principle $u_1 \geq 0$.

For the positivity of u_2 and u_4, we multiply (1.2) by $u_2(t)^- = (|u_2(t)| - u_2(t))/2$ and (1.3) by $u_3(t)^-$ and integrating over Ω to obtain

$$\int_\Omega u_2' u_2^- = \mu_2 \int_\Omega \partial_{xx} u_2 u_2^- - k_1 \int_\Omega u_1 u_2 u_2^- - 2k_6 \int_\Omega u_2^2 u_2^- - k_5 \int_\Omega u_2 u_3 u_2^- +$$

$$+ k_3 \int_\Omega u_3 u_2^- + k_8 \int_\Omega u_4 u_2^-$$

and

$$\int_\Omega u_3' u_3^- = \mu_3 \int_\Omega \partial_{xx} u_3 u_3^- + \int_\Omega u_1 u_2 u_3^- - k_5 \int_\Omega u_2 u_3 u_3^- -$$

$$- (k_2 + 2k_4) \int_\Omega u_3 u_3^- - 2k_3 \int_\Omega u_3^2 u_3^-,$$

where we have suppressed the dependence on t.

By the local existence, we clearly have

$$u_3 \in C^1((0,T); L^2(\Omega)) \quad \text{and} \quad u_3 \in C((0,T); H_0^1(\Omega)).$$

Thus $\int_\Omega u_3' u_3^- = -\frac{1}{2} d/dt \int_\Omega (u_3^-)^2$ and $\int_\Omega \partial_{xx} u_3 u_3^- = \int_\Omega |\partial_x u_3^-|^2$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The first formula above follows from the definition of u_3^- (multiply out $(u_3^-)^2$) and the second formula from well-known facts about $\partial_x u_3^-$ ([5], section 7.4). Once again we restrict ourselves to T_0. By the local existence, there exists a constant $C = C(T_0)$ such that

$$d/dt \int_\Omega (u_2(t)^-)^2 + (u_3(t)^-)^2 \leq 2C \int_\Omega (u_2(t)^-)^2 + (u_3(t)^-)^2$$

it follows that

$$\int_\Omega (u_2(t)^-)^2 + (u_3(t)^-)^2 \leq \int_\Omega (\phi_2^-)^2 + (\phi_3^-)^2.$$

By assumption, $\phi_2 \geq 0$ and $\phi_3 \geq 0$, so $\phi_2^- = 0$ and $\phi_3^- = 0$ for a.e. $x \in \Omega$ and $u_2 \geq 0$, $u_3 \geq 0$ for a.e. $x \in \Omega$. Since $T_0 > 0$ is arbitrary, the result is true for all $t \in [0, T)$.

3. Global existence and precompactness of trajectories

First, let us establish the following preparatory lemma.

Lemma 3.1. Denoting by $\| \ \|_1$ the $L^1(\Omega)$ norm, we have

(3.1) $$\|u_i(t)\|_1 \leq C, \quad i \in I,$$

(3.2) $$\int_0^t \|u_j(t)\|_1 \, d\sigma \leq C, \quad j = 3,4.$$

Proof. Integrating (1.1) and (1.3) over $Q_t = \Omega \times (0,t)$ and adding up, we find

$$\int_\Omega (u_1 + u_3) + k_3 \int_{Q_t} u_3^2 + (k_5 + 2k_4) \int_{Q_t} u_3 \leq -k_5 \int_{Q_t} u_2 u_3 + \int_\Omega (\phi_1 + \phi_3) \leq C,$$

hence, in particular, (3.1) for $i = 1,3$ and (3.2) for $j = 3$. Moreover, we have

(3.3) $$\int_{Q_t} u_3^2 \leq C.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On the other hand, integrating (1.1) over Q_t, we have

$$\int_\Omega u_1 + k_1 \int_{Q_t} u_1 u_2 \leq \int_\Omega \phi_1 + k_3 \int_{Q_t} u_3^2 \leq C,$$

hence, in particular

$$\int_{Q_t} u_1 u_2 \leq C. \tag{3.4}$$

Now, integrating (1.3) over Q_t, we get

$$\int_\Omega u_3 + (k_2 + k_4) \int_{Q_t} u_3 \leq \int_\Omega \phi_3 + k_1 \int_{Q_t} u_1 u_2,$$

hence, by (3.4)

$$\int_{Q_t} u_3 \leq C. \tag{3.5}$$

Finally, adding up (1.2) and (1.4) and integrating over Q_t, we find

$$\int_\Omega (u_2 + u_4) + k_7 \int_{Q_t} u_4 \leq \int_\Omega (\phi_2 + \phi_4) + k_3 \int_{Q_t} u_3 + 2k_3 \int_{Q_t} u_3^2.$$

Using (3.3) and (3.5), we deduce

$$\int_\Omega (u_2 + u_4) + k_7 \int_{Q_t} u_4 \leq C,$$

so, in particular (3.1) for i = 2, 4 and (3.2) for j = 4.

Theorem 3.2. Let U be a local solution of (1.1)-(1.6). Then we have

$$u_i \in C_B(R^+; C(\bar{\Omega})), \quad i \in I, \tag{3.6}$$

where $C_B(R^+; C(\bar{\Omega}))$ is the space of continuous bounded functions from R^+ to $C(\bar{\Omega})$.

Proof. We are going to show only how to obtain $u_2 \in C_B(R^+; C(\bar{\Omega}))$, because the same analysis works to obtain $u_i \in C_B(R^+; C(\bar{\Omega}))$, i=1,3,4. With $w = u_2$ and $\mu = \mu_2$, the equation (1.2) can be written in the following form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\partial_t w - (\mu \partial_{xx} - b)w = M(t,x), \tag{3.7}$$

where $M(t,x) = bu_2(t,x) + F_2(U(t,x))$, and $w(0,x) = \phi_2(x)$. Without loss of generality it may be assumed that $\phi_2 \in C^2(\bar{\Omega})$, for otherwise the initial value problem starting at $t = \delta > 0$ may be considered. There is a constant $C > 0$ such that

$$\|M(t)\|_\infty \leq C, \quad t \geq 0.$$

By [11, p.88], for some b, under homogeneous boundary conditions ∂_{xx}-bI generates an analytic semigroup in $L^p(\bar{\Omega})$ for $p > 1$, and with the associated operator A, there is a $\delta > 0$ such that $\mathrm{Re}\,\sigma(A) > \delta$ (where $\sigma(A)$ denotes the spectrum of A).

The integral solution of (3.7) is

$$w(t) = \exp(-At)\phi_0 + \int_0^t \exp(-A(t-s))M(s)ds. \tag{3.8}$$

From [7, p.26], for $\sigma > 0$, we have

$$\|A^\sigma \exp(-At)\| \leq C(\sigma)t^{-\sigma}\exp(-\delta t). \tag{3.9}$$

Taking some $\sigma \in (0,1)$ and applying A^σ to (3.8), we obtain

$$\|A^\sigma w(t)\|_\infty \leq \|A^\sigma \exp(-At)\|_\infty \|\phi_2\|_\infty + \int_0^t \|A^\sigma \exp(-A(t-s))\|_\infty \, ds \leq$$

$$\leq C(\sigma)t^{-\sigma}\exp(-\delta t)\|\phi_2\|_\infty + \max_{0\leq s\leq t} \|M(s)\|_\infty \int_0^t (t-s)^{-\sigma}\exp(-\delta(t-s))ds,$$

by (3.9). It follows from the bounds on ϕ_2 and M that there is a constant C such that

$$\|A^\sigma w(t)\|_\infty \leq C, \quad (t \geq 1). \tag{3.10}$$

From the definition of the fractional space $(L^p)^\sigma$ [7, p.29], and a standard imbedding theorem [7, p.39], we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.11) $$\|A^{\sigma} w(t)\| = \|w(t)\|_{(L^p)^{\sigma}},$$

(3.12) $$\|w(t)\|_{C^{\nu}} \leq k\|w(t)\|_{(L^p)^{\sigma}}, \quad 0 \leq \nu \leq 2\alpha - 1/p,$$

where k is independent of w. Taking $\alpha = 1/2$ and $p = 1$, $\nu = 0$ we obtain the result on combining (3.10), (3.11) and (3.12). By the alternative (Alt.), not only the solution exists for all $t > 0$ but also is bounded.

4. Large time behaviour

Lemma 4.1. If $\phi \in C^1(R^+)$, $\phi \geq 0$ is such that

$$\int_0^{\infty} \phi(t)\, dt < \infty \quad \text{and} \quad d/dt(\phi) \leq C < \infty, \ t \geq 0,$$

then $\lim_{t \to \infty} \phi(t) = 0$.

Proof. Let $\varepsilon > 0$, then there is $M = M(\varepsilon) > 0$ such that $\int_M^{\infty} \phi(t)dt \leq \varepsilon$. Let t and $t_e \geq M$ with $t > t_e$. Integrating by parts, we get

$$\int_{t_e}^{t} (s - t_e)\, d/ds(\phi) ds = (t - t_e)\, \phi(t) - \int_{t_e}^{t} \phi(s)\, ds,$$

hence

(*) $$(t - t_e)\, \phi(t) = \int_{t_e}^{t} \phi(s)\, ds + \int_{t_e}^{t} (s - t_e)\, d/ds(\phi)\, ds \leq$$

$$\leq \int_{t_e}^{t} \phi(s)\, ds + \frac{C}{2}(t - t_e)^2,$$

where $C > 0$ is a constant such that $d/ds\, \phi \leq C$. From (*) and choosing t_e such that $t - t_e = \varepsilon^{1/2}$ we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\phi(t) \leq \frac{1}{\sqrt{\varepsilon}} \int_M^\infty \phi(s)ds + \frac{C}{2}\frac{1}{\sqrt{\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}}\left(1 + \frac{C}{2}\right) \tag{**}$$

hence, $\varepsilon > 0$ given, there is $T_M = M + \frac{1}{\sqrt{\varepsilon}}$ such that (**) holds as soon as $t = t_e + \frac{1}{\sqrt{\varepsilon}} \geq M + \frac{1}{\sqrt{\varepsilon}}$.

Theorem 4.1. There holds $\lim_{t\to\infty} \|u_i(t)\|_\infty = 0$, $i \in I$.

Proof. Integrating (1.3) and (1.4) over Ω we get

$$d/dt \int_\Omega u_3 \leq k_1 \int_\Omega u_1 u_2 \quad \text{and} \quad d/dt \int_\Omega u_4 \leq 2k_3 \int_\Omega u_3^2.$$

As U is bounded, we can write

$$d/dt \int_\Omega u_3 \leq C \quad \text{and} \quad d/dt \int_\Omega u_4 \leq C.$$

On the other hand, from Lemma 3.1,

$$\int_\Omega u_3 \in L^1(R^+) \quad \text{and} \quad \int_\Omega u_4 \in L^1(R^+),$$

so, using Lemma 4.2, we obtain

$$\lim_{t\to\infty} \|u_j(t)\|_1 = 0, \quad j = 3,4. \tag{4.1}$$

Now multiplying (1.2) by u_2, integrating the result over Ω, we get

$$d/dt\left(\int_\Omega u_2^2\right) + 2\mu_2 \int_\Omega (\partial_x u_2)^2 \leq 2\int_\Omega (k_3 u_3 + k_8 u_4) u_2,$$

hence

$$d/dt\left(\int_\Omega u_2^2\right) \leq C.$$

On the other hand, integrating (1.2) over $\Omega \times (0,t)$ we can state that $\int_\Omega u_2^2(t) \in L^1(R^+)$. So, by Lemma 4.2, we find

$$\lim_{t\to\infty} \|u_2(t)\|_2 = 0. \tag{4.2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Finally, multiplying (1.1) by u_1 and integrating the result over Ω, we have

$$(4.3)\quad d/dt\left(\int_\Omega u_1^2\right) + 2\mu_1\int_\Omega (\partial_x u_1)^2 \le 2k_3\int_\Omega u_3^2 u_1 \le C_\varepsilon \int_\Omega u_3^4 + \varepsilon\int_\Omega u_1^2.$$

Using Poincaré's inequality, choosing ε, and integrating over $(0,t)$, we obtain by (3.3)

$$\int_\Omega u_1^2 + C_0\int_0^t\int_\Omega u_1^2 \le C\int_0^t\int_\Omega u_3^2 + \int_\Omega \phi_1^2 \le C.$$

Hence

$$\int_\Omega u_1^2 \in L^1(R^+).$$

Now, from (4.3), we have $d/dt\int_\Omega u_1^2 \le C$. So, by Lemma 4.2 we get

$$(4.4)\quad \lim_{t\to\infty} \|u_1(t)\|_2 = 0.$$

Finally, as the trajectories $\bigcup_{t>0}\{u_i(t)\}$, $i\in I$ are precompact in $C(\bar\Omega)$, so (4.1), (4.2) and (4.4) lead to

$$\lim_{t\to 0} \|u_i(t)\|_\infty = 0, \quad i\in I.$$

Theorem 4.2. Let U be a solution of (1.1)-(1.6). Then there exist strict positive constants $T, \alpha, K_i, i\in I, C_0, C_i$ with $C_0 < C_i$ such that for any $t \ge T$ we have $\|u_i(t)\|_\infty \le K_i \exp(-\alpha t)$, $i = 1,3,4$ and $C_0/t \le \|u_2(t)\|_\infty \le C_1/t$.

Proof. Let

$$A_i := \mu_i\partial_{xx}, \quad i = 1,2,$$

$$A_3 := \mu_3\partial_{xx} - (k_3+2k_4)\cdot Id =: \mu_3\partial_{xx} - \lambda_3 Id,$$

$$A_4 := \mu_4\partial_{xx} - (k_7+k_8)\cdot Id =: \mu_4\partial_{xx} - \lambda_4\cdot Id,$$

where Id := identity. It is simple to show for any $t \ge 0$ the following estimates

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|\exp(t A_1)\| \leq \exp(-(\pi/l)^2 t),$$

$$\|\exp(t A_2)\| \leq 1,$$

$$\|\exp(t A_i)\| \leq \exp(-\lambda_i t), \quad \text{for} \quad i=3,4.$$

So, if we set $\delta = \text{Min}\left\{(\pi/l)^2, \lambda_3, \lambda_4\right\} > 0$ we will have

$$\|\exp(t A_i)\| \leq \exp(-\delta t), \quad i = 1,3,4, \quad \text{for any } t \geq 0. \tag{4.5}$$

On the other hand, by Theorem 4.1, we state that for any $\varepsilon > 0$ there exists $T > 0$, such that for any $t \geq T$, we have

$$\|u_i(t)\|_\infty \leq \epsilon, \quad i \in I. \tag{4.6}$$

Using (4.6), the positivity of semigroups $\exp(t A_i)$, $i \in I$, and the integral formulation of the solution, we have for any $t \geq 0$

$$u_1(t) \leq \exp((t-T)A_1)\, u_1(T) + \varepsilon k_3 \int_T^t \exp((t-\sigma)A_1) u_3(\sigma) d\sigma,$$

$$u_3(t) \leq \exp((t-T)A_3)\, u_3(T) + \varepsilon k_1 \int_T^t \exp((t-\sigma)A_3) u_1(\sigma) d\sigma,$$

Now, setting $\psi(t) := \|u_1(t)\|_\infty + \|u_3(t)\|_\infty$ and using (4.5), we have

$$\exp(\delta t)\psi(t) \leq \psi(T) \exp(\delta T) + \varepsilon(k_1 + k_3) \int_T^t \exp(\delta\sigma)\psi(\sigma) d\sigma.$$

By Gronwall's lemma, we have

$$\psi(t) \leq \psi(T) \exp(-(t-T)(\delta - \varepsilon(k_1 + k_2))).$$

Hence, for any $t \geq T$

$$\|u_i(t)\|_\infty \leq K_i \exp(-\alpha t), \quad i = 1,3, \tag{4.7}$$

where we set

$$\alpha = \delta/2, \quad K_1 = K_3 = \exp((\delta - \varepsilon(k_1 + k_3))T)\psi(T), \quad \varepsilon = \delta(k_1+k_3)/2.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On the other hand, using (4.5), (4.6) and (4.7) yields

$$\|u_4(t)\|_\infty \leq \exp(\delta(t-T))\|u_4(T)\|_\infty + 2k_3k_1\int_T^t \exp(-\delta(t-\sigma/2))d\ .$$

Hence, for any $t \geq T$,

$$\|u_4(t)\|_\infty \leq K_4 \exp(-\alpha t). \tag{4.8}$$

Finally, for any $t \geq T$, there holds $\partial_t u_2 - \mu_2 \partial_{xx} u_2 \geq -2k_6 u_2^2$, so, u_2 is a sup-solution of

$$(d/dt)v(t) = -2k_6 v(t)^2, \quad v(T) = \min_x u_2(T,x).$$

A simple integration, yields $u_2(t,x) \geq v(t) \geq C_0/t$.

On the other hand, multiplying (1.2) by u_2^m, and integrating over Ω, one can find a constant $\gamma > 0$ such that

$$(1/m)x\left\{(d/dt\,\phi(t))/\phi(t)^{(m+1)/m}\right\} \leq -\gamma,$$

where we set $\phi(t) = \int_0^t u^m(t,x)dx$. Integrating this inequality over (t,T), one can find a positive constant C_1 such that $\|u_2(t)\|_\infty \leq C_1/t$ for any $t \geq T > 0$. Hence, we have found constants $T \geq 0$, $C_1 \geq C_0 \geq 0$ such that $C_0/t \leq \|u_2(t)\|_\infty \leq C_1/t$ for any $t \geq T$.

Remarks:

1) If instead of $F_2(U)$, we have $F_2(U) - a\,u_2$, for example, with $a > 0$, then we would have $\|u_2(t)\|_\infty \leq K_4 \exp(-\alpha t)$ for any $t \geq T$ and for certain positive constants K_4, α.

2) It will be interesting to have the same result for a domain Ω in R^2 or in R^3.

3) It will be interesting to have an estimate of the form $\|u_i - \beta_i\|_{L^\infty((0,+\infty;L^2(\Omega))} \leq C(\mu_i)$, $i = 1,2,3,4$, where β_i, $i \in I$, are the solutions of the corresponding ordinary differential system.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] H. Amman: Invariant sets and existence theorems for semilinear parabolic and elliptic systems. J. Math. Anal. Appl. 65 (1978) 432-467.

[2] A. Faucitano, A. Buttafava, F. Martinotti, V. Comincioli: Kinetic modelling and ESR study of solid state chemical reactions: The post irridiation oxydation of isotactic polypropylene (to appear).

[3] K. Deimling, V. Lakshmikhanthan: Quasi-solutions and their role in the theory of differential equation, Nonlinear Analysis T M. A. 4 (1980), 657-663.

[5] D. Gilbarg, N. Trudinger: Elliptic partial differential equation of second order, Springer-Verlag, Berlin (1977).

[6] A. Haraux, M. Kirane: Estimations C^1 pour des problèmes paraboliques semilinéaires, Ann. Fac. Sci. Toulouse, 5 (1983). 265-280.

[7] D. Henry: Geometric theory of semi-linear parabolic equations, Lect. Notes in Math. 842, Springer-Verlag (1981).

[8] J. Hernandez: Some existence and stability for reaction-diffusion systems with nonlinear boundary conditions in nonlinear differential equations: Invariance, Stability and Bifurcation, de Mottoni P. and Salvadori L. eds. New York, Academic Press (1981), 161-174.

[9] C.V. Pao: On nonlinear reaction-diffusion systems. J. Math. Anal. Appl. 87 (1982), 165-198.

[10] J. Stakgold: Reaction-diffusion problems in chemical engineering. Lect. Notes in Math. 1224, Springer-Verlag (1986).

[11] H. Tanabe: Equations of evolution. Pitman, London, (1979).

[12] W. Walter: Differential and integral inequalities. Springer-Verlag, Berlin-Heidelberg, (1970).

UNIVERSITÉ DE ANNABA, INSTITUT DE MATHÉMATIQUES, P.P.12, EL-HADJAR, ANNABA. ALGERIA;
UNIVERSITÉ PIERRE ET MARIE CURIE, LABORATOIRE D'ANALYSE NUMÉRIQUE (LA 189), 75230 PARIS cedex 05, FRANCE.

Received July 13, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Shyam L. Singh, Umesh C. Gairola

A COINCIDENCE THEOREM FOR THREE SYSTEMS OF TRANSFORMATIONS

With a view to generalizing the Banach contraction principle, Matkowski [5]-[6] proved a fixed point theorem (see Remark 1) for a system of n transformations, on a product of n metric spaces. This result has been extended and generalized by Czerwik [1]-[2], Reddy-Subrahmanyam [7] and Singh-Kulshrestha [9].

We follow the notations of Matkowski [4] and Czerwik [2]:

$$(1)\qquad c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for } i \neq k, \quad i,k = 1,\dots,n, \\ 1-a_{ik} & \text{for } i = k, \end{cases}$$

and $c_{ik}^{(t)}$ are defined recursively by

$$(2)\qquad c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)}\, c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)}\, c_{1,k+1}^{(t)}, & \text{for } i \neq k, \\ c_{11}^{(t)}\, c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)}\, c_{1,k+1}^{(t)}, & \text{for } i = k, \\ i,k=1,\dots,n-t-1, \quad t=0,1,\dots,n-2. \end{cases}$$

The following lemma is essentially due to Matkowski [5] (see also [2], [6] and [9]).

Lemma. Let $c_{ik}^{(0)} > 0$ for $i,k=1,\dots,n$, $n \geq 2$. The system of inequalities

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $$\sum_{k=1}^{n} a_{ik} r_k < r_i, \quad i=1,2,\ldots,n,$$

has a positive solution $r_1, r_2, \ldots, r_n$ if and only if the following inequalities hold:

(4) $$c_{ii}^{(t)} > 0, \quad i=1,2,\ldots,n-1, \ t=0,\ldots,n-1.$$

Let $A_1, A_2, \ldots, A_n$ be arbitrary (nonempty) sets, $A := A_1 \times A_2 \times \ldots \times A_n$ and $X_1, X_2, \ldots, X_n$ metric spaces (with metrics d_i, respectively). Assume that

(5) P_i, Q_i, T_i are transformations defined on the set A with values in X_i, $i=1,2,\ldots,n$;

(6) $P_i(A) \cup Q_i(A) \subset T_i(A)$, and $(T_i(A), d_i)$ are complete subspaces of X_i, $i=1,\ldots,n$;

(7) $$d_i(P_i(x_1,\ldots,x_n), Q_i(y_1,\ldots,y_n)) \leq$$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(T_k(x_1,\ldots,x_n), T_k(y_1,\ldots,y_n)) +$$

$$+ b\left[d_i(T_i(x_1,\ldots,x_n), P_i(x_1,\ldots,x_n)) + \right.$$

$$\left. + d_i(T_i(y_1,\ldots,y_n), Q_i(y_1,\ldots,y_n))\right]$$

for all $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)$ in A and some nonnegative number b.

The following is the main result of this paper.

Theorem. If there exist nonnegative numbers b and a_{ik}, $i,k = 1,2,\ldots,n$ such that (1)-(7), where

(8) $$0 \leq 2b < 1 - \lambda,$$

in which $$\lambda = \max_i \left(r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right),$$

then the system of equations

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(9) $$P_i(x_1,\ldots,x_n) = Q_i(x_1,\ldots,x_n) = T_i(x_1,\ldots,x_n)$$

has a solution in A. Moreover, if

$$P_i(v_1,\ldots,v_n) = Q_i(v_1,\ldots,v_n) = T_i(v_1,\ldots,v_n) := u_i$$

and $P_i(\bar{v}_1,\ldots,\bar{v}_n) = Q_i(\bar{v}_1,\ldots,\bar{v}_n) = T_i(\bar{v}_1,\ldots,\bar{v}_n) := \bar{u}_i$

then $u_i = \bar{u}_i$.

Proof. First we note that, in view of the system of inequalities (3), λ defined in (8) exists and $0<\lambda<1$ (cf. (Czerwik [2]). From the lemma and (8), we may choose positive numbers $r_1, r_2, \ldots, r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \leq \lambda r_i, \quad i=1,\ldots,n.$$

Pick x_i^0 in A_i, i=1,...,n. Taking a clue from [8], we construct sequences $\{x_i^m\}$ in A_i and $\{z_i^m\}$ in $T_i(A) \subset X_i$, i=1,...,n such that

$$P_i(x_1^{2m},\ldots,x_n^{2m}) = T_i(x_1^{2m+1},\ldots,x_n^{2m+1}) = z_i^{2m+1}$$

and

$$Q_i(x_1^{2m+1},\ldots,x_n^{2m+1}) = T_i(x_1^{2m+2},\ldots,x_n^{2m+2}) = z_i^{2m+2},$$

$m = 0,1,2,\ldots$. We can do this since $P_i(A) \subset T_i(A)$ and $Q_i(A) \subset T_i(A)$.

We may assume that

$$d_i(z_i^0, z_i^1) \leq r_i, \quad r_i \geq 1, \quad i=1,\ldots,n.$$

From (7),

$$d_i(z_i^1, z_i^2) = d_i(P_i(x_1^0,\ldots,x_n^0),\ Q_i(x_1^1,\ldots,x_n^1)) \leq$$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(T_k(x_1^0,\ldots,x_n^0),\ T_k(x_1^1,\ldots,x_n^1)) +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ b\left[d_i(T_i(x_1^0,\ldots,x_n^0),\ p_i(x_1^0,\ldots,x_n^0)) + \right.$$

$$\left. + d_i(T_i(x_1^1,\ldots,x_n^1),\ Q_i(x_1^1,\ldots,x_n^1))\right] =$$

$$= \sum_{k=1}^{n} a_{ik} d_k(z_k^0, z_k^1) + b d_i(z_i^0, z_i^1) + b d_i(z_i^1, z_i^2) \leq$$

$$\leq \sum_{k=1}^{n} a_{ik} r_k + b r_i + b d(z_i^1, z_i^2).$$

So

$$d_i(z_i^1, z_i^2) \leq q r_i$$

wherein $0 < q = (\lambda+b)/(1-b) < 1$. Similarly

$$d_i(z_i^2, z_i^3) \leq q^2 r_i.$$

Inductively

$$d_i(z_i^m, z_i^{m+1}) \leq q^m r_i, \qquad m=0,1,2,\ldots$$

This implies that $\left\{z_i^m\right\}$ is a Cauchy sequence and has a limit in $T_i(A)$, $i=1,\ldots,n$. Call it u_i, $i=1,\ldots,n$.

Now let $(v_1,v_2,\ldots,v_n)$ be a point in $T_i^{-1}u_i$. Thus $T_i(v_1,\ldots,v_n) = u_i$. Then from (7),

$$d_i(P_i(v_1,\ldots,x_n),\ Q_i(x_1^{2m+1},\ldots,x_n^{2m+1}) \leq$$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(u_k, z_k^{2m+2}) +$$

$$+ b\left[d_i(u_i,\ P(v_1,\ldots,v_n) + d_i(z_i^{2m+2}, z_i^{2m+3})\right].$$

Letting $m \to \infty$ yields

$$d_i(P_i(v_1,\ldots,v_n), u_i) \leq b d_i(u_i, P(v_1,\ldots,v_n),$$

that is $P_i(v_1,\ldots,v_n) = u_i$, $i=1,\ldots,n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Similarly $Q_i(v_1,\ldots,v_n) = u_i$, $i=1,\ldots,n$. This proved the existence of a solution of the system (9) in A.

To prove the last part of the theorem, let $(\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_n)$ in A be another solution of the system of equations (9), that is $P_i(\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_n) = Q_i(\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_n) = T_i(\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_n) = \bar{u}_i$, say. Then, since we can assume $d_i(u_i,\bar{u}_i) \leq r_i$, $i=1,\ldots,n$, we have by (7),

$$d_i(u_i,\bar{u}_i) = d_i(P_i(v_1,\ldots,v_n),\ Q_i(\bar{v}_1,\ldots,\bar{v}_n)) \leq$$

$$\leq \sum_{k=1}^{n} a_{ik} r_k \leq \lambda r_i .$$

Inductively

$$d_i(u_i,\bar{u}_i) \leq \lambda^m r_i .$$

Letting $m \to \infty$ gives $u_i = \bar{u}_i$. This completes the proof of the theorem.

Remarks: 1. Matkowski's fixed point theorem [5] is obtained as a corollary from the above theorem when $b = 0$, $A_i = X_i$, $P_i = Q_i$ and $T_i(x_1,x_2,\ldots,x_n) = x_i$ for each x_i in X_i, $i=1,\ldots,n$.

2. Kominek's coincidence theorem [4, Th.1] generalizing the coincidence theorem of Goebel [3] and Matkowski's theorem is obtained from the above theorem when $b = 0$ and $P_i = Q_i$, $i=1,\ldots,n$. Indeed, Kominek's theorem is proved using Matkowski's theorem while our approach is constructive and provides a method to construct a sequence converging to the coincidence value.

3. A fixed point theorem for two systems of transformations due to Singh-Kulshrestha [9] is obtained from our theorem when $A_i = X_i$ and $T_i(x_1,\ldots,x_n) = x_i$ for each x_i in X_i, $i=1,\ldots,n$.

REFERENCES

[1] S. Czerwik: Generalization of Edelstein's fixed point theorem, Demonstratio Math. 9 (1976), 281-285.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] S. Czerwik: A fixed point theorem for a system of multivalued transformations, Proc. Amer. Math. Soc. 55 (1976), 136-139.

[3] K. Goebel: A coincidence theorem, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 16 (1968), 733-735.

[4] Z. Kominek: A generalization of K. Goebel's and J. Matkowski's theorems, Univ. Śląski w Katowicach, Prace Nauk.-Prace Mat. No. 12(1982), 30-33.

[5] J. Matkowski: Some inequalities and a generalization of Banach's principle, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 21(1973), 323-324.

[6] J. Matkowski: Integrable solutions of functional equations, Dissertationes Math. Vol. 127 (Rozprawy), Warszawa, 1975.

[7] K.B. Reddy, P.V. Subrahmanyam: Extensions of Krasnoselskii's and Matkowski's fixed point theorems, Funcial. Ekv. 24(1981), 67-83.

[8] S.L. Singh: Some common fixed point theorems in L-spaces, Math. Sem. Notes Kobe Univ. 7(1979), 91-97.

[9] S.L. Singh, C. Kulshrestha: A common fixed point theorem for two systems of transformations, Pusan. Kyŏ. Math. J. 2 (1986), 1-8.

DEPARTMENT OF MATHEMATICS, GURUKUL KANGRI UNIVERSITY, HARDWAR 249404 INDIA

Received July 13, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Heinrich Begehr, Chen Xiu-qiu

MOMENT PROBLEM IN $\mathbb{R}^n$

By a counterexample from M. Sakai the classical moment problem in the complex plane $\mathbb{C}$ is known not to be uniquely solvable. The same construction is used here to show that this is true for the moment problem in $\mathbb{R}^n$ for any $n \geq 3$.

1. Moments in $\mathbb{R}^n$

The moments of a bounded domain D in the complex plane $\mathbb{C}$ defined by

$$m_k := \int_D z^k \, dxdy \qquad (k \in \mathbb{N}_0) \tag{1}$$

appear as the coefficients of the Taylor expansion of

$$\int_D \frac{d\xi d\eta}{z-\zeta} = \overline{\int_D \frac{z-\zeta}{|z-\zeta|^2} \, d\xi d\eta} \qquad (\zeta = \xi + i\eta) \tag{2}$$

at infinity. An analogue to this function in $\mathbb{R}^n$ is

$$\int_D \frac{x-y}{|x-y|^n} \, dy = \begin{cases} \nabla \frac{1}{2-n} \int_D |x-y|^{2-n} \, dy, & 3 \leq n, \\ \nabla \int_D \log |x-y| \, dy, & 2 = n. \end{cases} \tag{3}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the following only the case $3 \leq n$ will be considered. The coefficients of the power series development at infinity of the integral on the right-hand side of (3) are defined as the moments of D, see [1]. They are expressed by the Gegenbauer polynomials $C_k^{(\frac{n}{2}-1)}$ as

$$(4)\quad M_k(\xi) = \frac{n+k-2}{n-2} \int_D C_k^{(\frac{n}{2}-1)}\left(\xi \cdot \frac{}{|y|}\right)|y|^k \, dy \quad (k \in \mathbb{N}_0,\ \xi \in \mathbb{R}^n,\ |\xi| = 1)$$

which is suggested by

$$(5)\quad |x-y|^{2-n} = |x|^{2-n}\left(1 - 2\frac{x}{|x|}\cdot\frac{y}{|y|}\frac{|y|}{|x|} + \frac{|y|^2}{|x|^2}\right)^{1-\frac{n}{2}} =$$

$$= \sum_{k=0}^{\infty} C_k^{(\frac{n}{2}-1)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)|y|^k |x|^{2-n-k}.$$

The infinite series is convergent for $|y| < |x|$.

Moment problem. For a given sequence $(M_k(\xi))$ of real-valued functions on $|\xi| = 1$ find a bounded domain D in $\mathbb{R}^n$ with the moments $(M_k(\xi))$.

In the complex plane this problem was posed by H.S. Shapiro (see [4], p.193) and for domains with analytic boundaries under additional assumptions shown to be uniquely solvable by e.g. S. Richardson [2]. Later M. Sakai [3] gave an example of two different domains with the same moments.

2. Counterexample

Theorem. The moment problem is general is not uniquely solvable.

Proof. Let e be any unit vector, $|e| = 1$. The moments of the circles $\{|x \pm e| < r\}$ are by the mean value property of harmonic functions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\frac{n+k-2}{n-2}\,\omega_n\, r^n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot(\pm e)) \qquad (k\in N_0). \tag{6}$$

Observing the homogeneity of $C_k^{(\frac{n}{2}-1)}$ it follows that the moments of the rings $\left\{3<|x\mp e|<\sqrt[n]{3^n+1}\right\}$ and for $\left\{|x\mp e|<1\right\}$ are

$$(\pm 1)^k\,\frac{n+k-2}{n-2}\,\omega_n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot e) \qquad (k\in N_0). \tag{7}$$

Hence $\left\{|x\mp e|<1\right\}\cup\left\{3<|x\pm e|<\sqrt[n]{3^n+1}\right\}$ both have the same moments

$$\left[(+1)^k+(-1)^k\right]\frac{n+k-2}{n-2}\,\omega_n\, C_k^{(\frac{n}{2}-1)}\,(\xi\cdot e) \qquad (0\leqslant k). \tag{8}$$

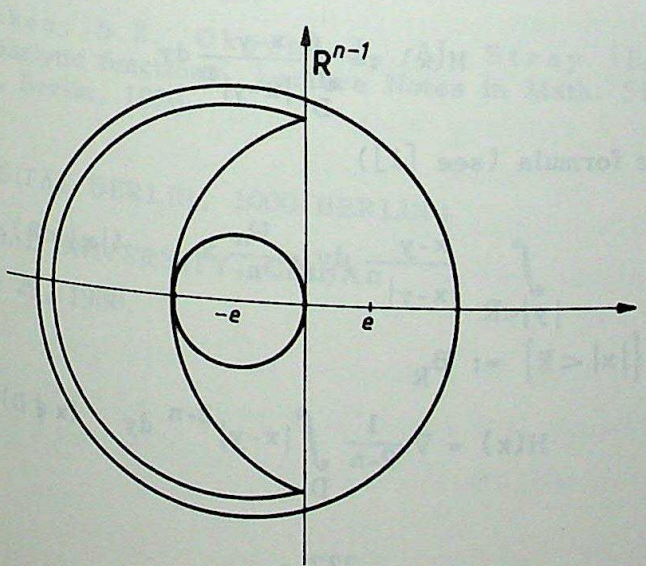
Because

$$\left\{1<|x-e|<3\right\}\cap\left\{1<|x+e|<3\right\} \tag{9}$$

is symmetric with respect to the (n-1)-dimensional plane through the origin with e as normal vector, the two domains

$$\left\{3<|x\mp e|<\sqrt[n]{3^n+1}\right\}\cup\left\{|x\pm e|<1\right\}\cup$$
$$\cup\left\{\left\{1<|x-e|<3\right\}\cap\left\{1<|x+e|<3\right\}\right\}\setminus\{0\} \tag{10}$$

have the same moments.



Figure

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The previous figure shows a crosscut of one of the domains. The other one is the reflection image of this one on the mentioned plane R^{n-1}.

3. Harmonic surfaces

A surface S in R^n is called a harmonic surface if it is representable by a function h harmonic in the neighbourhood of S such that

$$x = \nabla h(x) \qquad (x \in S). \tag{12}$$

The sphere $\{|x| = R\}$ for example is harmonic because

$$x = \nabla \frac{R^n}{2-n} |x|^{2-n} \quad \text{for} \quad |x| = R. \tag{13}$$

A harmonic surface is smooth while the domain constructed in Section 2 has a boundary which is not smooth everywhere. In [1] any smooth boundary of a bounded domain D in R^n is shown to be representable by a harmonic function h_i inside D and a harmonic function h_e outside D as

$$x = \nabla h_i(x) + \nabla h_e(x) \qquad (x \in \partial D). \tag{14}$$

The boundary will be harmonic if h_i and h_e may be harmonically continued across ∂D.

The proof of (14) simply follows from the continuity of

$$H(x) := \int_D \frac{x-y}{|x-y|^n} dy \tag{15}$$

in R^n, and the formula (see [1])

$$\int_{|y|<R} \frac{x-y}{|x-y|^n} dy = \frac{\omega_n}{n} x \qquad (|x|<R), \tag{16}$$

because if $\overline{D} \subset \{|x| < R\} =: B_R$

$$H(x) = \nabla \frac{1}{2-n} \int_D |x-y|^{2-n} dy \qquad (x \notin D), \tag{17}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$H(x) = \int_{|y|<R} \frac{x-y}{|x-y|^n}\,dy - \nabla \frac{1}{2-n} \int_{B_R \setminus D} |x-y|^{2-n}\,dy \quad (x \in D). \tag{18}$$

An example for a representation of kind (13) is for the surface of the ball $\{|x-x_0| < R\}$

$$x = \nabla x \cdot x_0 + \nabla \frac{R^n}{2-n} |x-x_0|^{2-n} \quad \text{for} \quad |x-x_0| = R. \tag{19}$$

Harmonic surfaces in $\mathbb{R}^n$ seem to have similar properties to analytic curves in $\mathbb{C}$. We therefore conjecture that the moment problem in the class of bounded domains with harmonic boundary is uniquely solvable under some additional conditions.

REFERENCES

[1] H. Begehr, R.P. Gilbert: Hele-Shaw type flows in $\mathbb{R}^n$, Nonlinear Anal. Theory, Methods and Appl. 10 (1986), 65-85.

[2] S. Richardson: Hele-Shaw flows with time-dependent free boundaries, J. Fluid Mech. 102 (1981), 263-278.

[3] M. Sakai: A moment problem on Jordan domains, Proc. Amer. Math. Soc. 70 (1978), 35-38.

[4] O.B. Bekken, B.K. Oksendal, A. Stray (Editors): Spaces of analytic functions. Lecture Notes in Math. 512, Springer-Verlag, Berlin, 1976.

FREIE UNIVERSITÄT BERLIN, 1000 BERLIN;

SICHUAN NORMAL UNIVERSITY, CHINA

Received August 29, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Cornelia-Livia Bejan

CR-SUBMANIFOLDS OF HYPERBOLICAL ALMOST HERMITIAN MANIFOLDS

Introduction

The aim of this paper is to study the class of CR-submanifolds of hyperbolical almost Hermitian manifolds, by following the same ideas of those used in the case of CR-submanifolds of almost Hermitian manifolds [5]. We mention that the class considered here is different of that studied in [2]. A corresponding notion of semi-invariant submanifolds of locally product Riemannian manifolds was given by A. Bejancu in [4], but the condition satisfied by the metric in our case lead to different results. Throughout the paper some examples are also included.

1. CR-submanifolds of hyperbolical almost Hermitian manifolds

We assume here $\tilde{M}$ to be a hyperbolical almost Hermitian manifold, i.e. $\tilde{M}$ is endowed with an almost product structure F (that is $F^2 = Id$ and $F \neq \pm Id$) and a semi-Riemannian metric g such that $g(FX,FY) = -g(X,Y)$ for $X,Y \in \Gamma(T\tilde{M})$. It follows that $\dim \tilde{M} = 2n$ and the index of g is n. We denote by $\tilde{\nabla}$ the Levi-Civita connection on $(\tilde{M},g)$.

Let M be a semi-Riemannian submanifold of $\tilde{M}$, i.e. M is a submanifold of $\tilde{M}$ on which g is nondegenerate and of constant index, [6]. Thus, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, we put:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$FX = fX + tX, \tag{1.1}$$

$$FV = BV + CV, \tag{1.2}$$

where BV, $fX \in \Gamma(TM)$ and $tX, CV \in \Gamma(T\tilde{M})$.

Also, by taking $\{N_i\}_{i=\overline{1,s}}$ to be a local orthonormal basis of $TM^{\perp}$ have locally

$$FX = fX + \sum_{i=1}^{s} \varrho_i(X)\, N_i, \tag{1.3}$$

where ϱ_i, $i = \overline{1,s}$ are some local 1-forms.

Let ∇, h, A and $\nabla^{\perp}$ be the induced covariant differentiation on M, the second fundamental form, the second fundamental tensor and the normal connection, respectively.

The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.4}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \quad \text{for } X,Y \in \Gamma(TM), \quad \xi \in \Gamma(TM^{\perp}), \tag{1.5}$$

The Codazzi equation is

$$[\tilde{R}(X,Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \quad \text{for } X,Y,Z \in \Gamma(TM), \tag{1.6}$$

where $\perp$ denotes the normal component and $\bar{\nabla}$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \quad \text{for } X,Y,Z \in \Gamma(TM).$$

We say that a semi-Riemannian submanifold M of $\tilde{M}$ is a CR-submanifold if it carries a differential distribution D which is nondegenerate and of constant index with respect to g (if D is non null), satisfying:

$$F(D_p) = D_p \quad \text{and} \tag{1.7}$$

$$F(D^{\perp}_p) \subset (T_p M)^{\perp} \quad \text{for } p \in M, \tag{1.8}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $D^{\perp}$ is the orthogonal complementary distribution of D with respect to g.

We denote by $\nu_p^{\perp} = F(D_p^{\perp})$ for $p \in M$ and we obtain that $\nu^{\perp}$ is a vector subbundle of $TM^{\perp}$. From the assumption made for D, it follows that if it is non null, then $D^{\perp}$ (resp. $\nu^{\perp}$, ν) is nondegenerate and of constant index with respect to g, where ν is the orthogonal complementary vector subbundle of $\nu^{\perp}$ in $TM^{\perp}$.

In particular, we say that M is:

1. invariant, when $D^{\perp} = \{0\}$,
2. anti-invariant, when $D = \{0\}$,
3. proper CR, when $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$,
4. generic, when $\dim D_p^{\perp} = \dim(T_pM)^{\perp} \neq 0$ for $p \in M$.

By a semi-Riemannian hypersurface of $\tilde{M}$, we mean a semi-Riemannian submanifold of codimension one.

In the case of locally product Riemannian manifolds, not all hypersurfaces are proper semi-invariant [4]. But similarly to the case of CR-submanifolds of almost Hermitian manifolds (see [5]), in our case, every semi-Riemannian hypersurface of $\tilde{M}$ is an example of a generic proper CR-submanifold of $\tilde{M}$ (for $n \geq 2$) and generic anti-invariant submanifold of $\tilde{M}$ (for $n = 1$), since no normal vector at a point of a semi-Riemannian hypersurface of $\tilde{M}$ can be an eigenvector of F.

We give now an example of a proper CR-submanifold which is not generic.

Let's take the 3-dimensional torous $T^3 = S^1 \times S^1 \times S^1$, endowed with the Riemannian metric g obtained as a product of the standard metric on S^1 and let $\{X_i\}_{i=\overline{1,3}}$ be an orthonormal basis giving a parallelization of T^3, with X_3 normal to T^2 and X_1 tangent to S^1, where $S^1 \times \{0\} \times \{0\} \hookrightarrow T^2 \{0\} \hookrightarrow T^3$. By taking $\overline{M} = T^3 \times T^3$ and $\overline{g} = \begin{pmatrix} -g & 0 \\ 0 & g \end{pmatrix}$, we define F pointwise by $F_{(p,q)} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$ with respect to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\left\{(X_i,0),\ i=\overline{1,3};\ (0,X_k),\ k=\overline{1,3}\right\}$ for $(p,q)\in\bar{M}$. Thus $T^2\times S^1$ is a proper CR-submanifold of the hyperbolical almost Hermitian manifold $(\bar{M},F,\bar{g})$, which is not generic.

Next we shall give a way to construct some proper CR-submanifolds of $\tilde{M}$.

Proposition 1.1. Let (L_i,F_i,g_i), $i=\overline{1,2}$ be two hyperbolic almost Hermitian manifolds with $\dim L_1>2$. Then $L=L_1\times L_2$ endowed with $F=\begin{pmatrix}F_1 & 0\\ 0 & F_2\end{pmatrix}$ and $g=\begin{pmatrix}g_1 & 0\\ 0 & g_2\end{pmatrix}$ is a hyperbolical almost Hermitian manifold and therefore any semi-Riemannian hypersurface of L_1 provides an example of a proper CR-submanifold of L which is not generic.

Remark The previous example is not a particular case of Proposition 1.1.

Proposition 1.2. Let M be a semi-Riemannian hypersurface of $\tilde{M}$ having a spacelike unit normal vector field N (i.e. $g(N,N)=1$). Then M is a hyperbolical almost paracontact manifold (said also almost paracohermitian manifold, see [3]).

Proof. Let $X\in\Gamma(TM)$. From (1.3), we get $f^2X=X-\eta(X)FN$ and $\eta(FN)=1$. We also have $g(fX,fY)=g(FX,FY)+\eta(X)\eta(Y)=$ $=-g(X,Y)+\eta(X)\eta(Y)$ for $X,Y\in\Gamma(TM)$.

As a consequence of this proposition, we get the following

Example. Let $(R^m_k,<.>)$ be the semi-Euclidean space and let $S^{m-1}_k(r)=\left\{x\in R^m_k \mid <x,x>=r^2\right\}$ be the pseudosphere, where $<,>=\begin{pmatrix}-I_k & 0\\ 0 & I_{m-k}\end{pmatrix}$ with respect to the standard basis of R^m, $m\geq 2$, $0<k\leq m$, see [6]. We remark that $(R^{2n}_n, F, <,>)$ and $S^{2n-1}_n(r)$ satisfy the conditions assumed in Proposition 1.2 for $\tilde{M}$ and M respectively, where F is given by $F=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}$ with respect to the standard

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

basis of R^{2n}, $n \geq 1$. Thus, we obtain that all $(2n-1)$-dimensional pseudospheres of index $n \geq 1$ are hyperbolical almost paracontact manifolds.

Now, we take M to be a CR-submanifold of $\tilde{M}$. For any $x \in \Gamma(TM)$, we put

$$X = PX + QX, \tag{1.9}$$

where $PX \in \Gamma(D)$ and $QX \in \Gamma(D^{\perp})$.

Remark. From (1.9) it follows that f is an f-structure on M [7]. M is said to be D-geodesic, $D^{\perp}$-geodesic or $(D,D^{\perp})$-geodesic if $h(D,D) = \{0\}$, $h(D^{\perp},D^{\perp}) = \{0\}$ or $h(D,D^{\perp}) = \{0\}$, respectively.

2. CR-submanifolds of hyperbolical Kählerian manifolds

We assume in this section $\tilde{M}$ to be a hyperbolical Kählerian manifold, i.e. $(\tilde{M},F,g)$ is a hyperbolical almost Hermitian manifold such that

$$\tilde{\nabla}F = 0, \tag{2.1}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of g. Now, from §1 it follows that $(R^{2n}_n, F, <,>)$ is a hyperbolical Kählerian manifold, $n \geq 1$. As it is well known that a connected, simply connected, complete m-dimensional semi-Riemannian manifold of index k and zero sectional curvature is isometric to R^m_k and as it can be proved similarly to the Kählerian case that a q-dimensional hyperbolical Kählerian manifold $(q>2)$ of constant sectional curvature c has $c = 0$, then it follows that $(R^{2n}_n, F, <,>)$ is the only one (up to isometry) among all connected, simply connected and complete 2n-dimensional hyperbolical Kählerian manifolds of constant sectional curvature, with $n > 1$. Let $H^{m-1}_{k-1}(r) = \left\{ x \in R^m_k \,\middle|\, <x,x> = -r^2 \right\}$ be the pseudohyperbolic space. Thus $S^{2n}_n(r)$ and $H^{2n}_n(r)$ can not be endowed with hyperbolical Kählerian

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

structure for $n>1$. For $n = 1$, we get that the 2-dimensional de Sitter space $(S_1^2(r), F_1, <,>)$ and $(H_1^2(r), F_2, <,>)$ are hyperbolical Kählerian manifolds, where $F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the orthonormal basis $\left\{ X_1 = (1/\sqrt{r^2+x_1^2})\ (-x_3e_2+x_2e_3);\ X_2=(1/r\sqrt{r^2+x_1^2})\ [(r^2+x_1^2)e_1 + + x_1x_2e_2 + x_1x_3e_3] \right\}$ on $S_1^2(r)$ and $F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the orthonormal basis $\left\{ Y_1 = (1/\sqrt{r^2+x_3^2})(x_2e_1-x_1e_2);\ Y_2 = (1/r\sqrt{r^2+x_3^2})x \right.$ $\left. x[x_1x_3e_1+x_2x_3e_2+(r^2+x_3^2)e_3] \right\}$ on $H_1^2(r)$, where $\left\{ e_i \right\}_{i=\overline{1,3}}$ denotes the standard basis of R^3.

From now on, we assume in this section M to be a CR-submanifold of $\tilde{M}$. If in Proposition 1.1 we take L_i, $i=\overline{1,2}$ to the hyperbolical Kählerian manifolds, then we get some examples of proper CR-submanifolds of hyperbolical Kählerian manifolds.

We deal here with the integrability of the distributions of M and we omit the proofs which are similar to those given in the Kählerian case, [5].

First, remark that from (1.9) it follows

$$\nabla_X FPY - A_{FQY}X = FP\nabla_X Y + Bh(X,Y), \tag{2.2}$$

$$h(X,FPY) + \nabla^{\perp}_X FQY = FQ\nabla_X Y + Ch(X,Y) \text{ for } X,Y \in \Gamma(TM). \tag{2.3}$$

From (2.2) and (2.3) we get the following

Proposition 2.1. a) D is integrable if and only if

$$B[h(X,FY) - h(Y,FX)] = 0 \quad \text{for} \quad X,Y \in \Gamma(D). \tag{2.4}$$

b) D is integrable and its leaves are totally geodesic in $\tilde{M}$ if and only if M is D-geodesic.

c) D is integrable and its leaves are totally geodesic in M if and only if $h(D,D) \subset \Gamma(\nu)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Remark. The relation (2.4) is equivalent with

$$h(X,FY) - h(Y,FX) = 0 \quad \text{for} \quad X,Y \in \Gamma(D).$$

By using (2.2) we get

Lemma 2.1. If $X,Y \in \Gamma(D^{\perp})$, then

$$A_{FX}Y = A_{FY}X. \tag{2.5}$$

Proposition 2.2. a) $D^{\perp}$ is always integrable.

b) $h(D,D) \subset \Gamma(\nu)$ if and only if the leaves of $D^{\perp}$ are totally geodesic in M.

In particular, when M is $(D,D^{\perp})$-geodesic, the leaves of $D^{\perp}$ are totally geodesic in M.

Proposition 2.3. The following assertions are equivalent:

a) D is parallel; b) $D^{\perp}$ is parallel; c) $h(X,Y) \in \Gamma(\nu)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$; d) D and $D^{\perp}$ are integrable and their leaves are totally geodesic in M; e) $(\nabla_X f)Y = 0$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Proof. We assume M to be a proper CR-submanifold, otherwise the assertion is trivial. We get a) $\Leftrightarrow$ b) since ∇ is the Levi-Civita connection of M. To prove b) $\Leftrightarrow$ c), we take $Z \in \Gamma(D^{\perp})$. From (2.2) we get $-A_{FZ}X = FP\nabla_X Z + Bh(X,Z)$. Since $g(A_{FZ}X,Y) = g(h(X,Y),FZ)$, then $h(X,Y) \in \Gamma(\nu) \Leftrightarrow A_{FZ}X \in \Gamma(D^{\perp}) \Leftrightarrow \nabla_X Z \in \Gamma(D^{\perp})$. Next, c) $\Leftrightarrow$ d) follows from Proposition 2.1 and 2.2. To prove c) $\Leftrightarrow$ e), we remark that for any $V,W \in \Gamma(TM)$, the relation (2.2) can be written as $(\nabla_V f)W - A_{FQW}V = Bh(V,W)$ from which we complete the proof.

Corollary 2.1. If $h(TM,D) \subset \Gamma(\nu)$, then

$$M = M_1 \times M_2 \quad \text{(locally)}, \tag{2.6}$$

where M_1 is a leaf of D and M_2 is a leaf of $D^{\perp}$.

3. Totally umbilical CR-submanifolds of hyperbolical Kählerian manifolds

In this section, we assume that M is a totally umbilical CR-submanifold of a hyperbolical Kählerian manifold $\tilde{M}$, i.e. M is a CR-sub-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

manifold of $\tilde{M}$ such that $h(X,Y) = g(X,Y)H$ for $X,Y \in \Gamma(TM)$, where $H \in \Gamma(TM^{\perp})$. Remark that $S_k^{2k-1}(r)$ and $H_{k-1}^{2k-1}(r)$ are, for instance, totally umbilical CR-submanifolds of $(R_k^{2k}, F, <,>)$, $k \geq 1$.

Proposition 3.1. If $\dim D^{\perp} > 1$, then either M is totally geodesic or M is anti-invariant.

Proof. We suppose $H \neq 0$. From Lemma 2.1 we get $A_{FX}BH = A_{FBH}X$ for $X \in \Gamma(D^{\perp})$. Thus $g(A_{FX}BH,X) = g(A_{FBH}X,X) \Leftrightarrow$ $\Leftrightarrow g(h(BH,X),FX) = g(h(X,X),FBH)$ and we get

$$[g(BH,X)]^2 = g(X,X)g(BH,BH) \quad \text{for} \quad X \in \Gamma(D^{\perp}). \tag{3.1}$$

We obtain that BH is isotropic, for if we suppose not, as $\dim D^{\perp} > 1$, we can take $Y \in \Gamma(D^{\perp})$ to be a unit vector field orthonormal to BH and from (3.1) we get that BH is isotropic again. Thus, from (3.1) it follows $g(BH,X) = 0$ for $X \in \Gamma(D^{\perp})$ and as $D^{\perp}$ is nondegenerate with respect to g, we get $BH = 0$. We have that M is anti-invariant, for if we suppose not, then $D \neq \{0\}$ and we take $Z \in \Gamma(D)$ to be a unit vector field (i.e. $|g(Z,Z)| = 1$). From (2.3) we get $g(Z,FZ)H = FQ\nabla_Z Z + g(Z,Z)CH$. We have $g(Z,FZ) = 0$, since F is sqew-symmetric with respect to g. As we have $FQ\nabla_Z Z \in \Gamma(\nu^{\perp})$, then $CH = 0$ and we complete the proof.

Corollary 3.1. If M is a proper CR-submanifold, with $\dim D^{\perp} > 1$, then M can be written as in (2.6).

Now, by using (1.6) and since $\tilde{R}(FX,FY) = -\tilde{R}(X,Y)$ for $X,Y \in \Gamma(T\tilde{M})$, we get $\tilde{K}(X \wedge Y) = 0$, where $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ span a nondegenerate plane with respect to g and $\tilde{K}$ is the sectional curvature of $\tilde{M}$. Thus we obtain

Proposition 3.2. There are no proper totally umbilical CR-submanifolds in any positively (or negatively) curved hyperbolical Kählerian manifold.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] T. Adati: Submanifolds of an almost product Riemannian manifold, Kodai Math. J. 4 (1981) 327-343.

[2] P. Amato: CR-sous-variétés d'une variété parakahlerienne possédant la propriété de Poisson, Rend. Math. 7, vol.4, 3(1984), p.351-355.

[3] C.L. Bejan: Almost parahermitian structures on the tangent bundle of an almost paracohermitian manifold, to appear in Proceedings of the 5-th Mat. Sem. on Finsler spaces, Brasov, Romania, 1988.

[4] A. Bejancu: Semi-invariant submanifolds of locally product Riemannian manifolds, An. Univ. Timisoara, 22, fasc.1-2(1984).

[5] A. Bejancu: Geometry of CR-submanifolds. Reidel Dordrecht, 1986.

[6] B. O'Neill: Semi-Riemannian geometry with applications to relativity. Academic Press, 1983.

[7] K. Yano: On a structure defined by a tensor field of type (1,1) satisfying $f^3 + f = 0$. Tensor N.S., 14(1963), 99-109.

SEMINARUL MATEMATIC, UNIVERSITATEA "AL.I.CUZA",
IASI 6600 ROMANIA

Received September 26, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol XXIII No 2 1990

Wiesława Chromik

EXTERNALLY COMPATIBLE IDENTITIES OF ALGEBRAS

0. Let $\tau: T \to N \cup \{0\}$ be a type of algebras and $\{f_t\}_{t\in T}$ is a family of fundamental operation symbols, associated with τ.

If K is a variety of algebras of type τ, we denote by E(K) the set of all identities of type τ satisfied in K.

For a set E of identities of type τ, we denote by V(E) the variety of type τ defined by E.

In this paper we introduce a new type of identities, namely so called externally compatible identities.

If K is an idempotent variety of algebras or a variety of Boole'an algebras, then we describe algebras from the variety V(Ex(K)), where Ex(K) is the set of all externally compatible identities belonging to E(K). We also construct equational base of V(Ex(K)) if an equational base of K is given.

For a term type τ we shall write $\phi(x_1,\dots,x_n)$ meaning that ϕ is a term on variables $x_1,\dots,x_n$ i.e. only variables from the set $\{x_1,\dots,x_n\}$ (not necessarily all) occur in ϕ.

1. Externally compatible identities

An identity $\phi = \psi$ of type τ is called externally compatible if it is of the form

(1)
$$x_i = x_i,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where x_i is a variable, or of the form

$$f_t(\phi_0,\dots,\phi_{\tau(t)-1}) = f_t(\psi_0,\dots,\psi_{\tau(t)-1}), \tag{2}$$

where $t \in T$ and $\phi_0,\dots,\phi_{\tau(t)-1},\psi_0,\dots,\psi_{\tau(t)-1}$ are terms of type τ.

For example, the identities $x_1 = x_1$, $x_1 + x_2 = x_2 + x_1$, $x_1 + (x_1 + x_2) = (x_1 + x_2) + x_1$, $0 = 0$, $x_1 \cdot x_2 + x_3 \cdot x_4 = x_5 + x_6$ are externally compatible, but the identities $(x_1 + x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3$, $x_1 + x_1 \cdot x_2 = x_1$ are not externally compatible.

Note that

(i) every identity provable by means of the set E of externally compatible identities is an externally compatible identity again.

Denote by $Ex(\tau)$ the set of all externally compatible identities of type τ.

Theorem 1. An algebra $\mathcal{U} = (A;\{f_t\}_{t\in T})$ belongs to $V(Ex(\tau))$ iff for every $t \in T$ there exist a fixed element $O_t(A)$ in A such that

$$f_t(a_0,\dots,a_{\tau(t)-1}) = O_t(A),$$

for every $a_0,\dots,a_{\tau(t)-1} \in A$.

Proof. The necessity follows from the fact that the identity

$$f_t(x_0,\dots,x_{\tau(t)-1}) = f_t(y_0,\dots,y_{\tau(t)-1})$$

belongs to $Ex(\tau)$, for each $t \in T$.

Since every realization of the left and right side of (2) in $\mathcal{U}$ is equal to $O_t(A)$, the sufficiency follows.

(ii) The variety $V(Ex(\tau))$ is nondegenerate.

Proof. In fact, consider an algebra $\mathcal{U}_0 = (\{0,1\};\{f_t\}_{t\in T})$, where for every $t \in T$ and $a_0,\dots,a_{\tau(t)-1} \in A$ we have

$$f_t(a_0,\dots,a_{\tau(t)-1}) = 0.$$

Then the algebra $\mathcal{U}_0$ belongs to $V(Ex(\tau))$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 2. For every variety K of type τ we have

$$V(Ex(K)) = V(Ex(\tau)) \vee K.$$

Proof. In fact,

$$E(V(Ex(\tau)) \vee K) = E(V(Ex(\tau))) \cap E(K) = Ex(\tau) \cap E(K) =$$

$$= Ex(K) = E(V(Ex(K))) \quad \text{by (i)}.$$

2. Idempotent varieties

In the sequel we shall consider algebras without nullary fundamental operation, i.e. we assume $O \notin \tau(T)$.

A variety K will be called an idempotent variety, if for every $t \in T$ the identity

$$f_t(x,x,\ldots,x) = x$$

belongs to E(K).

Remark 1. If we use an identity $\phi = \psi$ belonging to Ex(K), then we shall write $\underset{ex}{=}$ instead of $=$.

Theorem 3. Let K be an idempotent variety of type τ. Then an algebra $\mathcal{U} = (A; \{f_t\}_{t \in T})$ belongs to V(Ex(K)) iff there exists a congruence $\sim$ on $\mathcal{U}$, such that:

1° $\mathcal{U}/\sim \in K$,

2° for $a \in A$ the congruence class $[a]_\sim$, is a subalgebra of $\mathcal{U}$ belonging to V(Ex(τ)),

3° if $t \in T$ and $a_1,\ldots,a_{\tau(t)} \in A$, then in $\mathcal{U}$

$$f_t(a_1,\ldots,a_{\tau(t)}) = O_t([f_t(a_1,\ldots,a_{\tau(t)})]_\sim).$$

Proof Assume $\mathcal{U} \in V(Ex(K))$. Let t_0 be a fixed element of T. We define a relation $\sim$ putting

(c) $a \sim b \iff f_{t_0}(a,\ldots,a) = f_{t_0}(b,\ldots,b)$ for $a,b \in A$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since we have

$$f_{t_1}(a,\dots,a) \underset{ex}{=} f_{t_1}(f_{t_0}(a,\dots,a),\dots,f_{t_0}(a,\dots,a)) =$$

$$= f_{t_1}(f_{t_0}(b,\dots,b),\dots,f_{t_0}(b,\dots,b)) \underset{ex}{=} f_{t_1}(b,\dots,b),$$

the relation $\sim$ does not depend on the choice of t_0.

Obviously $\sim$ is an equivalence relation. We prove that $\sim$ is a congruence on $\mathcal{U}$.

Let $t \in T$ and $a_k, b_k \in A$, $k \in \{1,\dots,\tau(t)\}$. If $a_k \sim b_k$, $k \in \{1,\dots,\tau(t)\}$, then by (c)

$$f_{t_0}(f_t(a_1,\dots,a_{\tau(t)}),\dots,f_t(a_1,\dots,a_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f_{t_0}(f_t(f_{t_0}(a_1,\dots,a_1),\dots,f_{t_0}(a_{\tau(t)},\dots,a_{\tau(t)})),\dots$$

$$\dots,f_t(f_{t_0}(a_1,\dots,a_1),\dots,f_{t_0}(a_{\tau(t)},\dots,a_{\tau(t)}))) =$$

$$= f_{t_0}(f_t(f_{t_0}(b_1,\dots,b_1),\dots,f_{t_0}(b_{\tau(t)},\dots,b_{\tau(t)})),\dots$$

$$\dots,f_t(f_{t_0}(b_1,\dots,b_1),\dots,f_{t_0}(b_{\tau(t)},\dots,b_{\tau(t)}))) \underset{ex}{=}$$

$$\underset{ex}{=} f_{t_0}(f_t(b_1,\dots,b_{\tau(t)}),\dots,f_t(b_1,\dots,b_{\tau(t)})).$$

Therefore

$$f_t(a_1,\dots,a_{\tau(t)}) \sim f_t(b_1,\dots,b_{\tau(t)}).$$

Let $\phi = \psi$ belong to $E(K)$. Then $f_{t_0}(\phi,\dots,\phi) = f_{t_0}(\psi,\dots,\psi)$ belongs to $Ex(K)$. Hence $\phi^{\mathcal{U}} \sim \psi^{\mathcal{U}}$ for every substitution. Consequently $\mathcal{U}/\sim$ belongs to K.

We show that every $([a]_\sim; \{f_t\}_{t\in T})$ is a subalgebra of $\mathcal{U}$. Let

$$a_1,\dots,a_{\tau(t)} \in [a]_\sim.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By (c) we have

$$f_{t_0}(f_t(a_1,\dots,a_{\tau(t)}),\dots,f_t(a_1,\dots,a_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f_{t_0}(f_t(f_{t_0}(a_1,\dots,a_1),\dots,f_{t_0}(a_{\tau(t)},\dots,a_{\tau(t)})),\dots$$

$$\dots,f_t(f_{t_0}(a_1,\dots,a_1),\dots,f_{t_0}(a_{\tau(t)},\dots,a_{\tau(t)}))) =$$

$$= f_{t_0}(f_t(f_{t_0}(a,\dots,a),\dots,f_{t_0}(a,\dots,a)),\dots$$

$$\dots,f_t(f_{t_0}(a,\dots,a),\dots,f_{t_0}(a,\dots,a))) \underset{ex}{=}$$

$$\underset{ex}{=} f_{t_0}(f_{t_0}(a,\dots,a),\dots,f_{t_0}(a,\dots,a)) \underset{ex}{=} f_{t_0}(a,\dots,a),$$

so

$$f_t(a_1,\dots,a_{\tau(t)}) \sim a,$$

and consequently,

$$f_t(a_1,\dots,a_{\tau(t)}) \in [a]_\sim.$$

If $a_1,\dots,a_{\tau(t)}, b_1,\dots,b_{\tau(t)} \in [a]_\sim$, then by (c) we have

$$f_t(a_1,\dots,a_{\tau(t)}) \underset{ex}{=} f_t(f_{t_0}(a_1,\dots,a_1),\dots,f_{t_0}(a_{\tau(t)},\dots,a_{\tau(t)})) =$$

$$= f_t(f_{t_0}(b_1,\dots,b_1),\dots,f_{t_0}(b_{\tau(t)},\dots,b_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f_t(b_1,\dots,b_{\tau(t)}),$$

and thus, by Theorem 1, the algebra $([a]_\sim; \{f_t\}_{t\in T})$ belongs to $V(Ex(\tau))$.

We denote by

$$O_t([a]_\sim) = f_t(a_1,\dots,a_{\tau(t)})$$

for $a_1,\dots,a_{\tau(t)} \in [a]_\sim$.

Let $t \in T$ and $a_1,\dots,a_{\tau(t)} \in A$. Then $f_t(a_1,\dots,a_{\tau(t)}) \underset{ex}{=} f_t(f_t(a_1,\dots,a_{\tau(t)}),\dots,f_t(a_1,\dots,a_{\tau(t)})) = O_t([f_t(a_1,\dots,a_{\tau(t)})]_\sim) = O_t(f_t([a_1]_\sim,\dots,[a_{\tau(t)}]_\sim))$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This completes the proof of the necessity.

Suppose there is a congruence $\sim$ on $\mathcal{U}$ fulfilling the conditions 1° - 3°.

By an easy induction we can prove the following property.

If $\phi(x_1,\ldots,x_n)$ is a term of type τ on variables $x_1,\ldots,x_n$ and ϕ is different from a variable, then for $a_1,\ldots,a_n \in A$ we have

$$\text{(d)} \qquad \phi^{\mathcal{U}}(a_1,\ldots,a_n) = O_t(\phi^{\mathcal{U}/\sim}([a_1]_\sim,\ldots,[a_n]_\sim)),$$

where t is the index of the most external fundamental operation symbol in ϕ.

Consider an identity

$$\text{(e)} \qquad \phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_m)$$

belonging to Ex(K).

By the convention at the end of section 0, we xan assume $m = n$.

We prove that (e) is satisfied in $\mathcal{U}$. If (e) is of the form (1), then it holds true in $\mathcal{U}$.

Let (e) be of the form (2) and let t be the index of the most external fundamental operation symbol in ϕ and ψ.

Then, by (d), for $a_1,\ldots,a_n \in A$ we have

$$\phi^{\mathcal{U}}(a_1,\ldots,a_n) = O_t(\phi^{\mathcal{U}/\sim}([a_1]_\sim,\ldots,[a_n]_\sim)) =$$

$$= O_t(\psi^{\mathcal{U}/\sim}([a_1]_\sim,\ldots,[a_n]_\sim)) = \psi^{\mathcal{U}}(a_1,\ldots,a_n).$$

Thus (e) is satisfied in $\mathcal{U}$ and $\mathcal{U}$ belongs to V(Ex(K)).

Remark 2. If K is a variety defined by all identities $f_t(x,\ldots,x) = x$, then from Theorem 3 we obtain Theorem 1 from [1].

Now, given an equational base E of K, we construct an equational base of V(Ex(K)).

Let t_0 be a fixed index from T. We define a set E^* as follows:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(γ_1) for every $t,t' \in T$ the identity

$$f_t(x_1,\ldots,x_{\tau(t)}) = f_t(f_t(x_1,\ldots,x_1),\ldots,f_t(x_{\tau(t)},\ldots,x_{\tau(t)}))$$

belongs to E^*.

(γ_2) If $\phi = \psi$ belongs to E, then the identity

$$f_{t_0}(\phi,\ldots,\phi) = f_{t_0}(\psi,\ldots,\psi)$$

belongs to E^*.

(γ_3) for every $t \in T$ the identity

$$f_t(x_1,\ldots,x_{\tau(t)}) = f_t(f_t(x_1,\ldots,x_{\tau(t)}),\ldots,f_t(x_1,\ldots,x_{\tau(t)}))$$

belongs to E^*.

(γ_4) E^* consists only of the identities described in (γ_1), (γ_2), (γ_3).

Theorem 4. If E is an equational base of an idempotent variety K, then E^* is an equational base of $V(Ex(K))$.

Proof. We shall prove $V(E^*) = V(Ex(K))$. Obviously

$$E^* \subseteq Ex(K).$$

So

$$V(Ex(K)) \subseteq V(E^*).$$

To prove the converse inclusion define a relation $\sim$ in an algebra $\mathcal{U}^* = (A; \{f_t\}_{t\in T})$ belonging to $V(E^*)$ by putting

$$a \sim b \Leftrightarrow f_{t_0}(a,\ldots,a) = f_{t_0}(b,\ldots,b),$$

for $a,b \in A$. Using (γ_1), (γ_2), (γ_3) we prove that the conditions 1°, 2°, 3° from Theorem 3 hold true. Hence $\mathcal{U}^*$ belongs to $V(Ex(K))$.

Corollary 1. If K is finitely based and T is finite, then $V(Ex(K))$ is finitely based.

Example. Let K be a variety of distributive lattices with fundamental operation symbols $+$, $\cdot$; then $V(Ex(K))$ is finitely based.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Varieties with involution

A variety L of type τ will be called a variety with involution if there exists $t_0 \in T$ such that $\tau(t_0) = 1$, the identity

$$f_{t_0}(f_{t_0}(x)) = x$$

belongs to E(L), and all identities

$$f_t(x,\dots,x) = x, \quad t \in T \setminus \{t_0\}$$

belong to E(L).

We denote $f^2_{t_0}(x) = f_{t_0}(f_{t_0}(x))$.

Theorem 5. Let L be a variety with involution. Then an algebra $\mathcal{U} = (A; \{f_t\}_{t\in T})$ belongs to V(Ex(L)) iff there exists a congruence $\sim$ on $\mathcal{U}$ such that:

1° $\mathcal{U}/_\sim \in L$,

2° for every $a \in A$, the congruence class $[a]_\sim$, is a subalgebra of $\mathcal{U}^* = (A; \{f_t\}_{t\in T\setminus\{t_0\}} \cup \{f^2_{t_0}\})$ and the algebra $([a]_\sim; \{f_t\}_{t\in T\setminus\{t_0\}} \cup \{f^2_{t_0}\})$ satisfies the identities $f_t(x_1,\dots,x_{\tau(t)}) = f_t(y_1,\dots,y_{\tau(t)})$ for $t \in T\setminus\{t_0\}$, $f^2_{t_0}(x_1) = f^2_{t_0}(y_1)$, for every $x_1,\dots,x_{\tau(t)},y_1,\dots,y_{\tau(t)} \in [a]_\sim$,

3° for every $t \in T\setminus\{t_0\}$ and for every class $[a]_\sim$ an element $O_t([a]_\sim)$ is distinguished and for $a_1,\dots,a_{\tau(t)} \in A$ we have $f_t(a_1,\dots,a_{\tau(t)}) = O_t(f_t([a_1]_\sim,\dots,[a_{\tau(t)}]_\sim))$,

4° for every $a \in A$ and for every class $[a]_\sim$ an element $O^2_{t_0}([a]_\sim)$ is distinguished and we have

$$f_{t_0}(a) = O^2_{t_0}([f_{t_0}(a)]_\sim).$$

Proof. Assume $\mathcal{U} \in V(Ex(L))$. We define a relation $\sim$ in A as follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(f) $$a \sim b \iff f^2_{t_0}(a) = f^2_{t_0}(b), \quad \text{for} \quad a,b \in A.$$

Obviously $\sim$ is an equivalence in A. We show that $\sim$ is a congruence in $\mathcal{U}$. Assume that for $t \in T$, $a_1,\dots,a_{\tau(t)}$, $b_1,\dots,b_{\tau(t)} \in A$ we have $a_k \sim b_k$, $k \in \{1,\dots,\tau(t)\}$. Then

$$f^2_{t_0}(f_t(a_1,\dots,a_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f^2_{t_0}(f_t(f^2_{t_0}(a_1), f^2_{t_0}(a_2),\dots,f^2_{t_0}(a_{\tau(t)}))) =$$

$$= f^2_{t_0}(f_t(f^2_{t_0}(b_1), f^2_{t_0}(b_2),\dots,f^2_{t_0}(b_{\tau(t)}))) \underset{ex}{=}$$

$$\underset{ex}{=} f^2_{t_0}(f_t(b_1,\dots,b_{\tau(t)})).$$

So $\sim$ is a congruence on $\mathcal{U}$.

If $\phi = \psi$ belongs to $E(L)$, then $f^2_{t_0}(\phi) = f^2_{t_0}(\psi)$ belongs to $Ex(L)$. So for every substitution $\phi \sim \psi$.

Consequently $\mathcal{U}/\sim$ belongs to L. Assume that $a_1,\dots,a_{\tau(t)} \in [a]_\sim$. Then, by (f) for $t \in T \setminus \{t_0\}$ we have

$$f^2_{t_0}(f_t(a_1,\dots,a_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f^2_{t_0}(f_t(f^2_{t_0}(a_1),\dots,f^2_{t_0}(a_{\tau(t)}))) =$$

$$= f^2_{t_0}(f_t(f^2_{t_0}(a),\dots,f^2_{t_0}(a))) \underset{ex}{=} f^2_{t_0}(f^2_{t_0}(a)) \underset{ex}{=} f^2_{t_0}(a).$$

Hence $f_t(a_1,\dots,a_{\tau(t)}) \in [a]_\sim$. Since

$$f^2_{t_0}(f^2_{t_0}(a_1)) = f^2_{t_0}(f^2_{t_0}(a)) = f^2_{t_0}(a),$$

so $f^2_{t_0}(a_1) \sim a$, and $f^2_{t_0}(a_1) \in [a]_\sim$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $b_1,\dots,b_{\tau(t)} \in [a]_\sim$ and $c_1,\dots,c_{\tau(t)} \in [a]_\sim$. Then for $t \neq t_0$ we have

$$f_t(b_1,\dots,b_{\tau(t)}) \underset{ex}{=}$$

$$\underset{ex}{=} f_t(f^2_{t_0}(b_1),\dots,f^2_{t_0}(b_{\tau(t)})) =$$

$$= f_t(f^2_{t_0}(a),\dots,f^2_{t_0}(a)) = f_t(f^2_{t_0}(c_1),\dots,f^2_{t_0}(c_{\tau(t)})) \underset{ex}{=}$$

$$\underset{ex}{=} f_t(c_1,\dots,c_{\tau(t)}).$$

Further

$$f^2_{t_0}(b_1) \underset{ex}{=} f^2_{t_0}(f^2_{t_0}(b_1)) = f^2_{t_0}(f^2_{t_0}(c_1)) \underset{ex}{=} f^2_{t_0}(c_1).$$

We denote by $O_t([a]_\sim)$ the constant value of $f_t|_{[a]_\sim}$ for $t \neq t_0$, and by $O^2_{t_0}([a]_\sim)$ the constant value of $f^2_{t_0}|_{[a]_\sim}$. Now let $a_1,\dots,a_{\tau(t)} \in A$ and $t \in T \setminus \{t_0\}$. Then

$$f_t(a_1,\dots,a_{\tau(t)}) \underset{ex}{=}$$

$$\underset{ex}{=} f_t(f_t(a_1,\dots,a_{\tau(t)}),\dots,f_t(a_1,\dots,a_{\tau(t)})) =$$

$$= O_t([f_t(a_1,\dots,a_{\tau(t)})]_\sim) = O_t(f_t([a_1]_\sim,\dots,[a_{\tau(t)}]_\sim)).$$

Further for $t_0 \in T$ and $a \in A$ we obtain

$$f_{t_0}(a) = f_{t_0}(f^2_{t_0}(a)) = f^2_{t_0}(f_{t_0}(a)) = O^2_{t_0}([f_{t_0}(a)]_\sim).$$

Proof of the converse implication is analogous to that of Theorem 3.

Remark 3. Given an equational base of L. Similarly as in Theorem 4 we construct an equational base of V(Ex(L)).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Example. If B is a variety of Boole'an algebras with fundamental operations $+, \cdot, '$, then following system of identities is an equational base of $V(Ex(B))$

(b1) $x + y = x'' + y''$, where $x'' = (x')'$,

(b2) $x \cdot y = x'' \cdot y''$,

(b3) $x' = (x'')'$,

(b4) $(x + x)'' = x''$,

(b5) $(x \cdot x)'' = x''$,

(b6) $x + y = y + x$,

(b7) $x \cdot y = y \cdot x$,

(b8) $x + (y + z) = (x + y) + z$,

(b9) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(b10) $((x + y) \cdot z)'' = (x \cdot z + y \cdot z)''$,

(b11) $(x + x \cdot y)'' = x''$,

(b12) $((x + y)')'' = (x' \cdot y')''$,

(b13) $(x \cdot x')'' = (y \cdot y')''$,

(b14) $x + y = (x + y) + (x + y)$,

(b15) $x \cdot y = (x \cdot y) \cdot (x \cdot y)$.

Proof. Let B^* denote the variety of algebras with fundamental operation $+, \cdot, '$ defined by (b1)-(b15). Obviously $V(Ex(B)) \subseteq B^*$.

To prove the converse inclusion define relation $\sim$ in an algebra $\mathfrak{A} = (A; +, \cdot, ') \in B^*$ by putting

$$a \sim b \iff a'' = b'' \quad \text{for} \quad a, b \in A.$$

Then we easily show that conditions 1°-4° of Theorem 5 are satisfied.

REFERENCES

[1] J. Płonka: On the join of equational classes of idempotent algebras and algebras with constants, Colloq.Math. 23(1973), 193-195.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
45-052 OPOLE, POLAND

Received September 26, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol XXIII No 2 1990

B.G. Pachpatte

ON ABSTRACT SECOND ORDER DIFFERENTIAL EQUATIONS

1. Introduction

In this paper we study the problem

$$u''(t) + Au'(t) + Bu(t) = f(t,u(t)), \tag{1}$$

$$u(0) = \phi, \quad u'(0) = \psi, \tag{2}$$

in an arbitrary Banach space E with norm $\|\cdot\|$, where A, B are usually unbounded linear operators in E and $f \in C[I \times E, E]$, $I = [0,a]$, $a>0$. Equation (1) serves as an abstract model for the semilinear versions of the wave equation, the telegraph equation, and the equation of motion for a vibrating plate (see [1]). In [11] there was used a factoring technique and the method of successive approximations to study the existence, uniqueness and stability of solutions of (1), (2) (see also [1]-[4], [10]-[13], [15]).

In this paper we use the theory of infinitesimal generator of (C_o)-semigroup in a Banach space, a factoring technique and the general method of successive approximations based on the idea of Ważewski [14] (see also [6]-[8]). The results obtained here are nonlocal and the required conditions differ from those in [11].

2. Statement of results

The set of bounded linear operators $\{T(t);\ t \in R^+ := [0,\infty)\}$ is a (C_o)-semigroup on E, if

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $T(t+s) = T(t)T(s) = T(s)T(s)$, $t,s \geq 0$,

(ii) $T(0) = I_o$ (the identity operator),

(iii) $T(\cdot)$ is strongly continuous in $t \in R^+$,

(iv) $\|T(t)\| \leq Q\, e^{\mu t}$ for some $Q, \mu > 0$, $t \in R^+$.

The operator A is the generator of $T(\cdot)$, if

$$A\phi = \lim_{h \to 0^+} \left(\frac{(T(h) - T(0))}{h} \right) \phi$$

and D(A), the domain of A, is the set of $\phi \in E$ for which the limit exists. Formally $T(t)\phi$ satisfies the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = \phi. \tag{3}$$

If $\phi \in D(A)$, then $u(\cdot) \in C^1(R^+, E)$ and (3) holds. More generally, $u(t) = T(t)\phi$ is said to be a mild solution of (3) when $\phi \notin D(A)$. Following [11], we assume the existence of linear (possibly unbounded) operators A_1, A_2 such that $A_1 + A_2 = -A$, $A_2A_1 = B$ and A_j generates the (C_o)-semigroup T_j, $j = 1,2$. The operators A_1, A_2 need not commute. For the elementary properties of strongly continuous semigroups we refer to [5], [9].

We say that a function u is a mild solution of (1), (2), if u is continuous and satisfies

$$u(t) = u_o(t) + \int_0^t \int_0^\tau T_1(t-\tau) T_2(\tau - s) f(s, u(s)) ds\, d\tau, \tag{4}$$

where

$$u_o(t) = T_1(t)\phi + \int_0^t T_1(t-\tau) T_2(\tau)(\psi - A_1\phi) d\tau \tag{5}$$

and $\phi \in D(A_1)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By letting $z(t) = f(t,u(t))$, we rewrite (4) as

$$z(t) + f\left(t, u_o(t) + \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)z(s)ds\,d\tau\right). \tag{6}$$

Now, by substituting $F(t,r) = f(t,u_o(t) + r)$ in (6), we get an integral equation of the form

$$z(t) = F\left(t, \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)z(s)ds\,d\tau\right) =: Lz(t). \tag{7}$$

We make the following hypotheses used throughout this paper.

(H_1) Assume that:

(i) there exists a continuous function $g: I\times R^+ \longrightarrow R^+$ nondecreasing with respect to r such that $g(t,0) \equiv 0$,

(ii) for (t,y_1), $(t,y_2) \in I\times E$ there is

$$\|F(t,y_1) - F(t,y_2)\| \leq g(t,\|y_1 - y_2\|).$$

(H_2) There exists a continuous function $\bar{r} : I \longrightarrow R^+$ satisfying the inequality $Mr(t) + p(t) \leq r(t)$, where

$$Mr(t) = g\left(t,\ N^2 \int_0^t \int_0^\tau r(s)ds\,d\tau\right), \tag{8}$$

$$p(t) = \sup_{0\leq\xi\leq t} \|F(\xi,\ 0)\| \tag{9}$$

and $N>0$ is a constant.

(H_3) In the set of functions, satisfying the condition $0 \leq r(t) \leq \bar{r}(t)$ for $t \in I$, the function r such that $r(t) \equiv 0$ for $t \in I$ is the only measurable solution of the equation

$$r(t) = Mr(t), \quad t \in I. \tag{10}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Define the sequence $\{z_n\}$ by the relations

$$z_0(t) \equiv 0, \quad z_{n+1}(t) = Lz_n(t), \tag{11}$$

for $t \in I$, $n = 0,1,2,\ldots$.

To prove the convergence of $\{z_n\}$ to the solution $\bar{z}$ of (7), we define the sequence $\{r_n\}$ by the relations

$$r_0(t) = \bar{r}(t), \quad r_{n+1}(t) = Mr_n(t), \tag{12}$$

for $t \in I$, $n = 0,1,2,\ldots$, where $\bar{r}(t)$ is defined in hypothesis (H_2).

Theorem 1. Let the hypotheses (H_1)-(H_3) hold. Then there exists a continuous solution $\bar{z}(t)$, $t \in I$, of equation (7). The sequence $\{z_n\}$ defined by (11) converges uniformly to $\bar{z}$ in I and the following estimates

$$\|\bar{z}(t) - z_n(t)\| \leq r_n(t), \quad t \in I, \quad n = 0,1,2,\ldots, \tag{13}$$

$$\|\bar{z}(t)\| \leq \bar{r}(t), \quad t \in I, \tag{14}$$

hold. Moreover, the solution $\bar{z}$ of equation (7) is unique in the set of functions satisfying the condition (14).

The next theorem gives conditions under which equation (7) has at most one solution. These conditions do not guarantee the existence of a solution of equation (7).

Theorem 2. Let hypothesis (H_1) be fulfilled. If the function r, $r(t) \equiv 0$, $t \in I$, is the only nonnegative, finite and measurable solution of the inequality

$$r(t) \leq Mr(t), \quad t \in I, \tag{15}$$

then equation (7) has at most one solution in I.

Consider now the equation

$$v(t) = H\left(t, \int_0^t \int_0^\tau T_1(t-\tau)T_2(\tau-s)v(s)\,ds\,d\tau\right), \tag{16}$$

where $H \in C[IxE, E]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 3. Assume that the hypothesis (H_1) holds and

(i) $\bar{z}$ and $\bar{v}$ are solutions of equations (7) and (16), respectively,

(ii) the sequence $\{h_n(t)\}$, $t \in I$, defined by the relations

$$(17) \qquad \begin{cases} h_o(t) \geqslant \|\bar{z}(t)\| + \|\bar{v}(t)\|, \\ h_{n+1}(t) = Mh_k(t) + q(t). \end{cases}$$

for $t \in I$, $n = 0,1,2,\ldots$, where

$$(18) \qquad q(t) = \|L\bar{v}(t) - \bar{v}(t)\|,$$

has the limit $\bar{h}(t)$ for $t \in I$. Then we have

$$(19) \qquad \|\bar{z}(t) - \bar{v}(t)\| \leqslant \bar{h}(t), \qquad t \in I.$$

Remark 1. We note that Theorems 1-3 also yield the existence, uniqueness, error estimations and stability of the solutions of equivalent integral equation (4) and, consequently, of the mild solutions of the problem (1), (2). As shown in [11], Theorems 1-3 can be also extended to the initial value problems for a large class of partial differential equations.

3. Proofs of Theorems 1-3

Lemma 1. If the condition (i) of hypothesis (H_1) and hypotheses (H_2)-(H_3) are satisfied, then

$$(20) \qquad 0 \leqslant r_{n+1}(t) \leqslant r_n(t) \leqslant \bar{r}(t),$$

for $t \in I$, $n = 0,1,2,\ldots$ and $r_n \Longrightarrow 0$ for $n \longrightarrow \infty$, where $\Longrightarrow$ denotes the uniform convergence in I.

The relation (20) follows by induction, in view of (12). But (20) implies the convergence of $\{r_n\}$ to some nonnegative measurable function $m(t)$ such that $0 < m(t) \leqslant \bar{r}(t)$ for $t \in I$. By the Lebesque theorem and the continuity of g, the function $m(t)$ satisfies (10) and, by (H_3), we have $m(t) \equiv 0$, $t \in I$. The uniform convergence of $\{r_n\}$ in I follows from the Dini theorem. This completes the proof of Lemma 1.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In order to prove Theorem 1, first we prove that the sequence $\{z_n(t)\}$ defined by (11) satisfies the conditions

$$\|z_n(t)\| \leq \bar{r}(t), \quad t \in I, \quad n = 0,1,2,\ldots . \tag{21}$$

Obviously, $\|z_o(t)\| \equiv 0 \leq \bar{r}(t)$, $t \in I$. Furthermore, if we suppose that (21) is true for $n \geq 0$, then, by (H_1), (H_2) and by the fact that for $t \in I$, $\|Tj(t)\| \leq N$, $j = 1,2$, $N > 0$ with constant, we have

$$\|z_{n+1}(t)\| \leq M\|z_n(t)\| + p(t) \leq M\bar{r}(t) + p(t) \leq \bar{r}(t),$$

for $t \in I$. The relation (21) follows by induction. Next we prove that

$$\|z_{n+k}(t) - z_n(t)\| \leq r_n(t), \quad t \in I, \quad n, k = 0,1,2,\ldots . \tag{22}$$

By (21), we have

$$\|z_k(t) - z_o(t)\| = \|z_k(t)\| \leq \bar{r}(t) = r_o(t),$$

for $t \in I$, $k = 0,1,2,\ldots$. Suppose that (22) is true for n, $k \geq 0$; then, by the hypothesis (H_1) and by the fact that for $t \in I$, $\|T_j(t)\| \leq N$, $j = 1,2$; $N > 0$ with a constant, we have

$$\|z_{n+k+1}(t) - z_{n+1}(t)\| = \|Lz_{n+k}(t) - Lz_n(t)\| \leq$$

$$\leq M\|z_{n+k}(t) - z_n(t)\| < M\, r_n(t) = r_{n+1}(t),$$

and we obtain (22) by induction. By Lemma 1, $r_n(t) \Longrightarrow 0$ in I and so we have from (22) $z_n \Longrightarrow \bar{z}$ in I. The continuity of $\bar{z}$ follows from the uniform convergence of $\{z_n\}$ and from the continuity of all functions z_n. If $k \longrightarrow \infty$, then (22) gives estimation (13) and estimation (14) is implied by (21). It is obvious that $\bar{z}$ is a solution of (7).

To prove that the solution $\bar{z}$ of (7) is unique, let us suppose that there exists another solution $\hat{z}$ of (7) such that $\bar{z}(t) \neq \hat{z}(t)$ and $\|\hat{z}(t)\| \leq \bar{r}(t)$ for $t \in I$. By induction we get $\|\hat{z}(t) - z_n(t)\| \leq r_n(t)$, $t \in I$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$n = 0,1,2,\ldots$, and hence it follows that $\bar{z}(t) \equiv \hat{z}(t)$, $t \in I$. This contradiction proves the uniqueness of $\bar{z}$ in the set of functions satisfying relation (14). This completes the proof of Theorem 1.

To prove Theorem 2, let us suppose that there exist two solutions $\bar{z}$ and $\hat{z}$ of equation (7) in I, $\bar{z}(t) \not\equiv \hat{z}(t)$, $t \in I$. Now, by (H_1) and by the fact that for $t \in I$, $\|T_j(t)\| \leq N$, $j = 1,2$; $N > 0$ with a constant, we have

$$\|\bar{z}(t) - \hat{z}(t)\| \leq M\|\bar{z}(t) - \hat{z}(t)\|. \tag{23}$$

Taking $r(t) = \|\bar{z}(t) - \hat{z}(t)\|$, $t \in I$, in (23), we infer from (15) that $r(t) \equiv 0$ for $t \in I$, i.e. $\bar{z}(t) \equiv \hat{z}(t)$, $t \in I$. This contradiction completes the proof of Theorem 2.

To prove Theorem 3, let

$$h(t) = \|\bar{z}(t) - \bar{v}(t)\|, \quad t \in I, \tag{24}$$

then we have

$$h(t) \leq \|L\bar{z}(t) - L\bar{v}(t)\| + \|L\bar{v}(t) - \bar{v}(t)\| \leq \tag{25}$$

$$\leq M\|\bar{z}(t) - \bar{v}(t)\| + q(t) = Mh(t) + q(t).$$

From (25) and (17) we obtain

$$h(t) \leq \|\bar{z}(t)\| + \|\bar{v}(t)\| \leq h_o(t), \quad t \in I. \tag{26}$$

Now, by induction, we get

$$h(t) \leq h_n(t), \quad t \in I, \quad n = 0,1,2,\ldots. \tag{27}$$

The inequality (19) is implied by (27) as $n \to \infty$. This completes the proof of Theorem 3.

Remark 2. Theorems 1-3 can be easily extended to the Volterra integro-differential equation of the form

$$u''(t) + Au'(t) + Bu(t) = f\left(t, u(t), \int_0^t K[t,s,u(s)]ds\right), \tag{28}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

with the given initial conditions (2), where A, B are as in (1) and $K \in C[I\times I\times E, E]$, $f \in C[I\times E\times E, E]$. In [13] were obtained some results on the existence and uniqueness of the solutions slightly different than (28) by using cosine families and fixed point theorems.

4. Further applications

Consider now the problem

$$u''(t) + Au'(t) = f(t,u(t),u'(t)), \tag{29}$$

$$u(0) = u_o, \quad u'(0) = u_1, \tag{30}$$

where A generates a strongly continuous semigroup (T(t)) on a Banach space E.

Recently, in [2] was used the Banach fixed point theorem to the mild solutions of (29), (30). Particular examples of (29) are, among others, the strongly damped nonlinear Klein-Gordon equation and the vibrating beam equation (see [2]).

To solve (29), (30), consider the system of integral equations

$$\begin{cases} u(t) = u_o + u_1\int_0^t T(s)ds + \int_0^t\int_0^s T(s-\tau)f(\tau,u(\tau),v(\tau))d\tau ds, \\ v(t) = T(t)u_1 + \int_0^t T(t-s)f(s,u(s),v(s))ds. \end{cases} \tag{31}$$

If (u(t), v(t)) is a solution of (31), then $u(t) = u_o + \int_0^t v(s)ds$ is called a mild solution of (29), (30). By letting $z(t) = f(t,u(t),v(t))$, we rewrite (31) as

$$z(t) = f\Big(t, u_o + u_1\int_0^t T(s)ds + \int_0^t\int_0^s T(s-\tau)z(\tau)d\tau ds, \tag{32}$$

$$T(t)u_1 + \int_0^t T(t-s)z(s)ds\Big).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, by substituting

$$F(t,r_1,r_2) = f\left(t,u_o + u_1\int_0^t T(s)ds + r_1,\ T(t)u_1 + r_2\right).$$

in (32) we get

$$z(t) = F\left(t,\int_0^t\int_0^s T(s-\tau)z(\tau)d\tau ds,\ \int_0^t T(t-s)z(s)ds\right). \tag{33}$$

Theorems 1-3 can be naturally extended to (33) under some modificated hypotheses (H_1)-(H_3).

We also note that our method can be easily extended to the problem

$$u''(t) + Au'(t) = f\left(t,u(t),u'(t),\int_0^t K[t,s,u(s),u'(s)]ds\right), \tag{34}$$

$$u(0) = u_o,\quad u'(0) = u_1, \tag{35}$$

where A is from (29) and $K \in C[I\times I\times E\times E,\ E]$, $f \in C[I\times E\times E\times E,\ E]$. For the several interesting results on the existence and uniqueness of the solutions for slightly different versions of problems (29), (30) and (34), (35), by using different techniques, we refer to [12], [13].

REFERENCES

[1] T.K. Caughey, J. Ellison: Existence, uniqueness and stability of solutions of a class of nonlinear partial differential equations, J. Math. Anal. Appl. 51 (1975) 1-32.

[2] H. Engler, F. Neubrander, J. Sandefur: Strongly damped semilinear second order equations, Proc. First Howard Univ. Conf. on partial differential equations, Semigroups and Attractors, August 1985.

[3] W.E. Fitzgibbon: Strongly damped quasilinear evolution equations, J. Math. Anal. Appl. 79 (1981) 536-550.

[4] J.A. Goldstein: Semigroups and second-order differential equations, J. Functional Analysis 4 (1969) 50-70.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] E. Hille, R.S. Phillips: Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ. 31, Providence, R.I., 1957.

[6] T. Jankowski, M. Kwapisz: On the existence and uniqueness of solutions of systems of differential equations with a deviated argument, Ann. Polon. Math. 26 (1972) 253-277.

[7] M. Kwapisz, J. Turo: On the existence and convergence of successive approximations for some functional equations in a Banach space, J. Differential Equations 16 (1974) 298-318.

[8] B.G. Pachpatte: On Ważewski's method of successive approximations for certain boundary value problems, Demonstratio Math. 23 (1989) 955-966.

[9] A. Pazy: Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[10] J.T. Sandefur: Higher order abstract Cauchy problems, J. Math. Anal. Appl. 60 (1977), 728-742.

[11] J.T. Sandefur: Existence and uniqueness of solustions of second order nonlinear differential equations, SIAM J. Math. Anal. 14 (1983) 477-487.

[12] C.C. Travis, G.F. Webb: Cosine families and abstract nonlinear second order differential equations, Acta Math. Acad. Sci. Hungar. 32 (1978) 75-96.

[13] C.C. Travis, G.F. Webb: An abstract second order semilinear Volterra integro-differential equations, SIAM J. Math. Anal. 10 (1979) 412-424.

[14] T. Ważewski: Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. et Phys. 8 (1960) 45-52.

[15] G.F. Webb: Existence and asymptotic behavior for a strongly damped nonlinear wave equation, Canad. J. Math. 32 (1980) 631-643.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MARATHWADA UNIVERSITY, AURANGABAD 431004, MAHARASHTRA, INDIA

Received September 26, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Hanna Matuszczyk

VECTOR FIELDS AND DIFFERENTIAL FORMS ON PREMANIFOLDS

0. Preliminaries

In the papers [1] and [2] the author introduced the concept of a real premanifold and of a complex premanifold. We recall these concepts. Let $K = \mathbb{R}$ when we say about a real premanifold and $K = \mathbb{C}$ for a complex premanifold. Consider the set M of functions with values in K. Let $\underline{M}$ denote the union of all sets D_α, where $\alpha \in M$. Here D_α denotes the domain of α. The elements of the set $\underline{M}$ are said to be points of M. The smallest topology on $\underline{M}$ such that all the sets $\alpha^{-1}A$, where $\alpha \in M$ and A is open in K, are open is called the topology of M and denoted by topM. For any $S \subset \underline{M}$ we consider the set M_S of all functions β such that for any $p \in D_\beta$ there exist $U \in \mathrm{top}M$ and $\alpha \in M$ for which $U \subset D_\alpha$, $p \in U \cap S \subset D_\beta$ and $\beta|U \cap S = \alpha|U \cap S$. In particular, we have the set $M_{\underline{M}}$. Let anM denote the set of all functions $\omega(\alpha_1(\cdot),\ldots,\alpha_n(\cdot))$, where ω is an analytical function on the set D_ω open in K^n with values in K, $\alpha_1,\ldots,\alpha_n \in M$, n is an arbitrary natural number. We assume the set $\{p;\ p \in D_{\alpha_1} \cap \ldots \cap D_{\alpha_n}$ and $(\alpha_1(p),\ldots,\alpha_n(p)) \in D_\omega\}$ as the domain of $\omega(\alpha_1(\cdot),\ldots,\alpha_n(\cdot))$. We will often write $\omega(\alpha_1,\ldots,\alpha_n)$ instead of $\omega(\alpha_1(\cdot),\ldots,\alpha_n(\cdot))$. For any functions f and g we adopt $D_{g\circ f} = f^{-1}D_g$ and $(g\circ f)(x) = g(f(x))$ for $x \in D_{g\circ f}$.

The set M of functions with values in K satisfying the condition $M = M_{\underline{M}} = \mathrm{an}M$ is said to be K-premanifold or, shortly, premanifold.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $M(p)$ denote the set of all $\alpha \in M$ such that $p \in D_\alpha$ and let $T_p M$ denote the set of all $v: M(p) \longrightarrow K$ such that for any $\alpha, \beta \in M(p)$ and $a, b \in K$ we have $v(a\alpha+b\beta) = av(\alpha) + bv(\beta)$, $v(\alpha\cdot\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$. For any $v, w \in T_p M$ and $c \in K$ we set $(v+w)(\alpha) = v(\alpha) + w(\alpha)$ and $(c\cdot v)(\alpha) = cv(\alpha)$ for $\alpha \in M(p)$. So, we have the vector space $T_p M$ of the form $(T_p M, +, \cdot)$ called the tangent space to M at the point p.

1. The tangent bundle

The above definition of the tangent space $T_p M$ allows us to make some subtle consideration. From such a point of view we may treat as interesting the following theorem.

Theorem 1.1. For any points p and q of a premanifold M we have the following equivalent conditions:

(i) $M(p) = M(q)$;

(ii) $\{U;\ p \in U \in \operatorname{top} M\} = \{U;\ q \in U \in \operatorname{top} M\}$;

(iii) $T_p M = T_q M$;

(iv) $T_p M \cap T_q M \neq \emptyset$;

(iii*) $(T_p M)^* = (T_q M)^*$;

(iv*) $(T_p M)^* \cap (T_q M)^* \neq \emptyset$.

Proof. First, we prove that (i) implies

(i′) if $\alpha \in M$ then $\alpha(p) = \alpha(q)$.

Indeed, assuming that $\alpha(p) \neq \alpha(q)$ we take the set A open in K such that $\alpha(p) \in A$ and $\alpha(q) \notin A$. We get then the function $\alpha|\alpha^{-1}A$ belonging to M (see [1] and [2]) such that p belongs to its domain but q does not. Therefore $\alpha|\alpha^{-1}A \in M(p) - M(q)$, in spite of (i).

The implication (i) $\Longrightarrow$ (i′) yields (i) $\Longrightarrow$ (iii). The inverse one is obvious, because of $M(p) \neq \emptyset$. The equivalence of (iii) and (iv) is evident by (i) $\Longrightarrow$ (iii). To prove that (i) $\Longrightarrow$ (ii) we take any

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$U \in \mathrm{top}M$ such that $p \in U$. Then there exist $\alpha_1, \ldots, \alpha_n \in M$ and $A_1, \ldots, A_n$ open in K such that $p \in \bigcap_i \alpha_i^{-1} A_i \subset U$. Then $p \in D_{\alpha_i}$, $\alpha_i(p) \in A_i$ and $\alpha_i \in M$. Thus, $\alpha_i \in M(p)$ for $i \in \{1, \ldots, n\}$. From (i) it follows that $\alpha_i \in M(q)$ for $i \in \{1, \ldots, n\}$. Therefore $q \in D_{\alpha_i}$. Hence by (i') we get $\alpha_i(q) = \alpha_i(p)$, $i = 1, \ldots, n$. Thus, $q \in \bigcap_i \alpha_i^{-1} A_i \subset U$. This yields $q \in U \in \mathrm{top}M$. By symmetry we get (ii).

Now, assume (ii). Take any $\alpha \in M(p)$. If would be $\alpha \notin M(q)$ we would have $q \notin D_\alpha$, where $\alpha \in M$. Setting $U = D_\alpha$ we get a contradiction with (ii). The equivalence (iii), (iii*) and (iv*) is obvious. Q.E.D.

We set $\underline{TM} = \bigcup_{p \in \underline{M}} \underline{T_p M}$ and denote the set of all mappings $\pi: \underline{TM} \to \underline{M}$ such that $v \in \underline{T_{\pi(v)} M}$ for $v \in \underline{TM}$ by $\Pi(M)$.

Theorem 1.2. For any $\pi, \pi' \in \Pi(M)$ and any $v \in \underline{TM}$ we have $M(\pi(v)) = M(\pi'(v))$. Moreover, for any $\alpha \in M$ we have $\alpha \circ \pi = \alpha \circ \pi'$.

Proof. Taking any $v \in D_{\alpha \circ \pi}$, i.e. $v \in \pi^{-1} D_\alpha$, where $\alpha \in M$, and setting $\pi(v) = p$ and $\pi'(v) = q$ we get $v \in \underline{T_p M} \cap \underline{T_q M}$. By Theorem 1.1 we get $M(p) = M(q)$ and, by (i'), $\alpha(p) = \alpha(q)$. In other words, $\alpha(\pi(v)) = \alpha(\pi'(v))$. Q.E.D.

For any $\pi, \pi' \in \Pi(M)$ and any $\alpha \in M$, according to Theorem 1.2, we have $\pi^{-1} D_\alpha = \pi'^{-1} D_\alpha$. We have then the correct definition of the differential $d\alpha$ of $\alpha \in M$:

(1) $\quad (d\alpha)(v) = v(\alpha) \quad$ for $\quad v \in \pi^{-1} D_\alpha, \quad$ where $\quad \pi \in \Pi(M)$.

Consequently, we have the correct definition of the premanifold TM as the smallest premanifold (see [1] and [2]) containing the set

(2) $$\{ d\alpha;\ \alpha \in M \} \cup \{ \alpha \circ \pi;\ \alpha \in M \},$$

where $\pi \in \Pi(M)$, because the set (2) does not depend of the mapping π belonging to $\Pi(M)$. The premanifold TM will be called the tangent bundle of M.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1.3. For any premanifold M, topM is a T_o-topology if and only if $\Pi(M)$ has the only one element.

Proof. Assume that topM is a T_o-topology and that there exist $\pi, \pi' \in \Pi(M)$, $\pi \neq \pi'$. We have then $\pi(v) \neq \pi'(v)$ for some $v \in \underline{TM}$. We set $p = \pi(v)$ and $q = \pi'(v)$. Then $v \in \underline{T_pM} \cap \underline{T_qM}$. According to Theorem 1.1 we get $p \neq q$ and (ii) would be satisfied. Now, assume that topM is not a T_o-topology. Then would exist points $p, q \in \underline{M}$, $p \neq q$, for which we would have (ii). From the definition of TM it follows that $\Pi(M) \neq \emptyset$. Let $\pi \in \Pi(M)$. By Theorem 1.1 we have $T_pM = T_qM$. Setting

$$\pi_1(v) = \begin{cases} \pi(v) & \text{for } v \in \underline{TM} - \underline{T_pM}, \\ p & \text{for } v \in \underline{T_pM} \end{cases}$$

and

$$\pi_2(v) = \begin{cases} \pi(v) & \text{for } v \in \underline{TM} - \underline{T_qM}, \\ q & \text{for } v \in \underline{T_qM} \end{cases}$$

we get $\pi_1, \pi_2 \in \Pi(M)$ and $\pi_1 \neq \pi_2$. Q.E.D.

The above consideration allows us to adopt, in the case topM is T_o-topology, the following definition. The only element, π_M, of the set $\Pi(M)$ is called the natural projection of TM onto M.

Remark. From the definition of TM it follows that for any $\pi \in \Pi(M)$ we have the smooth mapping $\pi: TM \longrightarrow M$. In particular $\pi_M: TM \longrightarrow M$, when topM is T_o-topology.

2. Vector fields

Let us take any smooth mapping

(3) $$f: M \longrightarrow N.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Any function $X: \underline{M} \longrightarrow \underline{TM}$ such that $X(p) \in T_{f(p)}N$ for $p \in \underline{M}$ will be called an f-vector field on M. For any $\beta \in N$ we have a well defined function $\partial_X\beta$ on the set $f^{-1}D_\beta$ by the formula $(\partial_X\beta)(p) = X(p)(\beta)$ for $p \in f^{-1}D_\beta$. The vector field X is said to be smooth iff for any $\beta \in N$ we have $\partial_X\beta \in M$.

Theorem 2.1. If M is a premanifold such that topM is a T_o-topology, then the conditions:

(i) X is a smooth vector field;

(ii) $X: M \longrightarrow TN$ and $\pi_N \circ X = f$;

are equivalent.

Proof. Assume (i). For any $p \in \underline{M}$ we have $X(p) \in T_{f(p)}N$. Then $\pi_N(X(p)) = f(p)$. First, we take any $\beta \in N$. We have $D_{d\beta} = \pi_N^{-1}D_\beta$ and $(d\beta)(X(p)) = X(p)(\beta) = (\partial_X\beta)(p)$ for $p \in \underline{M}$. In other words, $d\beta \circ X = \partial_X\beta$. Therefore $d\beta \circ X \in M$. The equality $\pi_N \circ X = f$ yields $\beta \circ \pi_N \circ X = \beta \circ f \in M$ for $\beta \in N$. Hence it follows that $X: M \longrightarrow TN$. Now, assume (ii). Take any $\beta \in N$. Then $\partial_X\beta = d\beta \circ X \in M$. Thus, X is a smooth f-vector field. Q.E.D.

We often consider f-vector fields X, where $N = M$ or $N = M_U$, $U \in$ topM and $f = id_U$. Such f-vector fields are called vector fields in M. The set of all smooth vector fields in M will be denoted by $V_{loc}(M)$.

3. The cotangent bundle

For any K-premanifold M we consider the set $\underline{T^*M} = \bigcup_{p \in \underline{M}} \underline{(T_pM)^*}$. Assume that topM is T_o-topology. From Theorem 1.3 it follows that $\Pi(M)$ has the only one element. The equivalence of the conditions (iii) and (iii*) in Theorem 1.1 allows us to state that there exists the only one mapping π_M^* from $\underline{T^*M}$ onto $\underline{M}$ such that $w \in (T_{\underline{\pi_M^*(w)}}M)^*$ for $w \in \underline{T^*M}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For any $X \in V_{loc}(M)$ we have $D_X \in \text{top}M$. Let us set

$$(X)(w) = w(X(\pi_M^*(w))) \quad \text{for} \quad w \in \pi_M^{*-1} D_X. \tag{4}$$

We have then the function (X) with the domain $\pi_M^{*-1} D_X$ and values in K. The smallest K-premanifold containing the set

$$\left\{ \alpha \circ \pi_M^*;\ \alpha \in M \right\} \cup \left\{ (X);\ X \in V_{loc}(M) \right\}$$

will be called the cotangent bundle of premanifold M and denoted by T^*M. By this definition we have the smooth mapping $\pi_M^*: T^*M \longrightarrow M$. It will be called the natural projection from T^*M onto M. Any function $\omega: U \longrightarrow \underline{T^*M}$, where $U \in \text{top}M$, satisfying the condition $\omega(p) \in \underline{(T_p M)^*}$ for $p \in U$ will be called a 1-form in M. The form ω will be called smooth in M iff for any $X \in V_{loc}(M)$ the function $\omega(X)$ defined on the set $D_\omega \cap D_X$ by the formula

$$\omega(X)(p) = \omega(p)(X(p)) \text{ for } p \in D_\omega \cap D_X$$

belongs to M.

Theorem 3.1. A 1-form ω in M is smooth iff we have the smooth mapping

$$\omega : M_{D_\omega} \longrightarrow T^*M. \tag{5}$$

Proof. Assume that ω is a 1-form smooth in M. We have $\pi_M^*(\omega(p)) = p$ for $p \in D_\omega$. This yields $(\alpha \circ \pi_M^*) \circ \omega = \alpha \mid D_\omega \cap D_\alpha \in M$ for $\alpha \in M$. Taking any $X \in V_{loc}(M)$ we state that the domain of the function $(X) \circ \omega$ is $\omega^{-1} \pi_M^{*-1} D_X$. We have $\omega^{-1} \pi_M^{*-1} D_X = (\pi_M^* \circ \omega)^{-1} D_X = D_\omega \cap D_X$ and, by (4), $(X)(\omega(p)) = \omega(p)(X(p)) = \omega(X)(p)$ for $p \in D_\omega \cap D_X$. Thus, $(X) \circ \omega = \omega(X) \in M$. We have then the smooth mapping (5). Now, assume (5) and take any $X \in V_{loc}(M)$. Then we have $(X) \circ \omega \in M_{D_\omega}$. By the inclusion $M_{D_\omega} \subset M$ and the equality $\omega(X) = (X) \circ \omega$ we get the smoothness of the form ω. Q.E.D.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. n-dimensional premanifolds

Let us consider a premanifold M and assume that topM is T_o-topology. The system $(e_1,\dots,e_n)$ of vector fields $e_i: M_U \longrightarrow TM$, where $U \in \mathrm{top}M$, $i = 1,\dots,n$, satisfying the conditions:

1) $(e_1(p),\dots,e_n(p))$ is a base for T_pM, $p \in M$;

2) for any $X \in \mathsf{V}_{loc}(M)$ there exist functions $\alpha^i \in M_{U \cap D_X}$ such that

$$X \mid U \cap D_X = \alpha^i e_i \tag{6}$$

(summing for i) is said to be an n-dimensional local vector base for M. Let $(e^1(p),\dots,e^n(p))$ stands for the base dual to the base $(e_1(p),\dots,e_n(p))$, i.e. $e^j(p)(e_i(p)) = \delta^j_i$, $i,j = 1,\dots,n$. Taking any $X \in \mathsf{V}_{loc}(M)$ we get $e^j(X)(p) = e^j(p)(X(p)) = \alpha^j(p)$ for $p \in U \cap D_X$. We have then $e^j(X) = \alpha^j \mid (U \cap D_X)$. Hence e^j is smooth, $j = 1,\dots,n$.

A premanifold M such that for some natural n every point p has an n-dimensional local vector base for M defined in some neighbourhood of p is called an n-dimensional premanifold. For any $a \in K^n$ we write $a = (a^1,\dots,a^n)$.

Theorem 4.1. For any n-dimensional local vector base $(e_1,\dots,e_n)$ we have a diffeomorphism

$$((p,a) \longmapsto a^i e_i(p)): M_U \times K^n \longrightarrow (TM)_{\pi_M^{-1}U} \tag{7}$$

where $e_i: M_U \longrightarrow TM$, $i = 1,\dots,n$.

Proof. Let us set $e(p,a) = a^i e_i(p)$ for $p \in U$, $a \in K^n$. We have then $\pi_M(e(p,a)) = p$. Thus, $(\alpha \circ \pi_M \circ e)(p,a) = \alpha(p)$ and $(d\alpha \circ e)(p,a) = e(p,a)(\alpha) = a^i e_i(p)(\alpha) = a^i(\partial_{e_i}\alpha)(p)$ for $(p,a) \in U \times K^n$. So, the mapping (7) is smooth. Similarly, we check that the inverse mapping is smooth. Q.E.D.

As a corollary we get

Corollary 4.2. The tangent bundle of any n-dimensional premanifold is locally trivial.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] W. Waliszewski: Analytical premanifolds, Zeszyty Nauk. Politech. Śląsk. Ser. Mat.-Fiz. 48 (1986), 217-226.

[2] W. Waliszewski: Complex premanifolds and foliations. Seminar on Deformations, Lublin-Łódź 1985/87, D. Reidel Publishing Company, Dodrecht.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WROCŁAW, 50-370 WROCŁAW, POLAND

Received October 4, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Francisco Gómez

THE NUMBER OF GENERATORS OF THE ALGEBRA OF KÄHLER DIFFERENTIALS

In this paper R denotes an integrity domain (i.e. R is a commutative ring with unit without zero divisors), $\tilde{R}$ is the field of fractions of R and A is an associative and commutative R-algebra with a unit and containing R as a subalgebra.

Associated with A we have the A-module of Kähler differentials $\Omega_R(A)$ defined by

$$\Omega_R(A) = I/I^2$$

where I is the kernel of the multiplication $A \otimes_R A \longrightarrow A$.

The purpose of this paper is to find lower bounds for the number of generators of $\Omega_R(A)$. In particular we show that for most algebras of real continuous functions the cardinal of any set of generators of $\Omega_R(A)$ is at least that of the real numbers. This is in contrast to the A-module of derivations of A in A, $\mathrm{Der}_R(A,A)$, which is the dual of $\Omega_R(A)$ and is zero for $A = C_{\mathbb{R}}(X)$ (algebra of real continuous functions on any topological space X).

(1) Definition of $n(\mathfrak{p})$. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap R = 0$ and let $k_{\mathfrak{p}}$ be the field of fractions of $A/\mathfrak{p}$ (which is canonically isomorphic to the residue field of the local ring $A_{\mathfrak{p}}$). We define then $n(\mathfrak{p})$ as the degree of transcendence of the canonical inclusion $\tilde{R} \subset k_{\mathfrak{p}}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) Theorem. Let $\mathfrak{p}$ be as above, then $n(\mathfrak{p})$ is a lower bound for the cardinal of any set of generators of $\Omega_R(A)$.

Proof. Let $R_{\mathfrak{p}}$ be the algebraic closure of $\tilde{R}$ in $k_{\mathfrak{p}}$ and consider the homomorphisms of $k_{\mathfrak{p}}$-vector spaces

$$\Omega_R(k_{\mathfrak{p}}) \xrightarrow{\cong} \Omega_{\tilde{R}}(k) \longrightarrow \Omega_{R_{\mathfrak{p}}}(k_{\mathfrak{p}}).$$

As a consequence of the first fundamental exact sequence, see page 186 of [3], we obtain that the left map is an isomorphism (because $\Omega_R(\tilde{R}) = 0$) and the right map is an epimorphism, which is also an isomorphism whenever the characteristic of R is zero.

Therefore

$$\dim_{k_{\mathfrak{p}}} \Omega_R(k_{\mathfrak{p}}) \geq \dim \Omega_{R_{\mathfrak{p}}}(k_{\mathfrak{p}}) = n(\mathfrak{p}). \tag{3}$$

We consider next the composite of the following sequence of homomorphisms of $k_{\mathfrak{p}}$-vector spaces, which are defined in the obvious way,

$$k_{\mathfrak{p}} \otimes_A \Omega_R(A) \xrightarrow[(a)]{\cong} k_{\mathfrak{p}} \otimes_A (A/\mathfrak{p} \otimes_A \Omega_R(A)) \xrightarrow[(b)]{} k_{\mathfrak{p}} \otimes_A \Omega_R(A/\mathfrak{p}) \xrightarrow[(c)]{\cong} \Omega_R(k_{\mathfrak{p}}) \tag{4}$$

where (a) is clearly an isomorphism, (b) is an epimorphism because of the second fundamental exact sequence, see page 187 of [3], and (c) is an isomorphism since Ω_R commutes with formation of fractions.

Therefore (3) and (4) yield

$$\dim_{k_{\mathfrak{p}}} (k_{\mathfrak{p}} \otimes_A \Omega_R(A)) \geq n(\mathfrak{p}). \tag{5}$$

Finally, if $(\alpha_i)_{i\in I}$ is any set of generators of $\Omega_R(A)$ as an A-module we have an epimorphism $\bigoplus_{i\in I} A \longrightarrow \Omega_R(A)$ which sends $(a_i)_{i\in I}$ to $\sum_{i\in I} a_i\alpha_i$, and so we obtain an epimorphism of $k_{\mathfrak{p}}$ vector spaces $\bigoplus_{i\in I} k_{\mathfrak{p}} \longrightarrow k_{\mathfrak{p}} \otimes_A \Omega_R(A)$ and this, together with (5), shows that $|I| \geq n(\mathfrak{p})$, where $|I|$ denotes the cardinal of I. Q.E.D.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(6) Proposition. Let $\{a_i\}_{i\in I}$ be a set of elements of A such that the canonical map $R[(X_i)_{i\in I}] \longrightarrow R[(a_i)_{i\in I}]$ is an isomorphism (i.e. the elements a_i are algebraically independent over R). There exists then a prime ideal $\mathfrak{p}$ of A with $\mathfrak{p} \cap R = 0$ and such that the canonical image of the elements a_i in $k_{\mathfrak{p}}$ are algebraically independent over R and so $|I| \leq n(\mathfrak{p})$. In particular we have, because of theorem (2), that $|I|$ is a lower bound for the cardinal of any set of generators of $\Omega_R(A)$.

Proof. Let $\mathfrak{p}$ be an ideal of A which is maximal among those ideals $\mathfrak{p}$ of A such that $R[(a_i)_{i\in I}] \cap \mathfrak{p} = 0$. The ideal $\mathfrak{p}$ is clearly prime because of the hypothesis on $\{a_i\}_{i\in I}$ and the canonical images of the a_i in $k_{\mathfrak{p}}$ are algebraically independent over R because $\mathfrak{p} \cap R[(a_i)_{i\in I}] = 0$. Q.E.D.

Remark. The idea for considering $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal satisfying $\mathfrak{p} \cap R[(a_i)]_{i\in I} = 0$, was given to me by R. Swan.

(7) Proposition. If $da = 0$ for $a \in A$, then a is a zero of a nonzero polynomial in $R[X]$. Here d denotes the universal derivation on A, $d: A \longrightarrow \Omega_R(A)$, which sends x to dx = class of $(x\otimes 1 - 1\otimes x)$ modulo I^2 (with I = kernel of the multiplication $A \otimes_R A \longrightarrow A$).

Proof. If a was not a zero of any nonzero polynomial of $R[X]$, proposition (6) would imply the existence of a prime ideal $\mathfrak{p}$ of A such that the image $\bar{a}$ of a in $k_{\mathfrak{p}}$ would be transcendental over $\tilde{R}$. Therefore $d\bar{a} \neq 0$ and so $da \neq 0$, which would contradict our hypothesis.

(8) Theorem. Let A be a subalgebra of the algebra $C_R(X)$ of real continuous functions on a topological space X. Then $\Omega_R(A) = 0$ if and only if each element of A has finite image.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Assume $\Omega_R(A) = 0$. Thus $df = 0$ for all $f \in A$ and now (7) implies that we have $\lambda_o + \lambda_1 f + \ldots + \lambda_n f^n = 0$ with $\lambda_i \in R$ $(i=0,\ldots,n)$, $\lambda_n \neq 0$. This shows that f takes only a finite number of values.

Conversely assume that each element of A takes only a finite number of values. Let $\lambda_1,\ldots,\lambda_n$ be the nonzero values taken by $f \in A$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. The sets $X_i = \{x \in X | f(x) = \lambda_i\}$ $(i=1,\ldots,n)$ are both open and closed and so they are union of connected components of X. Then $1,f,\ldots,f^{n-1}$ is a basis of the real vector space $R[f]$ because of the following determinant (Vandermonde) being nonzero:

$$\begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & & \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} = \prod_{i>j} (\lambda_i - \lambda_j).$$

In particular the functions e_i $(i=1,\ldots,n)$ equal 1 on X_i and 0 on $X-X_i$ belong to $R[f]$ and they consitute a basis. Now, since $f = \sum_{i=1}^{n} \lambda_i e_i$ and $e_i^2 = e_i$ we have $df = 0$ in view of the following lemma. Q.E.D.

(9) Lemma. Let x be an idempotent element of A (i.e. $x^2 = x$), then $dx = 0$. In particular $\Omega_R(A) = 0$ if A is generated by idempotent elements.

Proof. $x \otimes 1 - 1 \otimes x = (x \otimes 1 - x \otimes x)^2 - (1 \otimes x - x \otimes x)^2 \in I^2$ where I is the kernel of the multiplication $A \otimes_R A \longrightarrow A$. Therefore $dx = 0$. Q.E.D.

(10) Theorem. Let A be a subalgebra of the algebra of real continuous functions $C_R(X)$, on an arbitrary topological space X and assume that there exists a function $f \in A$ satisfying the following three properties:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

a) f is nonnegative,

b) $f^r \in A$ for each positive real number r,

c) f takes arbitrarily small positive values.

In these hypotheses any set of generators of $\Omega_{\mathbb{R}}(A)$ has cardinal at least that of the real numbers.

Proof. The idea for this proof is taken from theorem on page 173 of [2].

Let $f \in A$ satisfying (a), (b) and (c) above and choose a family of positive real numbers $(r_i)_{i\in I}$ which are taken to be $\mathbb{Q}$-linearly independent and with I having the cardinal of the reals. We show now that the functions $(f^{r_i})_{i\in I}$ are $\mathbb{R}$-algebraically independent and this completes the proof by using proposition (6).

Let P be a real polynomial in the variables $(X_i)_{i\in I}$ and suppose $P \in \mathbb{R}[X_1,\ldots,X_q]$. Assume that P has m distinct monomials with non-zero coefficients $(m \geq 1)$:

$$P = \sum_{k=1}^{m} a_k X_1^{n_{k1}} \cdots X_q^{n_{k1}} \quad \text{with } a_k \neq 0 \quad (k=1,\ldots,m) \text{ and}$$

$$(n_{k1},\ldots,n_{kq}) = (n_{k'1},\ldots,n_{k'q}) \Longleftrightarrow k = k'.$$

We want to show that $P(f^{r_1},\ldots,f^{r_q}) \neq 0$. Let $x \in X$, we have then

$$P(f^{r_1},\ldots,f^{r_q})(x) = \sum_{k=1}^{m} a_k f(x)^{r_1 n_{k1}+\ldots+r_q n_{kq}}.$$

Set $s_k = r_1 n_{k1} + \cdots + r_q n_{kq}$ $(k = 1,\ldots,m)$.

The numbers s_k are clearly nonnegative and distinct because whenever $s_k = s_{k'}$, we have $\sum_{i=1}^{q} r_i(n_{ki}-n_{k'i}) = 0$ and so, the r_i being $\mathbb{Q}$-linearly independent, $n_{ki} = n_{k'i}$ $(i=1,..,,q)$. Thus $k = k'$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let s_j be the minimum of $\{s_1,\dots,s_m\}$ and write

$$\sum_{k=1}^{m} a_k f(x)^{s_k} = f(x)^{s_j}\left(a_j + \sum_{k\neq j} a_k f(x)^{s_k-s_j}\right) \text{ with } s_k-s_j>0 \text{ for all } k\neq j.$$

But now, since f takes arbitrarily small positive values, there exists $x\in X$ such that $f(x)>0$ and

$$\left|\sum_{k\neq j} a_k f(x)^{s_k-s_j}\right| < |a_j|.$$

Therefore, for this x, we have that $a_j + \sum_{k\neq j} a_k f(x)^{s_k-s_j}$ has the same sign as a_j. In particular it is nonzero and therefore $\sum_{k=1}^{m} a_k f(x)^{s_k} \neq 0$. Q.E.D.

(11) Theorem. If we replace condition (c) of theorem (10) by condition (d) below, the same result holds.

(d) $f(X)$ has nonempty interior in $\mathbf{R}$.

Proof. We follow word for word the proof of (10) until we have $\sum_{k=1}^{m} a_k f(x)^{s_k}$ with all $s_k\geq 0$ and distinct.

Condition (d) implies the existence of an interval $[a,b]$ contained in $f(X)$ and such that $a\geq 0$. Choose then a real number λ such that $0<\lambda<1$ and $\lambda^m > \frac{a}{b}$. Therefore

$$a < b\lambda^m < b\lambda^{m-1} < \dots < b\lambda < b.$$

But now, by hypothesis, there exist points $x_1,\dots,x_m$ in X such that $f(x_k) = b\lambda^k$ $(k=1,\dots,m)$ and we have

$$\begin{vmatrix} f(x_1)^{s_1} & \dots & f(x_1)^{s_m} \\ \vdots & & \vdots \\ f(x_m)^{s_1} & & f(x_m)^{s_m} \end{vmatrix} = b^{s_1+\dots+s_m}\lambda^{s_1+\dots+s_m}\cdot\prod_{i>j}(\lambda^{s_i}-\lambda^{s_j}) \neq 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Thus, $\sum_{k=1}^{m} a_k f(x_i)^{s_k} \neq 0$ for some i, because if not we had $a_k = 0$ $(k=1,\dots,m)$ contrary to our hypothesis. This shows that $P\left(f^{r_1},\dots,f^{r_q}\right) \neq 0$.

(12) Remark. Condition (d) of theorem (11) above can be, of course, replaced by the following weaker condition, which is really what is used in the proof:

(d') For each finite set of nonnegative distinct real numbers $s_1,\dots,s_m$, there exist real numbers $\lambda_1,\dots,\lambda_m$ in the image of f such that

$$\begin{vmatrix} \lambda_1^{s_1} & \dots & \lambda_1^{s_m} \\ \vdots & & \vdots \\ \lambda_m^{s_1} & \dots & \lambda_m^{s_m} \end{vmatrix} \neq 0.$$

(13) Definition. For each topological space X there exists a completely regular space X' and a surjective continuous map $\tau : X \longrightarrow X'$ such that $f \longrightarrow f\circ\tau$ is an isomorphism from $C_R X'$ onto $C_R X$ (see Theorem 3.9 page 41 of [2]). The space X' is the quotient of X by the equivalence relation $x \sim x' \Longleftrightarrow f(x) = f(x')$ for all $f \in C_R X$ (i.e. it identifies points that cannot be separated by real continuous functions). X' is endowed with the weak topology with respect to the family of maps $\left\{ f : X' \longrightarrow R \mid f\tau \in C_R(X) \right\}$ where $\tau : X \longrightarrow X'$ is the canonical map. We say that X' is the completely regular space associated to X.

(14) Corollary (of Theorem 10). Let X be a topological space whose completely regular space associated have infinite by many points, then any set of generators of $\Omega_R(C_R(X))$ has at least the cardinal of the reals.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. a) Suppose first that X is not pseudocompact, i.e. there exists a non bounded function $g \in C_R(X)$. Define then $f = \frac{1}{1+g^2}$. It is clear that f satisfies conditions (a), (b) and (c) of Theorem (10) and so the Corollary holds in this case.

b) Suppose now that X is pseudocompact, then its associate completely regular space X' is again pseudocompact and has an infinite number of points by hypothesis. Let $\beta X'$ be the Stone-Čech compactification of X' and we have isomorphisms

$$C_R(\beta X') \xrightarrow[i^*]{\cong} C_R(X') \xrightarrow[\tau^*]{\cong} C_R(X)$$

($i: X' \hookrightarrow \beta X'$ and $\tau: X \longrightarrow X'$ is the canonical projection). Let X_o be a connected component of $\beta X'$ having at least two points. It exists because if all components of $\beta X'$ were points, $\beta X'$ would have a finite number of points and the same would happen to X' contrary to our hypothesis. Let $f: \beta X' \longrightarrow R$ be continuous not vanishing identically on X_o, but having one zero in X_o. Clearly f^2 satisfies the hypothesis of Theorem (10) and so the Corollary also holds now.

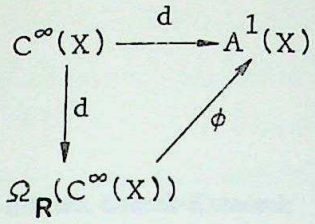
(15) Corollary (of Theorem 11). Let X be a C^∞-manifold of dimension $n \geq 1$. Then the cardinal of any set of generators of $\Omega_R(C^\infty(X))$ is at least that of the real numbers.

Proof. It is clear the existence of a function $f \in C^\infty(X)$ such that is positive and f(X) has nonempty interior. Then we apply theorem (11) since f^r is also C^∞ for all $r > 0$ since we are away from zero.

(16) Theorem. Let X be a C^∞ manifold of dimension $n \geq 0$ and let $\phi: \Omega_R(C^\infty(X)) \longrightarrow A^1(X)$ be the canonical epimorphism, where $A^1(X)$ denotes the C^∞ 1-forms on X and ϕ is the unique homomorphism of $C^\infty(X)$-modules making commutative the diagram

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri



Then ϕ is an isomorphism if and only if X consists of a finite number of points, and then both $\Omega_R(C^\infty(X))$ and $A^1(X)$ are zero.

Proof. It is an easy consequence of (14) and (15) together with the fact that $A^1(X)$ is a finitely generated $C^\infty(X)$-module when X is connected, see Corollary on page 107 of [1]).

(17) Remark. If X is a C^∞ manifold with an infinite number of components we have two differentials defined on $C^\infty(X)$: the algebraic differential $d: C^\infty(X) \to \Omega_R(C^\infty(X))$ and the usual exterior derivative $d: C^\infty(X) \to A^1(X)$. We have shown that the functions with zero differential are not the same for both, since if f is constant on each component but having an infinite image the algebraic differential is nonzero but the usual exterior derivative is zero.

REFERENCES

[1] W. Greub, S. Halperin, R. Vanstone: Connections, Curvature and Cohomology, vol. I, Academic Press (1972).

[2] L. Gillman, M. Gerison: Rings of continuous functions. Springer Verlag (1976).

[3] H. Matsumura: Commutative Algebra, 2nd Ed., Benjamin (1980).

DEPARTAMENTO DE ALGEBRA, GEOMETRIA Y TOPOLOGIA,
FACULTAD DE CIENCIAS, UNIVERSIDAD DE MALAGA,
29080 MALAGA, ESPANA

Received October 5, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Elżbieta Łuczak-Kumorek

THE EXISTENCE THEOREM FOR NEUTRAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS

1. Introduction

The aim of this paper is to present the existence theorem for functional-differential inclusions of the form

$$\frac{d}{dt} D(t,x_t) \in F(t,x_t)$$

where F is a multivalued mapping having a Carathéodory selector and taking as its values nonempty closed compact but not necessarily convex or nonempty closed convex subsets of R^n and D is a single-valued mapping with values in R^n. We extend the results of J.K.Hale [4] on the functional-differential inclusions of neutral type.

2. Notations and definitions

Suppose $r \geq 0$ is a given real number, $R = (-\infty, +\infty)$, R^n is a n-dimensional linear vector space with a norm $|\cdot|$, $C([a,b],R^n)$ is the Banach space of continuous functions endowed with the topology of uniform convergence. If $[a,b] = [-r,0]$ we let $C_{or} = C([-r,0],R^n)$ and denote the norm of an element $\phi \in C_{or}$ by $\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$.

If $\sigma \in R$, $a \geq 0$ and $x \in C([\sigma - r, \sigma + a], R^n)$, then for any $t \in [\sigma, \sigma + a]$, we define $x_t \in C_{or}$ by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By $P(X)$, $CCl(X)$ and $Comp(X)$ we denote the spaces of all nonempty, nonempty closed convex and nonempty closed compact, respectively, subsets of metric spaces X.

Let $(T,\mathcal{F})$ be a measurable space, X be a metric space. A set-valued function $F: T \longrightarrow P(X)$ is said to be measurable (weakly measurable), if $\{t \in T: F(t) \cap U \neq \emptyset\} \in \mathcal{F}$ for every closed (open) set $U \subset X$ (see [3]).

If $F: Y \longrightarrow P(X)$, where Y is a topological space, then the assertion that F is measurable (weakly measurable) means that F is measurable (weakly measurable) when Y is assigned by σ-algebra $\mathcal{B}(Y)$ of Borel subsets of Y.

If $F: T \times Y \longrightarrow P(X)$ then the various kinds of measurability of F are defined in terms of the product σ-algebra $\mathcal{F} \otimes \mathcal{B}(Y)$ on $T \times Y$. (For $T = R$ we will consider T together with the σ-algebra of all its Lebesgue measurable subsets).

A multivalued mapping $F: X \longrightarrow P(Y)$ is lower semicontinuous if the set $\{x \in X: F(x) \cap Z \neq \emptyset\}$ is open for every open subset Z of Y.

Let $CCl(R^n)$ be the family of all nonempty closed convex subsets of R^n, $\Omega \subseteq R \times C_{or}$ be an open set. Assume that $F: \Omega \longrightarrow CCl(R^n)$ satisfies the following conditions:

(i) $F: \Omega \longrightarrow CCl(R^n)$ is weakly measurable,

(ii) $F(t,\cdot)$ is lower semicontinuous for each fixed $t \in R$,

(iii) there is a Lebesgue integrable function $m: R \longrightarrow R^+$ such that $h(F(t,\phi), \{0\}) \leq m(t)$ for $(t,\phi) \in \Omega$, where h denotes the Hausdorff metric in $CCl(R^n)$.

Definition 1. Suppose Ω is an open set in $R \times C_{or}$, $D: \Omega \longrightarrow R^n$ is continuous, $\phi \longrightarrow D(t,\phi)$ has a continuous Fréchet derivative $D'_\phi(t,\phi)$ and

$$D'_\phi(t,\phi)\psi = \int_{-r}^{0} [d_\theta \eta(t,\phi,\theta)]\, \psi(\theta)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $(t,\phi)\in\Omega$, $\psi\in C_{or}$, where $\eta(t,\phi,\theta)$ is an $n\times n$ matrix function with elements of bounded variation in $\theta\in[-r,0]$. For any $\beta\in[-r,0]$ we say that $D(t,\phi)$ is atomic at β on Ω, if

$$\eta(t,\phi,\beta^+) - \eta(t,\phi,\beta^-) = A(t,\phi,\beta),\ \det A(t,\phi,\beta)\neq 0$$

for some $n\times n$ matrix function $A(t,\phi,\beta)$ continuous in (t,ϕ) and there is a scalar function $\gamma(t,\phi,s,\beta)$ continuous for $(t,\phi)\in\Omega$, $s\geq 0$, $\gamma(t,\phi,0,\beta)=0$ such that $\left|\int_{\beta-s}^{\beta+s}[d_\theta\eta(t,\phi,\theta)]\psi(\theta) - A(t,\phi,\beta)\psi(\beta)\right|\leq$ $\leq\gamma(t,\phi,s,\beta)\|\psi\|$ for $(t,\phi)\in\Omega$, $s\geq 0$, $\psi\in C_{or}$.

Definition 2. Suppose $\Omega\subseteq R\times C_{or}$ is an open set, $D:\Omega\to R^n$ is a given continuous function atomic at zero. The relation

$$\frac{d}{dt}D(t,x_t)\in F(t,x_t) \tag{1}$$

is called the neutral functional-differential inclusion (NFDI).

Definition 3. For a given NFDI a function $x:[\sigma-r,\sigma+a)\to R^n$ is said to be a solution of (1) if there are $\sigma\in R$, $a>0$, such that $x\in C([\sigma-r,\sigma+a),R^n)$, $(t,x_t)\in\Omega$, $t\in[\sigma,\sigma+a)$, $t\to D(t,x_t)$ is continuously differentiable and satisfies (1) a.e. on $[\sigma,\sigma+a)$.

Definition 4. For a given $\sigma\in R$, $\phi\in C_{or}$ and $(t,\phi)\in\Omega$ we say that x is a solution of inclusion (1) with initial value ϕ at σ or simply a solution through (σ,ϕ), if there is an $a>0$ such that x is a solution of (1) on $[\sigma-r,\sigma+a)$ and $x_\sigma=\phi$.

Definition 5. Let T be measurable space and X be a metric space. A function $f: T\times T\to R^n$ is said to be a Carathéodory selector of a set-valued function $F: T\times X\to CCl(R^n)$, if f is a selector for F and f is such that $f(\cdot,x)$ is measurable for $x\in X$ and $f(t,\cdot)$ is continuous for $t\in T$.

Definition 6. A set-valued function $F:T\times X\to P(R^n)$, where T and X are as above, is called an M-mapping if every its lower semi-continuous restriction has a continuous selector.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Existence theorem

Theorem 1. If $\Omega \subseteq R \times C_{or}$ is an open set, (1) is NFDI and $F: \Omega \to CCl(R^n)$ satisfies conditions (i)-(iii) then for any $(\sigma,\phi) \in \Omega$ there exists a solution of (1) through (σ,ϕ).

Proof. A function x is a solution of (1) through (σ,ϕ) if there is an $a>0$ such that $x \in C([\sigma-r,\sigma+a),R^n)$ and

$$(2) \quad \begin{cases} \frac{d}{dt} D(t,x_t) \in F(t,x_t) & \text{for a.e. } t \in [\sigma,\sigma+a], \\ x_\sigma = \phi. \end{cases}$$

Since $F: \Omega \to CCl(R^n)$ satisfies the conditions (i)-(iii) then, in virtue of [2], there exists a mapping $f: \Omega \to R^n$ such that:

(a) $f(t,\cdot)$ is continuous for each fixed $t \in R$,

(b) f is measurable,

(c) $f(t,z) \in F(t,z)$ for $(t,z) \in \Omega$.

Clearly, f is a Carathéodory selection for F. Then x is a solution of (1) through (σ,ϕ), whenever there is an $a>0$ such that x satisfies

$$(3) \quad \begin{cases} D(t,x_t) = D(\sigma,\phi) + \int_\sigma^t f(\tau,x_\tau)\,d\tau \\ \qquad\qquad\qquad\qquad \text{for a.e. } t \in [\sigma,\sigma+a], \\ x_\sigma = \phi, \end{cases}$$

where f is a Carathéodory selection of a set-valued map F. Let $\hat{\phi}: [-r,\infty) \to R^n$ be defined by

$$\hat{\phi}(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ \phi(0) & \text{for } t \in [0,\infty), \end{cases}$$

and let $C^o([\sigma,\sigma+a], R^n)$ denote the Banach space of all continuous functions $z: [\sigma,\sigma+a] \to R^n$ such that $z(\sigma) = 0$. For every $z \in C^o([\sigma,\sigma+a], R^n)$ let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\hat{z}(t) = \begin{cases} z(t) & \text{for } t \in [\sigma, \sigma+a], \\ 0 & \text{for } t \in [\sigma-r, \sigma]. \end{cases}$$

We have, of course, $\hat{z} \in C([\sigma-r, \sigma+a], R^n)$ and $\hat{z}\big|_{[\sigma,\sigma+a]} = z$.

Now, we can define for each fixed $\sigma \in R$ and $a > 0$ a mapping $\hat{\phi} \oplus \hat{z}$ by setting

$$(4) \qquad (\hat{\phi} \oplus \hat{z})(t) = \begin{cases} \hat{\phi}(t-\sigma) & \text{for } t \in [\sigma-r, \sigma], \\ \hat{\phi}(0)+\hat{z}(t) & \text{for } t \in [\sigma, \sigma+a]. \end{cases}$$

In what follows, we shall denote $\hat{\phi} \oplus \hat{z}$ by $\phi \oplus z$.

Let us observe that the functional-integral equation (3) is equivalent to the following one

$$(3') \qquad \begin{cases} D(t, (\phi \oplus z)_t) = D(\sigma, \phi) + \int_\sigma^t f(\tau, (\phi \oplus z)_\tau)\, d\tau \\ \qquad\qquad \text{for a.e. } t \in [\sigma, \sigma+a], \\ z_\sigma = 0. \end{cases}$$

In this way x is a solution of (1) through (σ, ϕ), if there is an $a > 0$ and $z \in C([\sigma, \sigma+a], R^n)$ such that z satisfies (3') and $x = \phi \oplus z$.

Since $D(t, \phi)$ is continuously differentiable in ϕ, then

$$(5) \qquad D(t, (\phi \oplus z)_t) = D(t, (\phi \oplus 0)_t) +$$

$$+ D'_\phi(t, (\phi \oplus 0)_t)\left[(\phi \oplus z)_t - (\phi \oplus 0)_t\right] + g(t, (\phi \oplus 0)_t, \left[(\phi \oplus z)_t - (\phi \oplus 0)_t\right]),$$

where $g(t, \varphi, 0) = 0$, $|g(t, \varphi, \psi) - g(t, \varphi, \xi)| \le \xi(t, \varphi, \delta)\|\psi - \xi\|$ for $(t, \varphi) \in \Omega$, $\|\psi\|, \|\xi\| \le \delta$ and $\xi(t, \varphi, \delta)$ is continuous in t, φ, δ for $(t, \varphi) \in \Omega$, $\delta \ge 0$ and $\xi(t, \varphi, 0) = 0$. By (4), we have $(\phi \oplus z)_t - (\phi \oplus 0)_t = z_t$. Hence, by (5), it follows that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$D'_{\Phi}(t,(\Phi \oplus 0)_t)z_t = D(t,(\Phi \oplus z)_t) - D(t,(\Phi \oplus 0)_t) - g(t,(\Phi \oplus 0)_t, z_t).$$

Therefore, by (3'), x is a solution of (1) through (σ,Φ), if $x(t) = (\Phi \oplus z)(t)$ for $t \in [\sigma-r,\sigma+a]$ and z satisfies

$$(6)\quad \begin{cases} D'_{\Phi}(t,(\Phi \oplus 0)_t)z_t = D(\sigma,\Phi) - D(t,(\Phi \oplus 0)_t) - \\ \qquad - g(t,(\Phi \oplus 0)_t, z_t) + \int_{\sigma}^{t} f(\tau,(\Phi \oplus z)_\tau)d\tau \text{ for a.e. } t \in [\sigma,\sigma+a], \\ z_\sigma = 0. \end{cases}$$

Using the fact that D is atomic at 0 on Ω, we have (as long as $(t,(\Phi \oplus 0)_t) \in \Omega$)

$$A(t,(\Phi \oplus 0)_t,0)z_t(0) + \int_{-r}^{0-} [d_\theta \ell(t,(\Phi \oplus 0)_t,\theta)]\, z_t(\theta) =$$

$$= D(\sigma,\Phi) - D(t,(\Phi \oplus 0)_t) - g(t,(\Phi \oplus 0)_t, z_t) + \int_{\sigma}^{t} f(\tau,(\Phi \oplus z)_\tau)\, d\tau.$$

Then, since $z_t(0) = z(t)$, we have

$$(7)\quad \begin{cases} z(t) = \left[A(t,(\Phi \oplus 0)_t,0)\right]^{-1} \left\{ - \int_{-r}^{0-} \left[d_\theta \ell(t,(\Phi \oplus 0)_t,\theta)\right] z_t(\theta) + \right. \\ \left. + D(\sigma,\Phi) - D(t,(\Phi \oplus 0)_t) - g(t,(\Phi \oplus 0)_t, z_t) + \int_{\sigma}^{t} f(\tau,(\Phi \oplus z)_\tau)d\tau \right\} \\ \qquad\qquad \text{for a.e. } t \in [\sigma,\sigma+a], \\ z_\sigma = 0. \end{cases}$$

If we let

$$(8)\quad (Tz)(t) = \begin{cases} 0 \text{ for } t \in [\sigma-r,\sigma], \\ \left[A(t,(\Phi \oplus 0)_t,0)\right]^{-1} \left\{ - \int_{-r}^{0-} \left[d_\theta \ell(t,(\Phi \oplus \theta)_t,\theta)\right] z_t(\theta) + \right. \\ \left. + D(\sigma,\Phi) - D(t,(\Phi \oplus 0)_t) - g(t,(\Phi \oplus 0)_t, z_t) \right\} \\ \text{for a.e. } t \in [\sigma,\sigma+a], \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(9)\qquad (Sz)(t) = \begin{cases} 0 & \text{for } t \in [\sigma-r,\sigma], \\ [A(t,(\varphi \oplus 0)_t,0)]^{-1} \int_\sigma^t f(\tau,(\varphi \oplus z)_\tau)\, d\tau & \\ \text{for a.e. } t \in [\sigma,\sigma+a], & \end{cases}$$

then (7) is equivalent to $z(t) = (Tz)(t) + (Sz)(t)$ where $z \in C([\sigma-r,\sigma+a],R^n)$, $z_\sigma = 0$.

The proof of Theorem 1 will be complete if we show the existence of a fixed point of $T + S$ in $\mathcal{A}(a,b) = \{\xi \in C([\sigma-r,\sigma+a],R^n): \xi_\sigma = 0, \|\xi_t\| \le b \text{ for } t \in [\sigma,\sigma+a]\}$. For this purpose we will show the following lemma.

Lemma 1. There are positive real numbers a, b and $\bar{a},\bar{b}, 0<\bar{a}<a$, $0<\bar{b}<b$, such that $T: \mathcal{A}(\bar{a},\bar{b}) \longrightarrow C([\sigma-r,\sigma+a],R^n)$ and $S: \mathcal{A}(\bar{a},\bar{b}) \longrightarrow C([\sigma-r,\sigma+a],R^n)$, have the properties

1° T is a contraction,

2° S is completely continuous,

3° $T + S : \mathcal{A}(\bar{a},\bar{b}) \longrightarrow \mathcal{A}(\bar{a},\bar{b})$,

where T and S are mappings defined by (8) and (9), respectively.

Proof. For any ν, $0<\nu<\frac{1}{4}$, there are $a>0$, $b>0$, such that $(t,(\varphi \oplus 0)_t) \in \Omega$ and

$$\left|[A(t,(\varphi \oplus 0)_t,0)]^{-1}\right| \xi(t,(\varphi \oplus 0)_t,b) < \nu,$$

$$\left|[A(t,(\varphi \oplus 0)_t,0)]^{-1}\right| \gamma(t,(\varphi \oplus 0)_t,a,0) < \nu$$

for $t \in [\sigma,\sigma+a]$ where $\gamma(t,\varphi,s,\beta)$ is the function from Definition 1 and $\xi(t,\varphi,\delta)$ is the function in the proof of Theorem 1. For any non-negative real $\bar{a}$ and $\bar{b}$ let

$$\mathcal{A}(\bar{a},\bar{b}) = \{\xi \in C([\sigma-r,\sigma+\bar{a}],R^n) : \xi_\sigma = 0, \|\xi_t\| \le \bar{b} \text{ for } t \in [\sigma,\sigma+a]\}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For any $0<\bar{b}<b$, there is an $\bar{a}$, $0<\bar{a}<a$, so that $\|(\varphi \oplus 0)_t - \varphi\| < b-\bar{b}$. Further, we restrict $\bar{a}$ so that for $t \in [\sigma, \sigma+\bar{a}]$ we have

$$|[A(t,(\varphi \oplus 0)_t, 0)]^{-1}| \, |D(\sigma,\varphi) - D(t,(\varphi \oplus 0)_t)| \leq \nu \bar{b},$$

$$|[A(t,(\varphi \oplus 0)_t, 0)]^{-1}| \, \left|\int_{\sigma}^{t} m(\tau)d\tau\right| \leq \nu \bar{b}.$$

We now show that T and S satisfy the conditions 1° - 3°. In fact, by (8) and the above restrictions on $\bar{a}$ and $\bar{b}$, for any $z,y \in \mathcal{A}(\bar{a},\bar{b})$ we have

$$|(Tz)(t) - (Ty)(t)| =$$

$$= \left| [A(t,(\varphi \oplus 0)_t, 0)]^{-1} \left\{ - \int_{-r}^{0-} [d_\theta \eta(t,(\varphi \oplus 0)_t, \theta)] z_t(\theta) - D(t,(\varphi \oplus 0)_t) + \right.\right.$$

$$+ D(\sigma,\varphi) - g(t,(\varphi \oplus 0)_t, z_t) + \int_{-r}^{0-} [d_\theta \eta(t,(\varphi \oplus 0)_t,\theta)] y_t(\theta) +$$

$$\left.\left. + D(t,(\varphi \oplus 0)_t) - D(\sigma,\varphi) + g(t,(\varphi \oplus 0)_t, y_t) \right\} \right| \leq$$

$$\leq \left| [A(t,(\varphi \oplus 0)_t, 0)]^{-1} \right| (\xi(t,(\varphi \oplus 0)_t, \bar{b}) \|z_t - y_t\| +$$

$$+ \gamma(t,(\varphi \oplus 0)_t, \bar{a}, 0) \cdot \|z_t - y_t\|) \leq \nu \|z_t - y_t\| + \nu \|z_t - y_t\| < \frac{1}{2} \|z_t - y_t\|.$$

Hence, T is a contraction.

Let $z,y \in \mathcal{A}(\bar{a},\bar{b})$ and let $\mu = Tz + Sy$. Since, for $t \in [\sigma - r, \sigma]$, by (8), (9), $(Tz)(t) + (Sy)(t) = 0$ and for $t \in [\sigma, \sigma+\bar{a}]$ we have

$$|\mu(t)| = \left| [A(t,(\varphi \oplus 0)_t, 0)]^{-1} \left\{ - \int_{-r}^{0-} [d_\theta \eta(t,(\varphi \oplus 0)_t, \theta)] \, z_t(\theta) + \right.\right.$$

$$\left.\left. + D(\sigma,\varphi) - D(t,(\varphi \oplus 0)_t) - g(t,(\varphi \oplus 0)_t, z_t) + \int_{\sigma}^{t} f(\tau,(\varphi \oplus y)_\tau)d\tau \right\} \right| \leq$$

$$\leq \nu\bar{b} + \nu\bar{b} + \nu\bar{b} + \nu\bar{b} < \bar{b},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

therefore $T + S : \mathcal{A}(\bar{a},\bar{b}) \to \mathcal{A}(\bar{a},\bar{b})$. It is not difficult to show that S is continuous. Moreover, S is compact, since for every $t_1, t_2 \in [\sigma - r, \sigma + \bar{a}]$

$$|(Sz)(t_1) - (Sz)(t_2)| \leq \frac{1}{2}\bar{b}.$$

This completes the proof of Lemma 1.

Lemma 1 yields the existence of a fixed point of T + S in $\mathcal{A}(\bar{a},\bar{b})$ (see [5], Lemma 2.1) and thus a solution of (1) through (σ,ϕ). Then the proof of Theorem 1 is completed.

Remark. Theorem 1 will be true in the case of the multivalued mapping $F : \Omega \to \mathrm{Comp}(R^n)$ if F is an M-mapping such that F is measurable on Ω, $F(t,\cdot)$ is lower semicontinuous for each fixed t and $\mathrm{pr}_R(\Omega)$ has a finite measure, where $\mathrm{pr}_R(\Omega)$ denotes the projection of Ω on the real line. Then, in virtue of [1] ΩF has a Carathéodory selector f and the proof of Theorem 1 will be analogous.

REFERENCES

[1] Z. Artstein, K. Prikry: Carathéodory selections and the Scorza Dragoni property, J. Math. Anal. Appl. 127 (1987) 540-547.

[2] C. Castaing: A proposé de l'existence des sections séparément mesurables et séparément continues d'une multi-application séparément mesurables et séparément semi-continue inférieurement, Séminaire d'Analyse Convexe, Montpellier 1976, Ex.6.

[3] C.J. Himmelberg: Measurable relations, Fund. Math. 87 (1975) 53-72.

[4] J.K. Hale: Forward and backward continuation for neutral functional-differential equations, J. Diff. Equations 9 (1971) 168-181.

[5] J.K. Hale, M.A. Cruz: Existence, uniqueness and continuous dependence for hereditary systems, Ann. Math. Pura Appl. 85 (1970) 63-82.

INSTITUTE OF MATHEMATICS AND PHYSICS, HIGHER COLLEGE OF ENGINEERING, 65-246 ZIELONA GÓRA, POLAND

Received October 13, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol XXIII No 2 1990

Jan Ambrosiewicz

POWERS OF SETS IN LINEAR GROUPS

It has been proved in the paper [1] that the set K_2^2 is not a subgroup of groups: $GL(n,F)$, $SL(n,F)$, $PSL(n,F)$ for $n>2$, where $K_2 = \{g \in G: o(g) = 2\}$. In the present paper we will prove: 1° $SL(n,F) = K_2^4$ for $2<n<|F|-1$, 2° $GL_1(n,F) = K_2^4$ for $n<|F|-1$, where $GL_1(n,F)$ denotes the group of all n by n matrices of determinant ± 1, 3° $SL(n,F) = C_V^4$ for $n<|F|-1$, where C_V denotes the conjugacy class of the matrix $V = \mathrm{diag}(v_1,\ldots,v_n)$, $v_i \neq v_j$ for $i \neq j$, 4° $PSL(n,F) = C_V^2 = K_2^4$ for $n<|F|-1$ (the result $PSL(n,F) = C_V^2$ was proved in [3] on another way), 5° $SL(3,F) = K_2^4$, 6° $PSL(n,F) = K_2^4$, 7° $GL_1(n,F) = K_2^4$. We will give also condition under which $SL(n,F) = C_V^2$ (see the question stated in [3], p.66).

Throughout this paper E' will denote the matrix $\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in M_{n\times n}$.

We will use the following lemmas.

Lemma 1 (see [1]). Let G be a group. An element $g \in K_2^2$ if and only if there is an element $x \in K_2$, $x \neq g^{-1}$ such that $(gx)^2 = 1$.

Lemma 2 (see [2]). If M is a non-empty subset of G, $M = M^{-1}$ and $xM \cap M \neq 0$ for each $x \in G$, then $M^2 = G$.

Lemma 3. Let $V = \mathrm{diag}(v_1,\ldots,v_n)$, $W = \mathrm{diag}(w_1,\ldots,w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$. If $\det A = \det V \det W$, $A \notin Z(GL(n,F))$ and A has a rational canonical form then there are matrices $X, Y \in GL(n,F)$ with arbitrary chosen $\det X$, $\det Y \in F^*$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(1) $$A = (X^{-1}V\,X)(Y^{-1}W\,Y).$$

The proof of Lemma 3, for $n = 3$ has been given in [2]. Using the same method one can prove Lemma 3 for any $n \geq 2$, (see [6]).

Lemma 4. Let $V = \mathrm{diag}(v_1,\ldots,v_n)$, $v_i \neq v_j$ for $i \neq j$. Let $v_n = 1$, $v_{2i-1}v_{2i} = 1$ for $i = 1,\ldots,\frac{n-1}{2}$ if n is odd. Let $v_{2i-1}v_{2i} = 1$ for $i = 1,\ldots,\frac{n}{2}$ if n is even. Then there exists $T \in SL(n,F)$ such that

(2) $$V\,T^{-1}V\,T = E,\ T^2 = E \text{ for } n > 2 \text{ and } T^2 = -E \text{ for } n = 2,\ T \neq V^{-1},$$

(3) $$xE \in C_V C_{xV} \quad \text{for} \quad x \in F^*,$$

(4) $$\text{If } n > 2, \text{ then } C_{xV} \subseteq K_2^2 \text{ iff } x_2 = 1.$$

Proof. If n is odd, we put $T = \mathrm{diag}\left(L_1,\ldots,L_{\frac{n-1}{2}}, L_{\frac{n+1}{2}}\right)$, $L_i = E_2'$, $i = 1,\ldots,\frac{n-1}{2}$, $L_{\frac{n+1}{2}} = [\pm 1]$ where we take $+1$ when $\frac{n-1}{2}$ is even and -1 when $\frac{n-1}{2}$ is odd; if n is even, we put $T = \mathrm{diag}(L_1,\ldots,L_{\frac{n}{2}})$, $L_i = E_2'$, $i = 1,\ldots,\frac{n}{2}$ when $n > 2$ and $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ when $n = 2$.

The relations (2) can be easy verified by direct computations. The relations (3) follows immediately from (2). By Lemma 1, $xV \in K_2^2$ iff there exists $T \in K_2$ such that $T^{-1} \neq xV$ and $T^{-1}(xV)T = x^{-1}V^{-1}$. But by (2) and by $x^2 = 1$ these relations hold for $n > 2$. The inclusion $C_{xV} \subseteq K_2^2$ results from the fact that K_2^2 is a normal subset.

Lemma 5. Let $V = \mathrm{diag}(v_1,\ldots,v_n)$, $v_i \neq v_j$ for $i \neq j$. If n is odd, we put $v_n = \pm 1$, $v_{2i-1}v_{2i} = 1$ for $i = 1,\ldots,\frac{n-1}{2}$; if n is even, we put (i) $v_{2i-1}v_{2i} = 1$ for $i = 1,\ldots,\frac{n}{2}$ or (ii) $v_1 = 1$, $v_2 = -1$, $v_{2i-1}v_{2i} = 1$ for $i = 2,\ldots,\frac{n}{2}$. Then there exists a matrix T such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$VT^{-1}VT = E,\ T^2 = E,\ T \neq V^{-1},\ \det T = \pm 1, \tag{5}$$

$$xE \in C_V C_V \quad \text{for} \quad x \in F^*, \tag{6}$$

$$C_{xV} \subseteq K_2^2 \ GL_1(n,F) \quad \text{iff} \quad x^2 = 1. \tag{7}$$

Proof. If n is odd, we put $T = \operatorname{diag}(L_1,\dots,L_{\frac{n-1}{2}},L_{\frac{n+1}{2}})$, $L_i = E_2'$, $i = 1,\dots,\frac{n-1}{2}$, $L_{\frac{n+1}{2}} = 1$; if n is even, we put $T = \operatorname{diag}(L_1,\dots,L_{\frac{n}{2}})$, $L_i = E_2'$, $i = 1,\dots,\frac{n}{2}$ in the case (i) and $T = \operatorname{diag}(L_1,\dots,L_{\frac{n}{2}})$, $L_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $L_i = E_2'$, $i = 2,\dots,\frac{n}{2}$ in the case (ii). By direct computation it is easy to verify the relations (5). The relation (6) follows from (5). The condition (7) holds by relations $T^{-1} \neq xV$, $T^{-1}(xV)T = x^{-1}V^{-1}$, $T \in K_2$ and by Lemma 1.

Lemma 6. $Z(SL(n,F)) \subseteq K_2^4$ except for $n = 2$ and char $F \neq 2$.

Proof. Let $a^n = 1$. In the proof we will distinguish two cases: (i) n is even, (ii) n is odd. If $n = 2k > 2$, then we put

$$A = \operatorname{diag}(a^2,a^4,\dots,a^{2k},a^2,a^4,\dots,a^{2k}),$$

$$B = \operatorname{diag}(a^{2k-1},a^{2k-3},\dots,a,a^{2k-1},a^{2k-3},\dots,a),$$

$S = \operatorname{diag}(E'_{k-1},-1,E'_{k-1},-1)$, $R = \operatorname{diag}(E'_k,E'_k)$. One can verify that the following relations $A,B,S,R \in SL(n,F)$; $aE = AB$; $S^{-1}AS = A^{-1}$; $R^{-1}BR = B^{-1}$; $S^2 = R^2 = E$; $S^{-1} \neq A$; $R^{-1} \neq B$ hold. Hence $A,B \in K_2^2$ by Lemma 1. Therefore $aE = AB \in K_2^4$. If $n = 2$, char $F = 2$ then obviously $Z(SL(n,F)) = \{E\} \subseteq K_2^2$. If $n = 2$, char $F \neq 2$ then $Z(SL(n,F)) = \{E,-E\} \not\subseteq K_2^4 = \{E\}$, (see [1]).

If n is odd, then we put $A = \operatorname{diag}(a^n,a^{n-1},\dots,a)$, $B = \operatorname{diag}(a,a^2,\dots,a^{n-1},a^n)$, $M = \operatorname{diag}(r,E'_{n-1})$, $N = \operatorname{diag}(E'_{n-1},r)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

 Digitized by Arya Samaj Foundation Chennai and eGangotri

where $r \in \{1,-1\}$. It is easy to see that we can choose r such that $M,N \in SL(n,F)$. One can verify that the following relations: $A,B \in SL(n,F)$; $AB = aE$; $M^{-1}AM = A^{-1}$; $N^{-1}BN = B^{-1}$; $M^2 = N^2 = E$; $M^{-1} \neq A$; $N^{-1} \neq B$; $M,N \in SL(n,F)$ hold for suitable $r \in \{1,-1\}$. Hence by Lemma 1 $A,B \in K_2^2$, so $aE = AB \in K_2^4$. Therefore $Z(SL(n,F)) \subseteq K_2^4$.

Lemma 7. $Z(GL_1(n,F)) \subseteq K_2^4$.

Proof. If $aE \in Z(GL_1(n,F))$ and $a^n = 1$, $n > 2$ then Lemma 7 is true by Lemma 6. If $n = 2$, $a^2 = 1$ then Lemma 7 is true by the identity $E_2'(-E_2)E_2' = -E_2$ and by Lemma 1 and by the fact that $E_2 \in K_2^4$.

Let $a^n = -1$. We will consider four cases: (i) $n = 2k$; k - even; (ii) $n = 2k$, $k > 1$, k - odd; (iii) $n = 2$; (iv) n - odd.

If $n = 2k$, k - even, then we put $A = \operatorname{diag}(a^2,-a^4,\ldots,-a^{2k},-a^2, a^4,-a^6,\ldots,a^{2k})$, $B = \operatorname{diag}(-a^{2k-1},a^{2k-3},\ldots,a,a^{2k-1},-a^{2k-3},\ldots,-a)$, $S = \operatorname{diag}(E'_{2k-1},1)$, $R = \operatorname{diag}(E'_k,E'_k)$. The following relations are easily checked: $AB = aE$, $S^{-1}AS = A^{-1}$, $R^{-1}BR = B^{-1}$, $S^2 = R^2 = E$, $S^{-1} \neq A$, $R^{-1} \neq B$ hold. Hence by Lemma 1 we have $A,B \in K_2^2$. Therefore $aE = AB \in K_2^4$. If $n = 2k$, k - odd, $k > 1$, we put $A = \operatorname{diag}(a^2,-a^4,\ldots,a^{2k},-a^2,a^4,\ldots,-a^{2k})$, $B = \operatorname{diag}(-a^{2k-1},a^{2k-3},\ldots,-a,a^{2k-1},-a^{2k-3},\ldots,a)$, $S = \operatorname{diag}(E'_{k-1},1,E'_{k-1},1)$, $R = E'_{2k}$. One can easy verify the following relations: $AB = aE$, $S^{-1}AS = A^{-1}$, $R^{-1}BR = B^{-1}$, $S^2 = R^2 = E$, $S^{-1} \neq A$, $R^{-1} \neq B$. Hence $A,B \in K_2^2$ by Lemma 1, so $aE = AB \in K_2^4$. If $n = 2$, then Lemma 7 follows from the following identities

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix} = aE, \quad \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^{-1}$$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix}^{-1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and by Lemma 1. If n is odd and $a^n = -1$, then Lemma 7 is true by the equation $(-a)^n = 1$ and by Lemma 6.

Theorem 1. If $V = \mathrm{diag}(v_1,\ldots,v_n)$, $W = \mathrm{diag}(w_1,\ldots,w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$ and $V,W \in SL(n,F)$, then $SL(n,F) = C_V C_W \cup Z(SL(n,F))$.

Proof. Let $B \in SL(n,F) - Z(SL(n,F))$. By Corollary 4.7 p.360 of [5] there exists $S \in GL(n,F)$ such that $S^{-1}B\,S = A \in SL(n,F)$ has a rational canonical form. It is clear that $\det B = \det V \det W$, so by Lemma 3, $S^{-1}B\,S = A = (X^{-1}V\,X)(Y^{-1}W\,Y)$ or

$$B = \left[(XS^{-1})^{-1}V(XS^{-1})\right]\cdot\left[(YS^{-1})^{-1}W(YS^{-1})\right]. \tag{8}$$

By Lemma 3 we can assume that $\det X = \det Y = \det S$. Hence $B \in C_V C_W$ by (8).

If we will put $V = W$ in Theorem 1, then we will receive

Corollary 1.1. If $V = \mathrm{diag}(v_1,\ldots,v_n)$, $v_i \neq v_j$ for $i \neq j$ and $V \in SL(n,F)$, then

$$SL(n,F) = C_V^2 \cup Z(SL(n,F)). \tag{9}$$

Theorem 2. If V is a matrix described in Lemma 4, then

$$SL(n,F) = \bigcup_{x \in Z(SL(n,F))} C_V C_{xV} \quad \text{for } n>2. \tag{10}$$

Proof. By Lemma 4, $xE \in C_V C_{xV}$, so $Z(SL(n,F)) \subseteq \bigcup_{x \in Z(SL(n,F))} C_V C_{xV}$.

By Corollary 1.1, $SL(n,F) - Z(SL(n,F)) = C_V C_V$. Hence the equality (10) holds.

Corollary 2.1. If $2<n<|F|-1$ and $Z(SL(n,F)) = E$ then there exists a matrix V such that $SL(n,F) = C_V^2$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let V be a matrix described in Lemma 4. An existence of V ensures the inequality $n<|F|-1$. By (2), $E \in C_V^2$. Hence by Theorem 2, $SL(n,F) = C_V^2$.

Corollary 2.2. If n,q are even numbers and $n<q$, then there exists $V \in SL(n,q)$ such that $SL(n,q) = C_V^2$.

Proof. Let V be a matrix described in Lemma 4. The existence of V ensures the inequality $n<q$. It is clear that Lemma 4 is true also for $n = 2$, because q is even. Now our Corollary results from Corollary 2.1, because $|Z(SL(n,Q))| = (n,q-1) = 1$, by assumption.

Corollary 2.3. Let $s = o(V)$ denotes an order of V described in Lemma 4, $r = |Z(SL(n,q))|$. If $r|s$, then $SL(n,q) = K_s^2$.

The proof results from Theorem 2 and from the fact that $o(Va) = o(V)$ provided $a \in F_q^*$.

Theorem 3. If $2<n<|F|-1$, then $SL(n,F) = K_2^4$.

Proof. Let V be a matrix described in Lemma 4. An existence of V follows by the inequality $n<|F|-1$. By Lemma 4, we have $C_V \subseteq K_2^2$, so $C_V^2 \subseteq K_2^4$. By Lemma 6, we obtain $Z(SL(n,F)) \subseteq K_2^4$. Hence $SL(n,F) \subseteq K_2^4$, by Corollary 1.1. An inverse inclusion is obvious.

Remark 2. If $n = 2$ and char $F \neq 2$, then Theorem 3 is not true, because in this case $K_2^2 = E$ (see [1], Theorem 3). If $n = 2$, char $F = 2$ and $|F|>2$, then $SL(n,F) = K_2^2$, (see [1], Theorem 2).

Corollary 3.1. If $n>2$, then $SL(n,F) = K_2^4$ where $F = Q,R,C$.

Theorem 4. If $n<|F|-1$, then $GL_1(n,F) = K_2^4$.

Proof. We will consider two cases: a) n is even, b) n is odd.

Ad a). Let V and W are matrices from Lemma 5 described in (i) and in (ii) respectively. The existence of matrices V and W ensures the inequality $n<|F|-1$. Hence we have $C_V, C_W \subseteq K_2^2$ by Lemma 5 and $Z(GL_1(n,F)) \subseteq K_2^4$, by Lemma 7. From (1) $GL_1(n,F) = C_V^2 \cup C_V C_W \cup Z(GL_1(n,F))$. Therefore $GL_1(n,F) \subseteq K_2^4$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Ad b). In this case a proof is similar to Ad a).

Corollary 4.1. If $F = Q,R,C$, then $GL_1(n,F) = K_2^4$.

Theorem 5. If n is odd and $|Z(SL(n,F))| = n$, then there exists $V \in SL(n,F)$ such that $SL(n,F) = C_V^2 = K_2^4$.

Proof. Let $a \in F^*$ be a nth root of 1. Then the matrices $V = diag(a^n, a^{n-1}, .., a)$, $a^iV = diag(\underbrace{a^{n+i}, a^{n-1+i}, \ldots, a^2, a,}_{i}$ $\underbrace{a^n, a^{n-1}, \ldots, a^{2+i}, a^{1+i}}_{n-i})$ have distinct entries for $i = 1, \ldots, n-1$ and we have V, $a^iV \in SL(n,F)$. Let $S = diag(r, E'_{n-1})$, $r = \pm 1$;

$$R = \begin{bmatrix} 0 & E_{n-i} \\ E_i & 0 \end{bmatrix}, \quad i=1,\ldots,n-1.$$

It is clear that $\det R = 1$ and that we can choose r such that $\det S = 1$. The following identities are easily checked: (i) $VS^{-1}VS=E$, (ii) $R\,a^iV\,R^{-1} = V$. By Corollary 1.1 we get $SL(n,F) = C_V^2 U Z(SL(n,F))$. From (i) results $(a^iV)S^{-1}V\,S = a^iE$. Hence, by (ii), $(R^{-1}VR)(S^{-1}VS) = a^iE$ for $i = 1,\ldots,n-1$. Since $E \in C_V^2$, by (i), then $Z(SL(n,F)) \subseteq C_V^2$. Therefore $SL(n,F) = C_V^2$, by Corollary 1.1. By (i) and by Lemma 1, $V \in K_2^2$, so $C_V^2 \subseteq K_2^4$, because the set K_2^2 is a normal set.

Theorem 5 is a partial answer on a question stated in paper 3, p.66.

Corollary 5.1. If n is odd, then there exists $V \in SL(n,C)$ such that $SL(n,C) = C_V^2$.

The proof results from Theorem 5 and from the fact that $|Z(SL(n,C))| = n$.

Theorem 6. If $n < |F| - 1$, $|F| > 2$, then there exists $V \in SL(n,F)$ such that $SL(n,F) = C_V^4$.

Proof. Let V be a matrix from Lemma 4. An existence of V follows from the inequality $n < |F| - 1$. By Corollary 1.1, $SL(n,F) =$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$= C_V^2 \cup Z(SL(n,F))$. To prove Theorem 6 it enough to show that $[SL(n,F)-Z(SL(n,F))]^2 = SL(n,F)$. Let $M = SL(n,F)-Z(SL(n,F))$. It is clear that $M = M^{-1}$. We will show that for any $g \in SL(n,F)$ we have $gM \cap M \neq 0$. Indeed, if this is false, there exists $g_o \in SL(n,F)$ such that $g_o M = Z(SL(n,F))$ i.e. $M = g_o^{-1} Z(SL(n,F))$ which means that $SL(n,F) = Z(SL(n,F)) \cup g_o^{-1} Z(SL(n,F))$. But this contradicts $|F|>2$.

The center $Z(SL(n,F))$ is an unity $\overline{E}$ of the group $PSL(n,F)$. Lemma 4 can be used to the group $PSL(n,F)$ without the assumption concerning n because for $n = 2$ we have $T^2 = -E \in Z = \overline{E}$. The equality (9) has now a form $\overline{V}\, T^{-1} \overline{V}\, T = \overline{E}$, so $\overline{E} \in C_{\overline{V}} C_{\overline{V}}$. Therefore by Lemma 4 and by Corollary 1.1, we obtain the theorem.

Theorem 7. If $V \in PSL(n,F)$ satisfies the assumption of Lemma 4, then $PSL(n,F) = C_V^2$.

Theorem 8. If $n < |F|-1$, then there exists $V \in PSL(n,F)$ such that $PSL(n,F) = C_V^2 = K_2^4$.

Proof. The assumption ensures the existence of the matrix V described in Lemma 4. Since $Z(PSL(n,F)) = \{\overline{E}\}$, we have $C_V \subseteq K_2^2$ by (4). Therefore by Theorem 7, $PSL(n,F) = C_V^2 = K_2^4$.

Theorem 9. $SL(3,F) = K_2^4$.

Proof. If $|F|>4$, then the proof follows from Theorem 3. If $|F| = 4$, then the proof follows from Theorem 5. If $|F| = 2$, then the proof follows from Theorem 8 because in this case $SL(3,2) \cong$ $\cong PSL(2.7)$. To prove Theorem 9 in the case $|F| = 3$, we will use character table of group $SL(3,3)$ (see [4], p.68). Using Burnside's formula on multiplication of conjugacy classes, it has been showed in [4] that for each class $C \neq \{E\}$ of the group $SL(3,3)$, we have $C^4 = SL(3,3)$. Hence in the special case $C \subseteq K_2$, we have $SL(3,3) =$ $= K_2^4$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Corollary 9.1. $GL_1(3,F) = K_2^4$.

Proof. We have $GL_1(3,F) = (-E)SL(3,F) \cup SL(3,F)$. If $A \in SL(3,F)$ then $A = A_1A_2A_3A_4$ with $A_i \in K_2 \subset SL(3,F)$, by Theorem 9. It is clear that $-A = (-EA_1)A_2A_3A_4$ and $(-EA_1), A_2, A_3, A_4 \in K_2 \subset GL_1(3,F)$. Hence $GL_1(3,F) = K_2^4$.

REFERENCES

[1] J. Ambrosiewicz: On the squares of linear groups, Rend. Sem. Mat. Univ. Padova, Vol.75 (1986).

[2] J. Ambrosiewicz: On the square of sets of the group SL(3,F), PSL(3,F), Demonstratio Math. 18 (1985) 963-968.

[3] J.L. Brenner: Covering Theorems for FINASIGS X. The group G=PSL(n,q) had a class C such that CC=G, ARS COMBINATORIA, Vol.16 (1983) pp.57-68.

[4] A. Dold, B. Eckmann, Z. Arad, M. Herzog: Products of conjugacy classes in groups. Springer-Verlag, Berlin Heidelberg (1985).

[5] T.W. Hungeerford: Algebra. Springer-Verlag, New York, Heidelberg, Berlin 1974.

[6] J. Ambrosiewicz: Products of sets in linear groups. Demonstratio Math. 22 (1989), (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARSAW,
BIAŁYSTOK BRANCH, 15-267 BIAŁYSTOK, POLAND

Received October 21, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Piotr Multarzyński, Wiesław Sasin

ON THE DIMENSION OF DIFFERENTIAL SPACES

A study of the dimension of the tangent space at a certain point of a differential space in the sense of Sikorski [6], [8] is presented. The aim of this paper is to give the sufficient conditions which permit us to determine this dimension basing only on properties of the algebra of smooth functions on a different space.

1. Preliminaries

Let M be any set and C be any non-empty set of real functions on M. By τ_C we shall denote the weakest topology on M in which all functions from C are continuous. For any subset $A \subset M$, by $C|A$ we denote the set of all functions of the form $\alpha|A$, where $\alpha \in C$. Let C_A be a set of all real functions β on A such that, for any point $p \in A$, there exists an open neighbourhood $U \in \tau_C$ of p and a function $\alpha \in C$ such that $\beta|A \cap U = \alpha|A \cap U$. By scC we shall denote the family of all real functions on M which assume the form $\omega \circ (\alpha_1, \ldots, \alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1, \ldots, \alpha_n \in C$, $n \in N$ and $\mathcal{E}_n := C^\infty(R^n)$.

A family C of real functions on M is called a differential structure (shortly a d-structure) on M if $C = C_M = scC$ [8]. The pair (M,C) is said to be a differential space (shortly d-space). The family C, with its natural addition and multiplication operations, turns out to be a real algebra called also a linear ring [8]. For an arbitrary set C_o of real functions on M, the set $(scC_o)_M$ is the smallest

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

differential structure on M containing C_o. A differential structure is said to be generated by C_o if $C = (scC_o)_M$. If (M,C) is a differential space and $A \subset M$ then (A,C_A) is also a differential space; it is called a differential subspace of (M,C), and C_A is called a differential structure induced on A from (M,C) [8]. Any C^∞ differential manifold may be considered as a differential space $(M,C^\infty(M))$, where $C^\infty(M)$ is the set of all smooth functions of the class C^∞ on M.

By a tangent vector to a d-space (M,C) at a point $p \in M$ we shall mean any linear mapping $v: C \longrightarrow \mathbb{R}$ which satisfies the condition

$$v(\alpha\cdot\beta) = v(\alpha)\cdot\beta(p) + \alpha(p)\cdot v(\beta),$$

for $\alpha,\beta \in C$. By T_pM we shall denote the linear space of all tangent vectors to (M,C) at $p \in M$, called a tangent space to (M,C) at $p \in M$. The C-module of all derivations in a linear ring C will be denoted by $\mathfrak{X}(M)$. We say that a differential space (M,C) is of the constant differential dimension n if $\dim T_pM = n$ for any $p \in M$ and the C-module $\mathfrak{X}(M)$ is locally free and has the rank n.

Let (M,C) and (N,D) be d-spaces and $f: M \longrightarrow N$ a map. The map f is said to be smooth if $f^*D \subset C$, where $f^*D := \{\alpha \circ f, \alpha \in D\}$.

2. Main results

Let (M,C) be any d-space.

Definition 1. A function $f \in C$ is said to be differentially dependent (shortly d-dependent) on functions $g_1,\ldots,g_n \in C$ at a point $p \in M$ if there exist a neighbourhood $U \in \tau_C$ of the point $p \in M$ and a function $\omega \in \mathcal{E}_n$ such that

$$f|U = \omega\cdot(g_1,\ldots,g_n)|U. \tag{1}$$

Definition 2. A set $\{f_1,\ldots,f_n\} \subset C$ is said to be differentially independent (shortly d-independent) at a point $p \in M$ if no function f_i, for $i \in \{1,\ldots,n\}$, depends differentially on the remaining ones at $p \in M$. Any set $\mathcal{F} \subset C$ is said to be differentially independent (d-independent)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

at $p \in M$ if every finite subset of $\mathcal{F}$ is d-independent at p. A set $\mathcal{F} \subset C$ is said to be d-dependent at $p \in M$ if $\mathcal{F}$ is not d-independent at p.

Example 1. Let us consider the n-dimensional Euclidean differential space $(R^n, \mathcal{E}_n)$. Then, the set $\{\pi_1, \ldots, \pi_n\} \subset \mathcal{E}_n$ of the projections is d-independent at any point $p \in R^n$.

One can prove

Lemma 1. Let (M, C) be a d-space, $\mathcal{F} \subset C$ a subset. The set $\mathcal{F}$ is d-dependent at $p \in M$ if and only if there exists an open neighbourhood $U \in \tau_C$ of the point p such that the set $\mathcal{F}|U$ is d-dependent at the point p. If the set $\mathcal{F} \subset C$ is d-dependent at $p \in M$ then for an arbitrary d-subspace (A, C_A) such that $p \in A$ the set $\mathcal{F}|A$ is d-dependent at the point $p \in A$.

Now we prove

Lemma 2. Let $M \subset R^n$ be a k-dimensional hypersurface. Then the set of the projections $\{\pi_1|M, \ldots, \pi_n|M\} \subset C^\infty(M)$ is d-independent at an arbitrary point $p \in M$ if and only if $k = n$.

Proof. ($\Longrightarrow$) Let us assume that the set $\{\pi_1|M, \ldots, \pi_n|M\}$ is d-independent at $p \in M$ and suppose that $k < n$. Then there exist a neighbourhood U of the point p open in M and C^∞-mapping $\psi: G \longrightarrow R^{n-k}$, $\psi = (\psi_1, \ldots, \psi_{n-k})$, where $G \subset R^k$ is an open subset, such that $U = \{(q, \psi(q)): q \in G\}$. Hence it follows

$$\pi_{k+s}|U = \psi_s \circ (\pi_1|U, \ldots, \pi_k|U) \quad \text{for} \quad s = 1, \ldots, n-k,$$

which contradicts our assumption. Therefore $k = n$.

($\Longleftarrow$) If $k = n$ then M is an open subset of R^n. Since the set of the projections $\{\pi_1, \ldots, \pi_n\}$ is d-independent at every point of R^n the set $\{\pi_1|M, \ldots, \pi_n|M\}$ is d-independent at every point $p \in M$.

Proposition 1. Let $M \subset R^n$ be any non-empty subset and $C = \mathcal{E}_{nM}$. The set of the projections $\{\pi_1|M, \ldots, \pi_n|M\}$ is d-independent at $p \in M$ if and only if $\dim T_p M = n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let us assume that the set $\{\pi_1|M,\dots,\pi_n|M\}$ is d-independent at $p\in M$ and suppose that $\dim T_pM = k<n$. Then there exist an open neighbourhood $V\in\tau_C$ of p and a k-dimensional hypersurface $\tilde{V}\subset R^n$ such that $C^\infty(\tilde{V})_V = C_V$ and $V = \tilde{V}\cap M$ [9], [2]. It follows from Lemma 2 that the set $\{\pi_1|\tilde{V},\dots,\pi_n|\tilde{V}\}$ is d-dependent at every point of V. In view of Lemma 1 the set $\{\pi_1|V,\dots,\pi_n|V\}$ is d-dependent at p or equivalently the set $\{\pi_1|M,\dots,\pi_n|M\}$ is d-dependent at p, which contradicts our assumption. Thus $\dim T_pM = n$.

Conversely, let $\dim T_pM = n$. Suppose that the set $\{\pi_1|M,\dots,\pi_n|M\}$ is d-dependent at p and assume, without loss of generality, that there exist $\omega\in\mathcal{E}_{n-1}$ and $V\in\tau_C$ such that $\pi_1|V = \omega\circ(\pi_2,\dots,\pi_n)|V$. Let us consider the mapping $\Phi: T_pM \longrightarrow R^n$ given by

$$\Phi(v) = (v(\pi_1|M),\dots,v(\pi_n|M)) \quad \text{for} \quad v\in T_pM. \tag{2}$$

It can be easily seen that $\dim T_pM = n$ iff Φ is an isomorphism of linear spaces. In our case, Φ is not an isomorphism because $v(\pi_1|M) = \sum_{i=2}^{n}\omega'_{1i}(p)\cdot v(\pi_i|M)$ for $v\in T_pM$. Thus $\dim T_pM<n$, which gives the contradiction. Thus the set $\{\pi_1|M,\dots,\pi_n|M\}$ is d-independent at p.

Proposition 2. Let (M,C) be a d-space with the d-structure C generated by the set of functions $\{f_1,\dots,f_n\}$. The set of functions $\{f_1,\dots,f_n\}$ is d-independent at $p\in M$ if and only if $\dim T_pM = n$.

Proof. If (M,C) is not a Hausdorff d-space then there exist a Hausdorff d-space (M_H,C_H), which is associated with (M,C) [8], [4] and a mapping $\pi_H: (M,C)\longrightarrow(M_H,C_H)$ such that $\pi_H^*: C_H\longrightarrow C$ is an isomorphism of rings. Then $\dim T_pM = \dim T_{\pi_H(p)}M_H$ for any point $p\in M$. Therefore, we may assume, without loss of generality, that (M,C) is a Hausdorff d-space. Then the map $\psi: M\longrightarrow R^n$ defined by the formula

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\psi(p) := (f_1(p),\dots,f_n(p)) \quad \text{for} \quad p \in M, \tag{3}$$

is a diffeomorphism of (M,C) onto the image $(\psi(M),\mathcal{E}_{n\psi(M)})$ [5]. Of course, $f_i = (\pi_i|\psi(M))\circ\psi$ for $i = 1,\dots,n$. Since ψ is a diffeomorphism, the set $\{f_1,\dots,f_n\}$ is d-independent at $p \in M$ iff the set $\{\pi_1|\psi(M),\dots,\pi_n|\psi(M)\}$ is d-independent at the point $\psi(p)$. From Proposition 1 it follows that the set $\{\pi_1|\psi(M),\dots,\pi_n|\psi(M)\}$ is d-independent iff $\dim T_{\psi(p)}\psi(M) = n$. Hence $\{f_1,\dots,f_n\}$ is d-independent at p iff $\dim T_{\psi(p)}\psi(M) = n$ or equivalently $\dim T_p M = n$.

Proposition 3. Let (M,C) be a d-space. A subset $\{f_1,\dots,f_n\} \subset C$ is d-independent at a point $p \in M$ if and only if for any function $\omega \in \mathcal{E}_n$ and any neighbourhood $U \in \tau_C$ of the point p the following implication is true:

$$(*) \quad \omega\circ(f_1,\dots,f_n)|U = 0 \Longrightarrow \bigvee_{1\le i\le n} \omega'_i(f_1(p),\dots,f_n(p)) = 0.$$

Proof. ($\Longrightarrow$) Let $\{f_1,\dots,f_n\}$ be d-independent at $p \in M$. Let us assume that $\omega\circ(f_1,\dots,f_n)|U = 0$ for a function $\omega \in \mathcal{E}_n$ and a neighbourhood $U \in \tau_C$ of p, and let us consider the d-space $(U, sc\{f_1,\dots,f_n\}_U)$. Then the set $\{f_1|U,\dots,f_n|U\}$ is d-independent at p, on the same space. Let $\tilde{\psi} := (f_1|U,\dots,f_n|U)$. From Proposition 2 it follows that $\dim T_{\tilde{\psi}(p)}\tilde{\psi}(U) = n$. Of course, $\omega\circ(\pi_1|\tilde{\psi}(U),\dots,\pi_n|\tilde{\psi}(U)) = 0$. Let $\partial_{iq} \in T_q\tilde{\psi}(U)$ be the tangent vector such that, for any function $\varphi \in \mathcal{E}_{n\tilde{\psi}(U)}$, the value $\partial_{iq}\varphi$ is the i-th partial derivative of φ at the point q, $i = 1,\dots,n$, $q = \tilde{\psi}(p)$. The vectors $\partial_{1q},\dots,\partial_{nq}$ are linearly independent and $\partial_{iq}(\pi_j|\tilde{\psi}(U)) = \delta_{ij}$ for $i,j = 1,\dots,n$. Thus for any $j \in \{1,\dots,n\}$

$$\partial_{jq}(\omega\circ(\pi_1|\tilde{\psi}(U),\dots,\pi_n|\tilde{\psi}(U))) = 0$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

or equivalently

$$\sum_{i=1}^{n} \omega'_{|i}(\tilde{\psi}(p)) \cdot \partial_{jq}(\pi_i | \tilde{\psi}(U)) = 0,$$

Therefore

$$\omega'_{|j}(f_1(p),\ldots,f_n(p)) = 0 \quad \text{for} \quad j = 1,\ldots,n,$$

which means that condition (*) is satisfied.

($\Longleftarrow$) Let $\{f_1,\ldots,f_n\} \subset C$ be a subset of the ring C that satisfies (*). Suppose that the set $\{f_1,\ldots,f_n\}$ is d-dependent at p and for the sake of concreteness assume that there exist a neighbourhood $U \in \tau_C$ of p and a function $\theta \in \mathcal{E}_{n-1}$ such that $f_1|U = \theta \circ (f_2,\ldots,f_n)|U$. Then

$$(f_1 - \theta \circ (f_2,\ldots,f_n))|U = 0.$$

Let $\omega \in \mathcal{E}_n$ be a function given by the formula

$$(4) \qquad \omega(x_1,\ldots,x_n) = x_1 - \theta(x_2,\ldots,x_n) \quad \text{for} \quad (x_1,\ldots,x_n) \in R^n.$$

Of course $\omega'_{|1}(x_1,\ldots,x_n) = 1$ for any $(x_1,\ldots,x_n) \in R^n$. Condition (*) is not satisfied if the set $\{f_1,\ldots,f_n\}$ is d-dependent at p. Therefore, the set $\{f_1,\ldots,f_n\}$ is d-independent at the point p.

Corollary 1. If tangent vectors $v_1,\ldots,v_k \in T_pM$ are linearly independent then any set of functions $\{f_1,\ldots.f_k\} \subset C$, such that

$$v_i(f_j) = \delta_{ij} \quad \text{for} \quad i,j = 1,\ldots,k,$$

is d-independent at p.

Proof. We shall show that the set $\{f_1,\ldots,f_k\}$ satisfies condition (*). Let $\omega \in \mathcal{E}_k$ be a function such that there exist an open neighbourhood $U \in \tau_C$ of p and $\omega \circ (f_1,\ldots,f_k)|U = 0$. We have

$$v_j(\omega \circ (f_1,\ldots,f_k)) = 0 \quad \text{for} \quad j = 1,\ldots,k.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Therefore,

$$\sum_{i=1}^{k} \omega'_{|i}(f_1(p),\ldots,f_k(p))\cdot v_j(f_i) = 0 \quad \text{for} \quad j = 1,\ldots,k.$$

From the assumption that $v_i(f_j) = \delta_{ij}$ it follows that

$$\omega'_{|j}(f_1(p),\ldots,f_k(p)) = 0 \quad \text{for} \quad j = 1,\ldots,k.$$

Since $\{f_1,\ldots,f_k\}$ satisfies condition $(*)$, on the strength of Proposition 3, this set of functions is d-independent.

Definition 3. We say that a subset $B \subset C$ reproduces the d-structure C at the point $p \in M$ if for any function $f \in C$ there exist an open neighbourhood $U \in \tau_C$ of p and functions $g_1,\ldots,g_n \in B$, $\omega \in \mathcal{E}_n$ such that

(5) $$f|U = \omega \circ (g_1,\ldots,g_n)|U.$$

Remark. Let (M,C) be a d-space which is locally finitely generated [5]. One can easily see that a subset $B \subset C$ reproduces C at a point $p \in M$ iff B locally generates the structure C in a certain neighbourhood of the point p.

Definition 4. A set $B \subset C$ is said to be a differential basis shortly (d-basis) of the d-structure C on M at a point $p \in M$ if B is a d-independent set at p and if B reproduces C at the point p.

Example 2. Let $(R^N, \mathcal{E}_\infty)$ be a d-space of sequences of real numbers with the d-structure generated by the set of projections $\{\pi_i : i \in N\}$, where $\pi_i : R^N \longrightarrow R$ is the projection of a sequence onto its i-th coordinate. The set $\{\pi_i : i \in N\}$ turns out to be a d-basis of the d-structure $\mathcal{E}_\infty$ at any point $p \in R^N$.

Example 3. Let C be a d-structure on the set R generated by the set of functions $\{f_a : a \in R\}$, where $f_a : R \longrightarrow R$ is given by

(6) $$f_a(x) := |x - a| \quad \text{for} \quad x \in R.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then, as it can be easily shown the set $\{f_a, f_b\}$ forms a d-basis for the d-structure C at the point a, for any $a, b \in R$, $a \neq b$.

Now we shall prove the following

Lemma 3. Let (M,C) be a d-space and B a d-basis of the d-structure C at $p \in M$. For each function $u_o: B \longrightarrow R$, there exists a unique tangent vector $u: C \longrightarrow R$ at the point p such that $u|B = u_o$.

Proof. Let $u: C \longrightarrow R$ be a mapping given by the formula

$$(7) \qquad u(\alpha) = \sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p), \ldots, \alpha_n(p)) \cdot u_o(\alpha_i) \quad \text{for} \quad \alpha \in C,$$

where $\alpha_1, \ldots, \alpha_n \in B$, $\omega \in \mathcal{E}_n$ are functions such that there exists a neighbourhood $U \in \tau_C$ of p for which $\alpha|U = \omega \circ (\alpha_1, \ldots, \alpha_n)|U$.

To verify the correctness of definition (7), let us assume that $\theta \in \mathcal{E}_n$ is another function such that $\alpha|U = \theta \circ (\alpha_1, \ldots, \alpha_n)|U$. Then

$$\omega \circ (\alpha_1, \ldots, \alpha_n)|U = \theta \circ (\alpha_1, \ldots, \alpha_n)|U.$$

Hence

$$(\omega - \theta) \circ (\alpha_1, \ldots, \alpha_n)|U = 0.$$

From condition (*), which is satisfied by the functions $\alpha_1, \ldots, \alpha_n$, it follows that

$$(\omega - \theta)'_{|j}(\alpha_1(p), \ldots, \alpha_n(p)) = 0 \quad \text{for} \quad j = 1, \ldots, n$$

or equivalently

$$\omega'_{|j}(\alpha_1(p), \ldots, \alpha_n(p)) = \theta_{|j}(\alpha_1(p), \ldots, \alpha_n(p))$$

for $j = 1, \ldots, n$. Finally, we obtain the following equality

$$\sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p), \ldots, \alpha_n(p)) \cdot u_o(\alpha_i) = \sum_{i=1}^{n} \theta'_{|i}(\alpha_1(p), \ldots, \alpha_n(p)) \cdot u_o(\alpha_i).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This equality demonstrates that definition (7) is correct. It can be easily shown that the function u given by (7) is a tangent vector and that u is unique vector from T_pM such that $u|B = u_o$.

Proposition 4. Let (M,C) be a d-space and $B \subset C$ a d-basis of the d-structure C at $p \in M$. Then the mapping $\lambda : T_pM \longrightarrow R^B$ given by the formula

$$\lambda(u) := u|B \quad \text{for} \quad u \in T_pM \tag{8}$$

is an isomorphism of linear spaces.

Proof. It can be easily seen that λ is a linear mapping. λ is also a surjection. Indeed, in view of Lemma 3, for each function $u_o \in R^B$, there exists a tangent vector $u \in T_pM$ such that $u|B = u_o$. Let us also notice that λ is a monomorphism. Indeed, suppose that $u \in T_pM$ is a vector such that $\lambda(u) = 0$. It means that $u|B = 0$. The null vector $0 \in T_pM$ has the property that $0|B = 0$. From Lemma 3, i.e. from the uniqueness of prolongation, it follows that $u = 0$. Therefore, λ is an isomorphism.

Corollary 2. Let (M,C) be a d-space and B be a d-basis of the d-structure C at $p \in M$. Then

(a) If Card $B < \infty$ then $\dim T_pM = \text{Card } B$,

(b) If Card $B = \infty$ then $\dim T_pM = 2^{\text{Card } B}$.

Now we prove

Proposition 5. Let (M,C) be a d-space. If for every point $p \in M$ there exists a neighbourhood $U \in \tau_C$ of p and a set of functions $\{f_1,\dots,f_n\} \subset C$ which is a d-basis of the d-structure C at any point of U, then the d-space (M,C) is of constant differential dimension.

Proof. Let $p \in M$ be any point and $U \in \tau_C$ its neighbourhood such that the set $\{f_1,\dots,f_n\} \subset C$ is a d-basis of C at any point $q \in U$. Let W_i for $i = 1,\dots,n$, be a vector field on U satisfying the condition

$$W_i(q)(f_j) = \delta_{ij} \quad \text{for} \quad q \in U, \quad j = 1,\dots,n.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The uniqueness of the fields $W_1,\dots,W_n$ follows from Lemma 3. We shall show that the vector fields $W_1,\dots,W_n$ are smooth. For an arbitrary function $\alpha \in C_U$ and any point $q \in U$, there exist a neighbourhood $V \in \tau_C$ of q and a function $\omega \in \mathcal{E}_n$ such that $\alpha|V = \omega \circ (f_1,\dots,f_n)|V$, $V \subset U$. It can be easily seen that

$$(W_i|V)(\alpha|V) = \omega'_{|i}(f_1,\dots,f_n)|V \quad \text{for} \quad i=1,\dots,n.$$

From this the smoothnes of the vector fields $W_1,\dots,W_n$ is obvious. Since any vector field $W \in \mathfrak{X}(U)$ can be uniquely represented as the sum $W = \sum_{i=1}^{n} \varphi_i W_i$, where $\varphi_i = W(f_i|U)$, $i = 1,\dots,n$, the d-space (M,C) is of constant differential dimension n.

Example 4. Let $(M,C^\infty(M))$ be a differential manifold and $x = (x^1,\dots,x^n)$ a map in the neighbourhood of $p \in M$. There exist functions $f_1,\dots,f_n \in C^\infty(M)$ and a neighbourhood V of p such that

$$f_i|V = x_i|V \quad \text{for} \quad i = 1,\dots,n.$$

The set $\{f_1,\dots,f_n\}$ is the d-basis of the structure $C^\infty(M)$ at every point $q \in V$.

Example 5. Let C be the d-structure on R generated by the family of functions $\{f_n : n \in N\} \cup \{g_n : n \in N\}$, where

$$f_n(x) := \sin\frac{x}{n} \quad \text{and} \quad g_n(x) := \cos\frac{x}{n} \quad \text{for} \quad x \in R \ [2].$$

For any $n \in N$, the function f_n is a d-basis of C at these points $x \in R$ for which $f'_n(x) \neq 0$ and analogously g_n forms a d-basis of C at these points $x \in R$ for which $g'_n(x) \neq 0$. For every point $x \in R$, there exist a neighbourhood $U \in \tau_C$ of x and a subset of C consisting of the one element which is a d-basis of C at every point of U. Therefore, on the strength of Proposition 5, the d-space (M,C) has the differential dimension equal to 1.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] A. Kowalczyk, J. Kubarski: A local property of the subspaces of euclidean differential spaces, Demonstratio Math., 11 (1978), 875-885.

[2] A. Kowalczyk: Tangent differential spaces and smooth forms, Demonstratio Math., 13 (1980), 893-905.

[3] W. Sasin: On some exterior algebra of differential forms over a differential space, Demonstratio Math., 19 (1986), 1063-1075.

[4] W. Sasin: On equivalence relation on a differential space, Comment. Math. Univ. Carolinae 29 (1988) 529-539.

[5] W. Sasin, Z. Żekanowski: On locally finitely generated differential spaces, Demonstratio Math., 20 (1987), 477-487.

[6] R. Sikorski: Abstract covariant derivative, Colloq. Math., 18 (1967), 251-272.

[7] R. Sikorski: Differential modules, Colloq. Math., 24 (1971), 45-70.

[8] R. Sikorski: Introduction to differential geometry. Warsaw 1972 (in Polish).

[9] P. Walczak: A theorem on diffeomorphisms in the category of differential space, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astronom. Phys., 21 (1973), 325-329.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received October 27, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Czesław Dyjak

BV-SOLUTIONS OF NONLINEAR FUNCTIONAL EQUATIONS OF HIGHER ORDER

Introduction

In [1] we have considered BV-solutions of the nonlinear functional equations of first order, i.e. the case $m = 1$. In this paper we shall be concerned with BV-solutions of a nonlinear functional equations of the order $m \geq 2$, i.e.

(1) $$\varphi(x) = h(x, \varphi[f_1(x)], \ldots, \varphi[f_m(x)]),$$

(2) $$\varphi[f^m(x)] = g(x, \varphi[f^{m-1}(x)], \ldots, \varphi[f(x)], \varphi(x)),$$

where φ is an unknown function and f^i denotes i-th iterate of the function f.

Let (X,d) be a metric space. By $P(I)$ we denote the set of all the finite partitions $p: x_o < x_1 < \ldots < x_s$ of an interval $I \subset R$, where $x_i \in I$, $i = 0,1,\ldots,s$. Similarly as in [1], for the function $\varphi: I \longrightarrow X$ we define

(3) $$\operatorname*{Var}_I \varphi := \sup_{P(I)} \sum_{i=1}^{s} d(\varphi(x_i), \varphi(x_{i-1})).$$

By $BV(I)$ we denote the set of all functions $\varphi: I \longrightarrow X$ such that $\operatorname{Var}_I \varphi < \infty$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1. In this section we assume that

(i) (X,d) is a complete metric space,

(ii) $h: I \times X^m \longrightarrow X$,

(iii) there are a function $H \in BV(I)$ and a positive constants l_i, $i = 1,2,\ldots,m$, such that $0 < \sum_{i=1}^{m} l_i < 1$ and $d(h(x,y_1,\ldots,y_m), h(\bar{x},\bar{y}_1,\ldots,\bar{y}_m)) \leq \sum_{i=1}^{m} l_i d(y_i,\bar{y}_i) + d(H(x),H(\bar{x}))$, for $(x,y_1,\ldots,y_m)$, $(\bar{x},\bar{y}_1,\ldots,\bar{y}_m) \in I \times X^m$,

(iv) $f_i: I \longrightarrow I$, $i = 1,2,\ldots,m$, are continuous and strictly increasing in I.

Theorem 1. If hypotheses (i)-(iv) are fulfilled then the equation (1) has a unique solution $\varphi \in BV(I)$ given by formula $\varphi = \lim_{n\to\infty} \varphi_n$ (uniformly in I), where

$$\varphi_{n+1}(x) = h(x,\varphi_n[f_1(x)],\ldots,\varphi_n[f_m(x)]), \quad x \in I, \quad n = 0,1,\ldots$$

and $\varphi_0 \in BV(I)$ is arbitrarily choosen. Moreover

$$\operatorname*{Var}_I \varphi \leq \frac{\operatorname*{Var}_I H}{1 - \sum_{i=1}^{m} l_i}. \tag{4}$$

Proof. Let us take an arbitrary $M \geq \frac{\operatorname*{Var}_I H}{1 - \sum_{i=1}^{m} l_i}$ and define $A_M := \left\{\varphi \in BV(I): \operatorname*{Var}_I \varphi \leq M\right\}$. The set A_M with the metric $\rho(\varphi_1,\varphi_2) := \sup_{x\in I} d(\varphi_1(x),\varphi_2(x))$, $\varphi_1,\varphi_2 \in A_M$ is a complete metric space. In the space A_M let us consider the transformation $\varphi \longrightarrow T[\varphi]$ defined by the formula

$$T[\varphi](x) = h(x,\varphi[f_1(x)],\ldots,\varphi[f_m(x)]), \quad x \in I. \tag{5}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We shall prove that (5) maps A_M into itself. By (iii) and (iv), we have

$$\operatorname*{Var}_I T[\varphi] = \sup_{P(I)} \sum_{j=1}^{s} d(h(x_j, \varphi[f_1(x_j)], \dots, \varphi[f_m(x_j)]),$$

$$, h(x_{j-1}, \varphi[f_1(x_{j-1})], \dots, \varphi[f_m(x_{j-1})])) \leq$$

$$\leq \sup_{P(I)} \sum_{j=1}^{s} \left\{ \sum_{i=1}^{m} l_i d(\varphi[f_i(x_j)], \varphi[f_i(x_{j-1})]) + d(H(x_j), H(x_{j-1})) \right\} \leq$$

$$\leq \sum_{i=1}^{m} l_i \sup_{P(I)} \sum_{j=1}^{s} d(\varphi[f_i(x_j)], \varphi[f_i(x_{j-1})]) +$$

$$+ \sup_{P(I)} \sum_{j=1}^{s} d(H(x_j), H(x_{j-1})) = \sum_{i=1}^{m} l_i \operatorname*{Var}_{f_i(I)} \varphi + \operatorname*{Var}_I H \leq$$

$$\leq \sum_{i=1}^{m} l_i \operatorname*{Var}_I \varphi + \operatorname*{Var}_I H \leq M \sum_{i=1}^{m} l_i + M - M \sum_{i=1}^{m} l_i = M.$$

Further, we shall prove that (5) is a contraction map. Let us take arbitrary $\varphi_1, \varphi_2 \in A_M$. Taking into account (iii) and (iv), we have

$$\rho(T[\varphi_1], T[\varphi_2]) = \sup_{x \in I} d(h(x, \varphi_1[f_1(x)], \dots, \varphi_1[f_m(x)]),$$

$$, h(x, \varphi_2[f_1(x)], \dots, \varphi_2[f_m(x)])) \leq$$

$$\leq \sum_{i=1}^{m} l_i \sup_{x \in I} d(\varphi_1(x), \varphi_2(x)) = \rho(\varphi_1, \varphi_2) \sum_{i=1}^{m} l_i.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Thus, in virtue of the Banach principle there exists a unique fixed point of (5) in the space A_M. In other words there exists a unique solution $\varphi \in A_M$ of the equation (1). Because M is arbitrarily large, this is a unique $BV(I)$ solution of equation (1).

Next, we shall prove that estimation (4) holds. Using successively hypotheses (iii) and (iv), we have

$$\operatorname*{Var}_I \varphi = \sup_{P(I)} \sum_{j=1}^{s} d(h(x_j,\varphi[f_1(x_j)],\dots,\varphi[f_m(x_j)]),$$

$$,h(x_{j-1},\varphi[f_1(x_{j-1})],\dots,\varphi[f_m(x_{j-1})]))\leqslant$$

$$\leqslant \sup_{P(I)} \sum_{j=1}^{s}\left\{\sum_{i=1}^{m} l_i\, d(\varphi[f_i(x_j)],\ \varphi[f_i(x_{j-1})]) + d(H(x_j),H(x_{j-1}))\right\}\leqslant$$

$$\leqslant \sum_{i=1}^{m} l_i \operatorname*{Var}_{f_i(I)} \varphi + \operatorname*{Var}_I H \leqslant \operatorname*{Var}_I \varphi \sum_{i=1}^{m} l_i + \operatorname*{Var}_I H,$$

which proves inequality (4). This completes the proof of Theorem 1.

2. Now let us consider equation (2) and assume that

(i) (X,d) is a complete metric space,

(ii) $g:\langle a,b\rangle \times X^m \longrightarrow X$,

(iii) there are function $G \in BV\langle a,b\rangle$ and positive constants l_i $(i=1,2,\dots,m)$ such that $d(g(x,y_1,\dots,y_m),g(\bar{x},\bar{y}_1,\dots,\bar{y}_m))\leqslant$ $\leqslant \sum_{i=1}^{m} l_i\, d(y_i,\bar{y}_i) + d(G(x),G(\bar{x}))$ for $(x,y_1,\dots,y_m),(\bar{x},\bar{y}_1,\dots,\bar{y}_m) \in \langle a,b\rangle \times X^m$,

(iv) $f:\langle a,b\rangle \longrightarrow \langle a,b\rangle$ is continuous and strictly increasing in $\langle a,b\rangle$, and $a<f(x)<x$ for $x \in \langle a,b\rangle$,

(v) all the roots of the polynomial $p(z) = z^m - l_1 z^{m-1} - \dots - l_m$ have absolute values less than 1.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 2. By hypotheses (i)-(v), the equation (2) has a solution $\varphi \in BV(a,b>$ depending on an arbitrary function. More precisely: for any system of functions $\{\varphi_i\}_{i=1,2,\dots,m}$ such that $\varphi_i \in BV(I_i)$, where $I_i = <f^i(b), f^{i-1}(b)>$ and fulfilling the conditions $\varphi_{i+1}[f^i(b)] = \varphi_i[f^i(b)]$, $i = 1,2,\dots,m-1$, there exists the unique function $\varphi \in BV(a,b>$ satisfying the equation (2) in $(a,b>$ and such that $\varphi(x) = \varphi_k(x)$ for $x \in I_k$, $k = 1,2,\dots,m$.

Proof. Let $I_i = <f^i(b), f^{i-1}(b)>$, $i = 1,2,\dots$ By (iv), we have $\bigcup_{i=1}^{\infty} I_i = (a,b>$. Let us take arbitrary functions $\varphi_i : I_i \to R$, $i = 1,2,\dots,m$, fulfilling the conditions of the theorem. Notice that if $x \in I_{1+m}$, then $f^{-i}(x) \in I_{1+m-i}$, $i = 1,2,\dots,m$. In particular $f^{-m}(x) \in I_1$. So let us define the function $\varphi_{1+m} : I_{1+m} \to R$ for $x \in I_{1+m}$ as follows

$$\varphi_{1+m}(x) = g(f^{-m}(x), \varphi_{1+m-1}[f^{-1}(x)], \dots, \varphi_1[f^{-m}(x)]). \tag{6}$$

We shall prove that $\varphi_{1+m} \in BV(I_{1+m})$. Using successively (6), (iii) and (iv), we obtain

$$\operatorname{Var}_{I_{1+m}} \varphi_{1+m} = \sup_{P(I_{1+m})} \sum_{j=1}^{s} d(g(f^{-m}(x_j), \varphi_{1+m-1}[f^{-1}(x_j)], \dots, \varphi_1[f^{-m}(x_j)]),$$

$$g(f^{-m}(x_{j-1}), \varphi_{1+m-1}[f^{-1}(x_{j-1})], \dots, \varphi_1[f^{-m}(x_{j-1})])) \leqslant$$

$$\leqslant \sup_{P(I_{1+m})} \sum_{i=1}^{m} \sum_{j=1}^{s} l_i \, d(\varphi_{1+m-i}[f^{-i}(x_j)], \varphi_{1+m-i}[f^{-i}(x_{j-1})]) +$$

$$+ \sup_{P(I_{1+m})} \sum_{j=1}^{s} d(G[f^{-m}(x_j)], G[f^{-m}(x_{j-1})]) \leqslant \sum_{i=1}^{m} l_i \operatorname{Var}_{I_{1+m-i}} \varphi_{1+m-i} + \operatorname{Var}_{I_1} G.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Hence $\varphi_{1+m} \in BV(I_{1+m})$. More generally, let us take a function $\varphi_{k+m-1}: I_{k+m-1} \longrightarrow R$ and let $\varphi_{k+m-1} \in BV(I_{k+m-1})$, $k = 2,3,\ldots$. Evidently, if $x \in I_{k+m}$ then $f^{-m}(x) \in I_k$. Therefore the function $\varphi_{k+m}: I_{k+m} \to R$ given by the formula

$$(7) \quad \varphi_{k+m}(x) = g(f^{-m}(x), \varphi_{k+m-1}[f^{-1}(x)], \ldots, \varphi_k[f^{-m}(x)]), \quad k=2,3,\ldots$$

is well defined. Moreover, by a similar calculation as above, one can prove that $\varphi_{k+m} \in BV(I_{k+m})$ and that

$$(8) \quad \operatorname*{Var}_{I_{k+m}} \varphi_{k+m} \leq \sum_{i=1}^{m} l_i \operatorname*{Var}_{I_{k+m-i}} \varphi_{k+m-i} + \operatorname*{Var}_{I_k} G, \quad k = 2,3,\ldots$$

Now we introduce the following notations

$$a_k := \operatorname*{Var}_{I_k} \varphi_k, \quad b_k := \operatorname*{Var}_{I_k} G, \quad k = 1,2,\ldots$$

Using these notations we can write (8) as follows

$$(9) \quad a_{k+m} \leq \sum_{i=1}^{m} l_i a_{k+m-i} + b_k, \quad k = 1,2,\ldots$$

We define the function $\varphi: (a,b\rangle \to R$ in the following way

$$(10) \quad \varphi(x) = \begin{cases} \varphi_i(x) & \text{for } x \in I_i, \quad i=1,2,\ldots,m, \\ \varphi_i(x) & \text{for } x \in I_i, \quad i=m+1,m+2,\ldots \end{cases}$$

From (6), (7) and (10) it follows that φ satisfies equation (2) in $(a,b\rangle$. The uniqueness of the solution is obvious. To prove that (10) belongs to the class $BV(a,b\rangle$, we write $\operatorname*{Var}_{(a,b\rangle} \varphi = \sum_{n=1}^{\infty} a_n$. Since the sequence a_n, $n \in N$, satisfies inequality (9) it follows from Lemma 5.1 in [2] that the series $\sum_{n=1}^{\infty} a_n$ converges. This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] Cz. Dyjak, J. Matkowski: BV-solutions of a nonlinear functional equation, Glasnik Mat. 22 (1987), 335-342.

[2] J. Matkowski: Integrable solutions of functional equations, Dissertationes Math. CXXVII, Warszawa 1975, 68 pp.

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY,
42-201 CZĘSTOCHOWA, POLAND

Received November 7, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Michał Kisielewicz, Jerzy Motyl

CONTROLLABILITY OF THE NEUTRAL FUNCTIONAL-DIFFERENTIAL INCLUSION WITH M-DISSIPATIVE RIGHT HAND SIDE

1. Notation and definitions

Let C_{or}, (resp. AC_{or}, L_{or}) denote the space of all continuous (resp. absolutely continuous, integrable) functions x acting from the interval $[-r,0]$ to R^n with the standard norm:

$$\|x\|_C = \sup_{-r\leq t\leq 0} |x(t)|$$

$$(\text{resp. } \|x\|_{AC} = x(-r) + \int_{-r}^{0} |\dot{x}(t)|\,dt$$

$$\text{and } \|x\|_L = \int_{-r}^{0} |x(t)|\,dt).$$

We introduce the following measurability condition (M):

Let D be a nonempty open subset of $R \times C_{or} \times L_{or}$ and $F: D \to \text{Comp}(R^n)$ be a set-valued function such that:

(M) For every interval $I \subset R^1$ and sets $K \subset C_{or}$, $S \subset L_{or}$ with $I \times K \times S \subset D$ and for every $\varepsilon > 0$ there exists a compact set $E \subset I$ with $\mu(I \setminus E) \leq \varepsilon$ such that the restriction of F to $E \times K \times S$ is Borel measurable.

The space of all set-valued functions $F: D \to \text{Comp}(R^n)$ satisfying condition (M) will be denoted by $\mathcal{M}(D, \text{Comp}(R^n))$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For every x belonging to $C([\delta,\delta+\rho],R^n)$, $x(\delta) = 0$, every z belonging to $L([\delta,\delta+\rho],R^n)$ and Φ from AC_{or} we define $\Phi \oplus x$ and $\dot{\Phi} \boxplus z$ by setting:

$$(\Phi \oplus x)(t) = \begin{cases} \Phi(t-\delta) & \text{for } t \in [\delta-r,\delta] \\ \Phi(0)+x(t) & \text{for } t \in [\delta,\delta+\rho], \end{cases}$$

$$(\dot{\Phi} \boxplus z)(t) = \begin{cases} \dot{\Phi}(t-\delta) & \text{for } t \in [\delta-r,\delta] \\ z(t) & \text{for } t \in [\delta,\delta+\rho]. \end{cases}$$

A pair (D,Ω) of nonempty sets $D \subset R \times C_{or} \times L_{or}$ and $\Omega \subset R \times AC_{or}$ will be called conformable if for every $(t,x),(t,x),(t,y) \in \Omega$ one has $(t,x,\dot{y}) \in D$.

Given a conformable pair (D,Ω), a set $C \subset R \times C_{or} \times R \times C_{or}$ and a set-valued function $F: D \longrightarrow \mathrm{Comp}(R^n)$ we say that $\mathrm{NFDI}(D,F)$ is (Ω,C)-controllable if there exist numbers $\delta \in R$, $\rho > 0$ and an absolutely continuous function $x: [\delta-r,\delta+\rho] \longrightarrow R^n$ such that

$$\begin{cases} \dot{x}(t) \in F(t,x_t,\dot{x}_t) & \text{for a.e. } t \in [\delta,\delta+\rho] \\ (t,x_t) & \text{for a.e. } t \in [\delta,\delta+\rho] \\ (\delta,x_\delta,\delta+\rho,x_{\delta+\rho}) \in C \end{cases}$$

where $x_t(s) = x(t+s)$ for $t \in [\delta,\delta+\rho]$, $s \in [-r,0]$.

We say that (D,Ω,C) has a nonempty intersection property if the following conditions are satisfied:

(i) (D,Ω) is a conformable pair,

(ii) $(\Omega \times \Omega) \cap C$ is nonempty,

(iii) there are $(\delta,\Phi) \in \Omega$ and a number $\rho > 0$ such that the set $K_\Phi(\Omega,C) = \{ x \in AC([\delta,\delta+\rho],R^n): x(\delta) = 0,\ (t,(\Phi \oplus x)_t) \in \Omega$ for $t \in [\delta,\delta+\rho]$ and $(\delta,x_\delta,\delta+\rho,x_{\delta+\rho}) \in C \}$ is nonempty.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For $F:D \longrightarrow Comp(R^n)$ we define set-valued functions G^F and $\mathcal{F}(G^F)$ by the formulas:

$$G^F(t,x) = F(t,(\phi \oplus x)_t,(\dot{\phi} \boxplus \dot{x})_t) \quad \text{for} \quad t \in [\delta,\delta+\rho] \text{ and } x \in K_{\bar{\phi}}(\Omega,C).$$

$$\mathcal{F}(G^F)(x) = \left\{ f \in L([\delta,\delta+\rho],R^n) : f(t) \in G^F(t,x) \text{ for a.e. } t \in [\delta,\delta+\rho] \right\}$$

$$\text{Dom } \mathcal{F}(G^F) = \left\{ x \in K_{\bar{\phi}}(\Omega,C) : \mathcal{F}(G^F)(x) \text{ is nonempty} \right\}.$$

We will say that $\mathcal{F}(G^F)$ has strong-weak closed graph if for every sequence (x_n) in Dom $\mathcal{F}(G^F)$ and every sequence (u_n) of $L([\delta,\delta+\rho],R^n)$ such that $u_n \in \mathcal{F}(G^F)(x_n)$ for $n = 1,2,\ldots$, x_n tending to x strongly and u_n tending to u weakly in $L([\delta,\delta+\rho],R^n)$ we also have $u \in \mathcal{F}(G^F)(x)$.

Definition. Let $\Omega \subset R \times AC_{or}$ and $C \subset R \times C_{or} \times R \times C_{or}$ be given. A set-valued function $F \in \mathcal{M}(D, Comp(R^n))$ is said to be m-dissipative with respect to (Ω,C) if (D,Ω,C) has a nonempty intersection property and the following hold:

(i) for every $x,y \in$ Dom $\mathcal{F}(G^F)$, every $u \in \mathcal{F}(G^F)(x)$, $v \in \mathcal{F}(G^F)(y)$ and $\lambda > 0$ we have $\|x-y\|_{L_\delta} \leq \|x-y-\lambda(u-v)\|_{L_\delta}$,

(ii) $\text{Rang}(I-\mathcal{F}(G^F)) = L([\delta,\delta+\rho],R^n)$,

where I denotes the identity operator and $\|\ \|_{L_\delta}$ denotes the norm in the space $L([\delta,\delta+\rho],R^n)$.

2. Main result

Let us collect together some known consequences of m-dissipativity in the following Lemma 1 (see e.g. [1]).

Lemma 1. Let F be m-dissipative with respect to (Ω,C). For every $\lambda > 0$ let us define the following operators

$$J_\lambda(x) = (I - \lambda \mathcal{F}(G^F))^{-1}(x),$$

$$\mathcal{F}_\lambda(x) = \frac{1}{\lambda}(J_\lambda - I)(x).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then:

(i) J_λ and $\mathcal{F}_\lambda$ are well defined single-valued operators acting from whole space $L([\delta,\delta+\rho],R^n)$ to itself.

(ii) $\mathcal{F}_\lambda(x) \in \mathcal{F}(G^F)(J_\lambda(x))$.

(iii) If $\operatorname{dist}(0,\mathcal{F}(G^F)(x)) = \inf\{\|y\|_{L_\delta} : y \in \mathcal{F}(G^F)(x)\}$ is bounded by K for each x then

$$\lim_{\lambda\to 0} \|J_\lambda(x) - x\|_{L_\delta} = 0 \quad \text{uniformly with respect to } x.$$

(iv) $\|\mathcal{F}_\lambda(x)\|_{L_\delta} \le \operatorname{dist}(0, \mathcal{F}(G^F)(x))$ for $x \in \operatorname{Dom} \mathcal{F}(G^F)$.

Lemma 2. Let $F \in \mathcal{M}(D,\operatorname{Comp}(R^n))$ be bounded and such that $\mathcal{F}(G^F)$ has a strong-weak closed graph. Then for every $\rho>0$,

$$\mathcal{F}(G^F_\rho):\operatorname{Dom}\mathcal{F}(G^F) \longrightarrow 2^{L([\delta,\delta+\rho],R^n)} \quad \text{with} \quad G^F_\rho(t,x) = G^F(t,x) + \rho B$$

(where B is a closed unit ball) has also a strong-weak closed graph.

Proof. Suppose (x_n) and (v_n) are arbitrary sequences in $\operatorname{Dom}\mathcal{F}(G^F)$ and $L([\delta,\delta+\rho],R^n)$ respectively and such that $v_n \in \mathcal{F}(G^F_\rho)(x_n)$ for $n = 1,2,\ldots, \|x_n - x\|_{L_\delta} \longrightarrow 0$ and v_n tends weakly to v in $L([\delta,\delta+\rho],R^n)$.

Since $v_n \in \mathcal{F}(G^F_\rho(x_n))$ then $\operatorname{dist}(v_n(t), G^F(t,x_n)) \le \rho$ for a.e. $t \in [\delta,\delta+\rho]$ and $n = 1,2,\ldots$. It is known ([2]) that for every $n = 1,2,\ldots$ there exists $u_n \in \mathcal{F}(G^F)(x_n)$ such that

$$|v_n(t) - u_n(t)| = \operatorname{dist}(v_n(t), G^F(t,x_n)) \le \rho \quad \text{for a.e. } t \in [\delta,\delta+\rho].$$

By the boundedness of F the set $\bigcup_{n=1}^{\infty} \mathcal{F}(G^F)(x_n)$ is uniformly integrable on $[\delta,\delta+\rho]$.

Therefore, by Dunford Theorem the set $\{u_n\}_{n=1}^{\infty}$ is relatively weakly compact in $L((\delta,\delta+\rho],R^n)$. Thus there exists u and a subsequence, say (u_k) of (u_n) such that u_k tends weakly to u as k tends to ∞.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We have of course $|v_k(t) - u_k(t)| \leq \varrho$ for $k = 1,2,\ldots$ and a.e. $t \in [\delta, \delta+\rho]$.

By Banach-Mazur Theorem there exists a set of real numbers $C_{n_k} \geq 0$, $k=1,2,\ldots,N$, $N=1,2,\ldots$ with $\sum_{k=1}^{N} C_{n_k} = 1$ and such that $\sum_{k=1}^{N} C_{n_k}\left[v_{n_k}(t) - u_{n_k}(t)\right]$ tends to $v(t) - u(t)$ for a.e. $t \in [\delta, \delta+\rho]$ as $N \to \infty$. But for every $N = 1,2,\ldots$ and a.e. $t \in [\delta, \delta+\rho]$ we have

$$\left| \sum_{k=1}^{N} C_{n_k}\left[v_{n_k}(t) - u_{n_k}(t)\right] \right| \leq \sum_{k=1}^{N} C_{n_k} \varrho = \varrho .$$

Then we obtain $|v(t) - u(t)| \leq \varrho$ for a.e. $t \in [\delta, \delta+\rho]$.

Now by the assumption on the graph of $\mathcal{F}(G^F)$ we have $u \in \mathcal{F}(G^F)(x)$, i.e. $u(t) \in G^F(t,x)$ for a.e. t. Therefore for a.e. $t \in [\delta, \delta+\rho]$ we have $v(t) \in G^F(t,x) + \varrho B$ i.e. $v \in \mathcal{F}(G^F_\varrho)(x)$.

Before formulating our main result we introduce the following notation

$$\Lambda_\alpha = \left\{ u \in L([\delta, \delta+\rho], R^n) : |u(t)| < \alpha \quad \text{almost everywhere} \right\},$$

$K_\alpha = \mathcal{T}(\Lambda_\alpha)$ where $\mathcal{T}$ is an operator defined by the formula

$$(\mathcal{T}u)(t) = \int_\delta^t u(t)dt;$$

$$T_H(u) = \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left[\frac{1}{h}(H-u) + \varepsilon B\right]$$

is Bouligand's contingent cone to H at $u \in H$.

The following viability theorem given in [3] will be used in the proof of our main result:

Viability Theorem ([3], Th.4): Let $\Phi \in AC_{or}$ and a compact set $H \subset R^n$ be such that $\mathcal{T}(u)(t) \in H - \Phi(0)$ for every $t \in [\delta, \delta+\rho]$ and $u \in \Lambda_\alpha$ with some constant $\alpha = 1+M$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $F : [\delta,\delta+\rho]\times C_{or}\times L_{or} \to \mathrm{Comp}\ R^n$ be such that:

a) there exist a constant $\rho>0$ and a continuous function $q : K_\alpha \to L([\delta,\delta+\rho],R^n)$ such that $q(x)(t)\in G^F_\rho(t,x)$ for a.e. $t\in[\delta,\delta+\rho]$, $x\in K_\alpha$ and such that a function $g : [\delta,\delta+\rho]\times K_\alpha \to R^n$ defined by $g(t,x) = q(x)(t)$ has Volterra's property with respect to $x\in K_\alpha$,

b) $G^F([\delta,\delta+\rho]\times K_\phi(\Omega,C))\subset MB$,

c) for every $x\in K_\alpha$ and a.e. $t\in[\delta,\delta+\rho]$ we have

$$G^F(t,x)\subset T_H[(\phi\oplus x)_t(0)] + \rho B.$$

Then for a given $\rho>0$ there exists $x^\rho\in K_\alpha$ such that $x^\rho(t)\in H-\phi(0)$ and $\dot{x}^\rho(t)\in G^F(t,x^\rho)$ for a.e. $t\in[\delta,\delta+\rho]$.

Theorem. Let $\Omega\subset R\times AC_{or}$, $C\subset R\times C_{or}\times R\times C_{or}$ and $F\in\mu(D,\mathrm{Comp}(R^n))$ be such that the following conditions (i)-(iv) hold:

(i) F is bounded and m-dissipative with respect to (Ω,C),

(ii) $\mathcal{F}(G^F)$ has a strong-weak closed graph,

(iii) there exist numbers $l>0$, $M>0$ and a compact set $H\subset R^n$ such that

(a) $\mathcal{T}(u)(t)\in H-\phi(0)$ for every $t\in[\delta,\delta+\rho]$ and $u\in\Lambda_\alpha$ with $\alpha = l+M$,

(b) $G^F([\delta,\delta+\rho]\times K_\phi(\Omega,C))\subset MB$,

(c) $K_\alpha\subset K_\phi(\Omega,C)$,

(d) for every $\rho>0$ there exists $\delta_\rho>0$ such that for every $x\in K_\alpha$ and $y\in K_\phi(\Omega,C)$ satisfying $\|x-y\|_L\le\delta_\rho$ we have

$$F(t,(\phi\oplus y)_t,(\dot{\phi}\boxplus\dot{y})_t)\subset T_H[(\phi\oplus x)_t(0)] + \rho B \text{ for a.e. } t\in[\delta,\delta+\rho],$$

(iv) for every $x,y\in K_\phi(\Omega,C)$, every $\lambda>0$ and $u\in\mathcal{F}(G^F)(x)$, $v\in\mathcal{F}(G^F)(y)$ we have for all $t\in[\delta,\delta+\rho]$

$$\sup_{\delta\le\tau\le t}|x(\tau)-y(\tau)| \le \sup_{\delta\le\tau\le t}|x(\tau)-y(\tau)-\lambda(u(\tau)-v(\tau))|.$$

Then NFDI(D,F) is (Ω,C)-controllable.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let J_λ and $\mathcal{F}_\lambda$ be operators defined in Lemma 1. For every fixed $\lambda>0$ we define $f_\lambda:[\delta,\delta+\rho]\times K_\phi(\Omega,C) \longrightarrow R^n$ by setting $f_\lambda(t,x) = \mathcal{F}_\lambda(x)(t)$. We have $f_\lambda(\cdot,x) = \mathcal{F}_\lambda(x) \in L([\delta,\delta+\rho],R^n)$ for every $x \in K_\phi(\Omega,C)$.

Now let $z_1 = J_\lambda(x)$ and $z_2 = J_\lambda(y)$ for fixed $x,y \in K_\phi(\Omega,C)$. By the definition of J_λ we obtain $z_1,z_2 \in K_\phi(\Omega,C)$ and $x \in z_1-\lambda\mathcal{F}(G^F)(z_1)$, $y \in z_2-\lambda\mathcal{F}(G^F)(z_2)$. Select now $u_1 \in \mathcal{F}(G^F)(z_1)$ and $u_2 \in \mathcal{F}(G^F)(z_2)$ such that $x = z_1 - \lambda u_1$, $y = z_2 - \lambda u_2$.

By (iv) for fixed $t \in [\delta,\delta+\rho]$ we obtain

$$\sup_{\delta\le\tau\le t} |z_1(\tau)-z_2(\tau)| \le \sup_{\delta\le\tau\le t} |z_1(\tau)-z_2(\tau)-\lambda(u_1(\tau)-u_2(\tau))|$$

$$\text{i.e. } \sup_{\delta\le\tau\le t} |J_\lambda(x)(\tau) - J_\lambda(y)(\tau)| \le \sup_{\delta\le\tau\le t} |x(\tau)-y(\tau)|.$$

Hence, by the definition of $\mathcal{F}_\lambda$ if follows that

$$\sup_{\delta\le\tau\le t} |\mathcal{F}_\lambda(x)(\tau) - \mathcal{F}_\lambda(y)(\tau)| = \frac{1}{\lambda}|[J_\lambda(x)(\tau)-J_\lambda(y)(\tau)]-[x(\tau)-y(\tau)]| \le$$

$$\le \frac{2}{\lambda} \sup_{\delta\le\tau\le t} |x(\tau)-y(\tau)| \quad \text{for } t \in [\delta,\delta+\rho].$$

Therefore

$$|f_\lambda(t,x)-f_\lambda(t,y)| \le \frac{2}{\lambda}|x-y|_t \quad \text{where} \quad |x-y|_t = \sup_{\delta\le\tau\le t} |x(\tau)-y(\tau)|.$$

Hence, in particular it follows that f_λ is for every fixed $\lambda>0$ continuous and has Volterra's property with respect to its second variable (it means that for every $u,v \in K_\phi(\Omega,C)$ and $t \in [\delta,\delta+\rho]$ such that $u(\tau) = v(\tau)$ for $\tau \in [\delta,t]$ we have $f_\lambda(t,u) = f_\lambda(t,v)$).

By virtue of Lemma 1 (ii) and (iv) we have $f_\lambda(\cdot,x) \in \mathcal{F}(G^F)(J_\lambda(x))$ and $|f_\lambda(t,x)| \le K$ for $x \in K_\phi(\Omega,C)$ and a.e. $t \in [\delta,\delta+\rho]$. By assumption (iii)(b) we can use Lemma 1 (iii) to obtain that for every $\delta_o>0$ there is $\lambda_o>0$ such that $\|J_\lambda(x)-x\|_{L_\delta} \le \delta_o$ for $x \in K_\phi(\Omega,C)$ and $\lambda \in (0,\lambda_o)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now let $\varepsilon>0$ be given and suppose $\delta_o>0$ is such taken that

$$F(t,(\phi\oplus y)_t,(\dot{\phi}\boxplus\dot{y})_t)\subset T_H[(\phi\oplus x)_t(0)] + \varepsilon B$$

for a.e. $t\in[\delta,\delta+\rho]$, every $x\in K_\alpha$ and $y\in K_\phi(\Omega,C)$ satisfying $\|x-y\|_{L_\delta}\leq\delta_o$. Then we have $G^F(t,J_\lambda(x))\subset T_H[(\phi\oplus x)_t(0)] + \varepsilon B$ and $f_\lambda(t,x)\in T_H[(\phi\oplus x)_t(0)] + \varepsilon B$ for every $x\in K_\alpha$, $\lambda\in(0,\lambda_o)$ and a.e. $t\in[\delta,\delta+\rho]$.

Now by Viability Theorem 4 given in [3] we obtain that for every $\varepsilon>0$ and every $\lambda\in(0,\lambda_o)$ there is $x_\lambda\in K$ such that

$$\dot{x}_\lambda(t)\in G^F(t,x_\lambda) + \varepsilon B \quad \text{for a.e.} \quad t\in[\delta,\delta+\rho].$$

Let $\lambda=\frac{1}{k}$ and select N such that $\frac{1}{k}<\lambda_o$ for $k\geq N$. We put $x^k=x_{1/k}$ for $k\geq N$. Since K_α is bounded, closed and uniformly absolutely continuous in AC then it is compact in C and therefore in L^1 too. Since $|x(t)|<1+M$ for almost all t and $k\geq N$ then the set $\{\dot{x}^k\}$ is uniformly integrable so by Dunford's theorem L^1-relative sequential weakly compact. Therefore there exists $x_\varepsilon\in K_\alpha$ and a subsequence, say again (x^k) of (x^k) such that $\|x^k-x_\varepsilon\|_{L_\delta}\longrightarrow 0$ and $\dot{x}^k\longrightarrow\dot{x}_\varepsilon$ weakly in $L([\delta,\delta+\rho],R^n)$.

We have $\dot{x}^k\in\mathcal{F}(G^F_\varepsilon)(x^k)$ for $k=1,2,\ldots$ where $G^F_\varepsilon(t,x)=$ $=G^F(t,x)+\varepsilon B$ for $(t,x)\in I\times K$.

By virtue of Lemma 2 it follows that $\dot{x}_\varepsilon\in\mathcal{F}(G^F_\varepsilon)(x_\varepsilon)$ for every $\varepsilon>0$.

Taking ε tending monotonically to zero we can again find $x\in K_\alpha$ and a subsequence of (x_ε) say (x_{ε_n}) such that $\|x_{\varepsilon_n}-x\|_{L_\delta}\longrightarrow 0$ and $\dot{x}_{\varepsilon_n}\longrightarrow\dot{x}$ weakly in $L([\delta,\delta+\rho],R^n)$. Since $\dot{x}_{\varepsilon_n}\in\mathcal{F}(G^F_{\varepsilon_n})(x_{\varepsilon_n})$ for every $n=1,2,\ldots$ then $\dot{x}\in\mathcal{F}(G^F_{\varepsilon_n})(x)$ for every ε_n. Therefore

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\dot{x} \in \mathcal{F}(G^F)(x)$. Put now $y = \Phi \oplus x$. We have $\dot{y}(t) \in F(t, y_t, \dot{y}_t)$ for $t \in [\delta, \delta+\rho]$, $(t, y_t) \in \Omega$ and $(\delta, y_\delta, \delta+\rho, y_{\delta+\rho}) \in C$ because $x \in K_\alpha \subset K_\Phi(\Omega, C)$. Then NFDI(D,F) is (Ω, C)-controllable.

REFERENCES

[1] V. Barbu: Nonlinear semigroups and differential equations in Banach spaces. Noordhoff, Intern. Publishing, Leyden 1976.

[2] C.J. Himmelberg: Measurable relations, Fundamenta Math. 87 (1975), 53-72.

[3] M. Kisielewicz: Viability theorems for neutral-functional-differential inclusions, Funkcial Ekvac. 33 (1) (to appear).

DEPARTMENT OF MATHEMATICS, HIGHER ENGINEERING COLLEGE, 65-245 ZIELONA GÓRA, POLAND

Received November 30, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Andrzej Gębarowski

ON A CLASS OF HOMOGENEOUS CONFORMALLY RECURRENT MANIFOLDS

1. Introduction

Let (M,g) be a Riemannian manifold with a (possibly indefinite) metric g. A tensor field $T^{i_1\dots i_p}_{\quad j_1\dots j_q}$ of type (p,q) on M will be called recurrent if

$$T^{h_1\dots h_p}_{\quad t_1\dots t_q} T^{i_1\dots i_p}_{\quad j_1\dots j_q,k} = T^{h_1\dots h_p}_{\quad t_1\dots t_q,k} T^{i_1\dots i_p}_{\quad j_1\dots j_q}, \tag{1}$$

where the comma denotes covariant differentiation with respect to g. Relation (1) states that at any point $x \in M$ such that $T(x) \neq 0$ there exists a (unique) covariant vector $\varkappa$ (called the recurrence vector of T) which satisfies the condition

$$T^{i_1\dots i_p}_{\quad j_1\dots j_q,k}(x) = \varkappa_k T^{i_1\dots i_p}_{\quad j_1\dots j_q}(x). \tag{2}$$

A Riemannian manifold (M,g) will be called recurrent (Ricci-recurrent) if its curvature tensor (Ricci tensor) is recurrent. Throughout this paper we assume that the Ricci tensor of a Ricci-recurrent manifold is not parallel.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

According to Adati and Miyazawa [1], an n-dimensional ($n \geq 4$) Riemannian manifold (M,g) will be called conformally recurrent if its Weyl conformal curvature tensor

$$(3)\quad C_{hijk} = R_{hijk} - \frac{1}{n-2}\left(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}\right) + \\ + \frac{R}{(n-1)(n-2)}\left(g_{hk}g_{ij} - g_{ik}g_{hj}\right)$$

is recurrent.

If $C_{hijk,l} = 0$ everywhere on M and dim $M \geq 4$, then (M,g) is said to be conformally symmetric [2]. Such a manifold is called essentially conformally symmetric if it is neigher conformally flat ($C_{hijk} = 0$) nor locally symmetric ($R_{hijk,l} = 0$). Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as recurrent manifolds of dimension $n \geq 4$. The existence of essentially conformally recurrent manifolds, i.e. of conformally recurrent manifolds which lie beyond the two classes mentioned above, has been established in [8].

Let (M,g) be an n-dimensional Riemannian manifold whose metric g need not be definite. If $\overline{g}$ is another metric on M, and there exists a function p on M such that $\overline{g} = (\exp 2p)g$, then g and $\overline{g}$ are said to be conformally related or conformal to each other. A special class of conformally recurrent manifolds which is a natural extension of the class of essentially conformally symmetric ones have been defined and studied by W. Roter in his paper [11]. Namely, a conformally recurrent manifold (M,g) is called simple (s.c.r. in short) if its metric is locally conformal to a non-conformally flat conformally symmetric one, i.e. if for each point $x \in M$ there exists a neighbourhood V of x and a function p on V such that $\overline{g} = (\exp 2p)g$ is a non-conformally flat conformally symmetric metric. In [11] W. Roter gave a characterization of s.c.r. manifolds (Theorem 1) and proved

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(Theorem 5) that every non-locally symmetric s.c.r. manifold is Ricci-recurrent or it admits a unique recurrent absolute exterior 2-form ω satisfying

$$C_{hijk} = e\omega_{hi}\omega_{jk} \tag{4}$$

with $|e| = 1$, rank $\omega = 2$ and $\omega_{ri}\omega^{r}_{j} = 0$.

Although the Ricci-recurrent ones do not exhaust the whole class of essentially s.c.r. manifolds (see [5]), they form a remarkable subclass.

We shall restrict our consideration to manifolds which are essentially s.c.r. and whose Weyl conformal curvature tensor is not of the form (4). In [11] W. Roter proved the existence and gave a complete local description (at generic points) of those manifolds. The present paper deals with a global classification problem for homogeneous s.c.r. manifolds of class C^{∞} or analytic. First we consider homogeneous s.c.r. manifolds $\bar{M}^{n}$ (determined in Theorem 1), which are universal in the sense that the pseudo-Riemannian universal coverings of their homogeneous open submanifolds exhaust, up to isometry, all simply connected homogeneous s.c.r. manifolds of the considered type. Next using the same method as in [3] we obtain some information about the global structure of essentially s.c.r. manifolds. We prove there (Theorem 3) that such a manifold is always diffeomorphic to a product $R^{2}\times M$, M being flat and homogeneous.

Throughout this paper, by a manifold we shall mean a connected paracompact manifold either of class C^{∞} or analytic. Concerning Riemannian manifolds, we shall often write M instead of (M,g).

2. The general form of universal models

In this section each Latin index runs over 1,2,...,n, and each Greek index - over 2,3,...,n-1. Given a pseudo-Riemannian manifold (M,g), by a local isometry of M we shall mean any isometry between open connected subsets of M. First we formulate certain important

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

examples of simply connected, complete, essentially simple conformally recurrent manifolds whose Weyl conformal curvature tensor is not of the form (4) described as follows.

Theorem 1. (i) Let $\bar{M}$ denote the Euclidean n-space ($n \geq 4$) endowed with the metric $\bar{g}$ given by

$$\bar{g}_{ij}dx^i dx^j = \Phi(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2\,dx^1 dx^n,$$

where the function Φ is defined by

$$\Phi(x^1,\ldots,x^n) = (A(x^1)k_{\lambda\mu} + B(x^1)c_{\lambda\mu})x^\lambda x^\mu,$$

A, B being a non-constant analytic functions on R and $[k_{\lambda\mu}]$, $[c_{\lambda\mu}]$ non-zero symmetric matrices such that $[k_{\lambda\mu}]$ is non-singular and $k^{\lambda\mu}c_{\lambda\mu} = 0$, rank $c_{\lambda\mu} > 1$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$. Then $\bar{M}$ is a simply connected, complete, analytic, n-dimensional simple conformally recurrent manifold whose Weyl conformal curvature tensor is not of the form (4).

(ii) In the Cartesian coordinates, any local isometry $f = (f^1,\ldots,f^n)$ of $\bar{M}$ is of the form

$$(5)\quad \begin{cases} f^1(x^1,\ldots,x^n) = \varepsilon x^1 + T \\ f^\lambda(x^1,\ldots,x^n) = H^\lambda_\mu x^\mu + C^\lambda(x^1), \quad \lambda = 2,\ldots,n-1 \\ f^n(x^1,\ldots,x^n) = -\varepsilon k_{\lambda\mu}\dot{C}^\lambda(x^1)\left[H^\mu_\nu x^\nu + \frac{1}{2}C^\mu(x^1)\right] + \varepsilon x^n + z \end{cases}$$

where $[H^\lambda_\mu]$ is an $(n-2)\times(n-2)$ matrix and z, ε, T are real numbers satisfying the conditions

$$(6)\qquad |\varepsilon| = 1, \quad A(t) = A(\varepsilon t + T),$$

$$(7)\quad \begin{cases} \text{a)} \quad k_{\nu\tau}H^\nu_\lambda H^\tau_\mu = k_{\lambda\mu} \\ \text{b)} \quad c_{\lambda\mu}B(t) = B(\varepsilon t + T)\,c_{\nu\tau}H^\nu_\lambda H^\tau_\mu \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for any real t, and the functions C^λ, $\lambda = 2,3,\ldots,n-1$, form a solution of the following system of ordinary differential equations

$$\ddot{E}^\lambda(t) = A(t)E^\lambda(t) + B(t)k^{\lambda\nu}\, c_{\nu\tau} E^\tau(t), \tag{8}$$

with $E^\lambda(t) = H^\lambda_\tau C^\tau(t)$.

(iii) Conversely, given z, ε, T and H^λ_μ, C^λ satisfying (6)-(8), formulae (5) define a global isometry of $\overline{M}$ onto itself.

Proof. (i) From Roter's consideration (cf. [11], proof of Th.7) it follows that $\overline{M}$ is a non-conformally symmetric s.c.r. Ricci-recurrent manifold whose Weyl conformal curvature tensor is not of the form (4). By an explicit computation we verify that the geodesic equations for $\overline{M}$ reduce to a system of linear differential equations, so that $\overline{M}$ is complete.

(ii) Let f be a local isometry of $(\overline{M},\overline{g})$. The differential dx^1 is the unique (up to a constant factor) parallel covariant vector field in $\overline{M}$ (cf. [9] p.54). Hence $f^*dx^1 = \varepsilon dx^1$, ε being a non-zero constant. Raising indices, we obtain $f_* \frac{\partial}{\partial x^n} = \varepsilon^{-1} \frac{\partial}{\partial x^n}$ i.e. $\partial_n f^n = \varepsilon^{-1}$ and $\partial_n f^i = 0$ for $i<n$. One can explicity compute that, the essential components of Ricci tensor R and ∇R are

$$R_{11} = (n-2)A, \qquad \nabla_j R_{11} = (n-2)\,\nabla_j A$$

(see [11], formulae (36)) so that the relation $f^*\nabla R = \nabla R$ yields $f^1 = x^1 \circ f = \varphi(x^1)$. Moreover, f leaves invariant the orthogonal complement D of $\frac{\partial}{\partial x^n}$, which is an integrable codimension one distribution on $\overline{M}$. This parallel $(n-1)$-plane field determines a foliation whose leaves are totally geodesic submanifolds of $\overline{M}$. Any leaf $\mathcal{L}$ of D is given by $x^1 = \text{const}$ and inherits from $\overline{M}$ a symmetric connection (as a totally geodesic submanifold), which is flat since $dx^2,\ldots,dx^n$ are parallel along $\mathcal{L}$ (see [7], pp.56-59) so that $x^2,\ldots,x^n$ are affine co-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ordinates for α. Our local isometry f, whenever defined, sends (local) leaves of D affinely into leaves. Thus, f^λ and f^n are affine functions of $x^2,\dots,x^n$, and f is of the form

$$(9)\qquad \begin{cases} f^1(x^1,\dots,x^n) = \varphi(x^1), \\ f^\lambda(x^1,\dots,x^n) = f^\lambda_\mu(x^1)x^\mu + C^\lambda(x^1), \\ f^n(x^1,\dots,x^n) = f^n_\mu(x^1)x^\mu + \varepsilon^{-1}x^n + C^n(x^1), \end{cases}$$

with

$$(10)\qquad \det\left[f^\lambda_\mu(x^1)\right] \neq 0.$$

Comparing now the components of $\bar{g}$ with those transformed by f, we obtain

$$1 = \bar{g}_{1n} = (f^*\bar{g})_{1n} = \varepsilon^{-1}\dot{\varphi}(x^1),$$

which implies $f^1(x^1,\dots,x^n) = \varepsilon x^1 + T$ for some real T.

Next we have

$$0 = \bar{g}_{1\lambda} = (f^*\bar{g})_{1\lambda} = k_{\nu\mu}\dot{f}^\nu_\rho(x^1)x^\rho f^\mu_\lambda(x^1) + k_{\nu\mu}\dot{C}^\nu(x^1)f^\mu_\lambda(x^1) + \varepsilon f^n_\lambda(x^1).$$

The right-hand side of this equality is a polynomial in variables $x^2,\dots,x^{n-1}$, so that $k_{\lambda\mu}\dot{f}^\lambda_\rho(x^1)f^\mu_\nu(x^1) = 0$, hence $\dot{f}^\lambda_\mu = 0$ by (10) i.e. f^λ_μ is constant, say $f^\lambda_\mu(x^1) = H^\lambda_\mu$, and

$$(11)\qquad f^n_\lambda(x^1) = -\varepsilon^{-1}k_{\nu\mu}\dot{C}^\nu(x^1)H^\mu_\lambda.$$

It is also easy to see that $k_{\lambda\mu} = (f^*\bar{g})_{\lambda\mu} = k_{\nu\rho}H^\nu_\lambda H^\rho_\mu$, which implies (7)(a). Evaluating the equality $f^*\bar{g} = \bar{g}$ for the component $\bar{g}_{11}$ we obtain certain equality between polynomials in variables x^λ, $\lambda = 2,\dots,n-1$, namely

$$\left[A(x^1)k_{\lambda\mu} + B(x^1)\,c_{\lambda\mu}\right]x^\lambda x^\mu = \varepsilon^2\left[A(\varepsilon x^1 + T)k_{\nu\tau} + B(\varepsilon x^1 + T)\,c_{\nu\tau}\right] \cdot \left[H^\nu_\lambda H^\tau_\mu x^\lambda x^\mu + 2H^\nu_\lambda C^\tau(x^1)x^\lambda + C^\nu(x^1)C^\tau(x^1)\right] + 2\varepsilon\dot{f}^n_\mu(x^1)x^\mu + 2\varepsilon\dot{C}^n(x^1) + k_{\lambda\mu}\dot{C}^\lambda(x^1)\dot{C}^\mu(x^1),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and by comparing their coefficients and using (7) a) we get

$$A(x^1)k_{\lambda\mu} + B(x^1)\, c_{\lambda\mu} = \varepsilon^2\left[A(\varepsilon x^1+T)k_{\lambda\mu} + B(\varepsilon x^1+T)\, c_{\nu\tau}H^{\nu}_{\lambda}H^{\tau}_{\mu}\right], \tag{12}$$

$$\dot{f}^{n}_{\mu}(x^1) + \varepsilon\left[A(\varepsilon x^1+T)k_{\nu\tau} + B(\varepsilon x^1+T)\, c_{\nu\tau}\right]H^{\nu}_{\mu}C^{\tau}(x^1) = 0, \tag{13}$$

$$2\varepsilon\dot{C}^{n}(x^1) + k_{\lambda\mu}\dot{C}^{\lambda}(x^1)\dot{C}^{\mu}(x^1) + \varepsilon^2\left[A(\varepsilon x^1+T)k_{\nu\tau} + B(\varepsilon x^1+T)\, c_{\nu\tau}\right]C^{\nu}(x^1)C^{\tau}(x^1) = 0. \tag{14}$$

Transvecting (12) with $k^{\lambda\mu}$ and taking account of $k^{\lambda\mu}c_{\lambda\mu} = 0$ and $k^{\lambda\mu}H^{\nu}_{\lambda}H^{\tau}_{\mu} = k^{\nu\tau}$, we have

$$A(t) = \varepsilon^2 A(\varepsilon t + T) \tag{15}$$

for any real t. Using the very same argument as in [4] (proof of Theorem 2) we obtain $|\varepsilon| = 1$ and (15) implies (6). Obvious consequence of (12) is formula (7) b).

Combining (11) with (13) and taking into account (6) and 7 a) b) we obtain

$$H^{\lambda}_{\tau}\ddot{C}^{\tau}(t) = A(t)H^{\lambda}_{\tau}C^{\tau}(t) + B(t)k^{\lambda\nu}\, c_{\nu\tau}H^{\tau}_{\sigma}C^{\sigma}(t),$$

which by setting $E^{\lambda}(t) = H^{\lambda}_{\tau}C^{\lambda}(t)$ yields (8).

From (14) and (8) it follows immediately that

$$d\left(\dot{C}^{n}(t) + \frac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^{\lambda}(t)C^{\mu}(t)\right)\Big/dt = d\left(\dot{C}^{n}(t) + \frac{1}{2}\varepsilon k_{\lambda\mu}\dot{E}^{\lambda}(t)E^{\mu}(t)\right)\Big/dt = 0$$

hence

$$C^{n}(t) = -\frac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^{\lambda}(t)C^{\mu}(t) + z$$

for some real z. In view of (11), this completes the proof of assertion (ii).

(iii) As for the inverse one, we can immediately verify that formulae (5) together with (6)-(8) define global isometry of $\overline{M}$ onto itself. This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the sequel we shall need the following lemma:

Lemma 1 (see [3], Lemma 1). Let (M,g) and $(\bar{M},\bar{g})$ be two homogeneous pseudo-Riemannian manifolds, locally isometric to each other. If M is simply connected and $\bar{M}$ has the property

(16) any local isometry of $\bar{M}$ can be extended to a global isometry of $\bar{M}$ onto itself,

then there exists an isometric immersion $f: M \longrightarrow \bar{M}$ such that

(i) the image $f(M)$ is homogeneous (as an open submanifold of $\bar{M}$),

(ii) $f: M \longrightarrow f(M)$ is a covering.

As mentioned in the introduction the name "universal model" for $\bar{M}^n$ can now be justified as follows.

Theorem 2. Any simply connected homogeneous s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4) is isometric to the pseudo-Riemannian universal covering of an open homogeneous submanifold of universal model.

Proof. Our argument is a replica of the proof of Theorem 2 in [3]. To prove the statement, observe that any homogeneous simply connected s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4) is locally isometric to a universal model $\bar{M}$ (cf. [11] Theorem 7). Theorem 1 implies clearly that $\bar{M}$ enjoys the property (16). By Lemma 1, there exists an isometric immersion $f: M \longrightarrow f(M) \subset \bar{M}$ which is nothing but the universal covering projection. This completes the proof.

3. Some global properties

We are now going to derive some consequences of the above results. It will be convenient to adopt the following notations and conventions. Each element h of the isometry group $I(\bar{M})$ determined in Theorem 1 will be identified with quintuples $h = (\varepsilon, T, H, C(t), z)$ where (as we know) ε belongs to the multiplicative group $Z_2 = \{-1,1\}$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

T satisfies (6) and so its range is a discrete subset of R, $H = \left[H^{\lambda}_{\mu}\right] \in G$ the group of all $(n-2)\times(n-2)$ matrices satisfying (7), the curve $t \to C(t) = \left[C^2(t), \ldots, C^{n-1}(t)\right]$ in R^{n-2} is an element of the vector space V of all solutions of (8) and z is an arbitrary real number. On the space V we define the exterior 2-form ω by

$$\omega(C_1, C_2) = \frac{1}{2} k_{\lambda\mu}\left[\dot{C}^{\lambda}_1(t) C^{\mu}_2(t) - C^{\lambda}_1(t)\dot{C}^{\mu}_2(t)\right].$$

Differentiating $\omega(E_1, E_2)$ and taking into account (8) we get

$$d\omega(E_1, E_2)/dt = \frac{1}{2} k_{\lambda\mu}\left[\ddot{E}^{\lambda}_1(t) E^{\mu}_2(t) - E^{\lambda}_1(t)\ddot{E}^{\mu}_2(t)\right] = 0,$$

which implies that ω is a constant independent of t. The 2-form ω is the group operation of $I(\overline{M})$, namely

$$(\varepsilon_1, T_1, H_1, C_1(t), z_1)(\varepsilon_2, T_2, H_2, C_2(t), z_2) =$$

$$= \left[\varepsilon_1\varepsilon_2,\ \varepsilon_1 T_2 + T_1,\ H_1 H_2,\ H_1 C_2(t) + C_1(\varepsilon_2 t + T_2),\right.$$

$$\left.\varepsilon_1\varepsilon_2\omega(H_1 C_2(t),\ C_1(\varepsilon_2 t + T_2)) + \varepsilon_1 z_2 + z_1\right],$$

where the courves $t \to H_1 C_2(t)$ and $t \to C_1(\varepsilon_2 t + T_2)$ are easily seen to lie in space V again.

Points of our manifold $\overline{M}$, whose underlying set is just R^n, will be described as triples (x, w, u), $x, u \in R$, $w \in R^{n-2}$, so that for an isometry $h = (\varepsilon, T, H, C(t), z) \in I(\overline{M})$ we have

(17) $$h(x, w, u) = (\varepsilon x + T,\ Hw + C(x),\ -\varepsilon\langle\dot{C}(x),\ Hw + \frac{1}{2}C(x)\rangle + \varepsilon u + z),$$

$\langle\cdot,\cdot\rangle$ being the (possibly indefinite) inner product in R^{n-2} determined by $k_{\lambda\mu}$.

Going on to a further study of the homogeneous s.c.r. manifolds we start with the following auxiliary fact (see [3], Lemma 4):

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 2. Given real vector spaces V and W with a not necessarily definite inner product in W, let G be a Lie group transformations of $V \times W \times V$, each of which is of the form

$$(18)\quad (v_1, w, v_2) \longrightarrow (\varepsilon v_1 + T,\ Aw + C(v_1),\ P(v_1)w + \varepsilon v_2 + S(v_1))$$

for some linear isometry A of W, $|\varepsilon| = 1$, $T \in V$ and C^∞-mappings $C: V \longrightarrow W$, $P: V \longrightarrow L(W,V)$ and $S: V \longrightarrow V$. Then

(i) any open orbit U of G is diffeomorphic to the product $V \times U_o \times V$, U_o being an open subset of W on which a certain group G_o of affine isometries acts transitively;

(ii) any open orbit of G coincides with $V \times W \times V$ whenever the inner product in W is definite.

Using Lemma 2, we now proceed to prove

Theorem 3. Let (M,g) be an n-dimensional simply connected homogeneous s.c.r. manifold whose Weyl conformal curvature tensor is not of the form (4). Then M is diffeomorphic to $R^2 \times M_1$, where M_1 is a simply connected homogeneous flat pseudo-Riemannian manifold with a metric of index k, k = index g - 1.

Proof. By Theorem 2, M is the universal covering of an open homogeneous submanifold U of a universal model $\bar{M}$. Let G be a group of isometries acting on U transitively. G may be assumed to be connected (if necessary we take its connected component of the identity). We are now in the conditions of Lemma 2. In fact, we have the natural decomposition $\bar{M} = R \times R^{n-2} \times R$ with the inner product $\sum_{\lambda,\mu} k_{\lambda\mu} dx^\lambda dx^\mu$ in R^{n-2}, and, by (17) the transformations of G are of form (18). Therefore, by Lemma 2, U is diffeomorphic to $R^2 \times U_o$, U_o being a flat homogeneous manifold with a metric of index equal to k, k = index g-1. Hence M is diffeomorphic to $R^2 \times M_1$, M_1 being the universal covering of U_o. This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] T. Adati, T. Miyazawa: On a Riemannian space with recurrent conformal curvature, Tensor, New Series, 18 (1967), 348-354.

[2] M.C. Chaki, B. Gupta: On conformally symmetric spaces, Indian J. Math. 5 (1963) 113-122.

[3] A. Derdziński: On homogeneous conformally symmetric pseudo-Riemannian manifolds, Colloq. Math. 40 (1978) 167-186.

[4] A. Derdziński: On conformally symmetric Ricci-recurrent manifolds with Abelian fundamental groups, Tensor N.S. 34 (1980) 21-29.

[5] A. Gębarowski: On the existence of simple conformally recurrent non-Ricci-recurrent manifolds, Tensor, N.S. 43 (1986) 95-97.

[6] S. Kobayashi, K. Nomizu: Foundations of differential geometry, vol. I, Interscience, New York (1963).

[7] S. Kobayashi, K. Nomizu: Foundations of differential geometry, vol. II, Interscience, New York (1969).

[8] W. Roter: On the existence of conformally recurrent Ricci-recurrent spaces, Bulletin de l'Académie Polonaise des Sciences. Série des sciences mathématiques, astronomiques et physiques, 24 (1976) 937-979.

[9] W. Roter: On conformally recurrent Ricci-recurrent manifolds, Colloq Math. 46 (1982) 45-58.

[10] W. Roter: On conformally related conformally recurrent metrics I. Some general results, Colloq. Math. 47 (1982) 39-46.

[11] W. Roter: On a class of conformally recurrent manifolds, Tensor, New Series, 39 (1982).

MATHEMATICAL INSTITUTE, COLLEGE OF PEDAGOGICS,
35-950 RZESZÓW, POLAND

Received December 1st, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

James G. Raftery, Teo Sturm

COMPLETIONS OF UNIFORM UNIVERSAL ALGEBRAS AND STRONGLY UNIFORM BCK-ALGEBRAS

Let $\mathcal{U}$ be a uniformity on a universal algebra $(A; F)$. $\mathcal{A} = (A; F; \mathcal{U})$ is called a uniform algebra if each operation $f \in F$ is uniformly continuous on A (with respect to $\mathcal{U}$). It is shown that $\mathcal{A}$ is embeddable, as a dense subalgebra, in a complete uniform algebra of the same type as $\mathcal{A}$. If $(A; \mathcal{U})$ is Hausdorff, then $\mathcal{A}$ has a Hausdorff completion $\bar{\mathcal{A}}$, which is unique up to an isomorphism which fixes A pointwise, and the embedding $\mathcal{A} \to \bar{\mathcal{A}}$ may be described as a category reflection.

Special consideration is given to the case of (Hausdorff) uniform BCK-algebras and to conditions under which their (Hausdorff) completions are also BCK-algebras.

0. Introduction and preliminaries

We assume a familiarity with the theory of BCK-algebras (as general references, we recommend [2] and [3]). The binary operation on a BCK-algebra will be denoted by juxtaposition.

Let $(A; ., 0)$ be a BCK-algebra. We denote by Id(A) (resp. Con(A)) the complete lattice of all ideals (resp. all $\{.,0\}$-congruen-

AMS (MOS) 1980 Subject Classification: 03G25, 08A99, 98B99, 54E15, 54E52, 54H99.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ces) of A. If $I \in Id(A)$ and $\theta_I = \{(x,y) \in A \times A : x,y\ yx \in I\}$ then $\theta_I \in Con(A)$ and A/θ_I is a BCK-algebra (see [3, p.12]). Conversely if $\theta \in Con(A)$ and $\theta(0) = \{x \in A : (x,0) \in \theta\}$ then $\theta(0) \in Id(A)$. If A is a member of a $\{.,0\}$-variety of BCK-algebras then the mappings $I \longmapsto \theta_I$, $\theta \longmapsto \theta(0)$ are mutually inverse lattice isomorphisms between $Id(A)$ and $Con(A)$ [2,p.108]. It should be noted that the class of all BCK-algebras is not a $\{.,0\}$-variety [8] and that there exist BCK-algebras A for which $Id(A)$ and $Con(A)$ are not isomorphic [5, Remark 9].

Throughout this paper, R shall denote the classical linearly ordered group of real numbers and ω the set of all non-negative integers. 0 shall denote both the real zero and the zero element of a BCK-algebra. If S and T are sets then for each $i \in T$, π_i shall denote the i-th projection map $S^T \longrightarrow S$.

We assume a familiarity with the theory of uniform spaces. In this regard, our terminology accords with that of [4]. If $(S, \mathcal{U})$ is a uniform space and $U, V \in \mathcal{U}$, we define

$$U \circ V = \{(x,y) \in S \times S : \text{there exists } z \in S \text{ such that } (x,z) \in U \text{ and } (z,y) \in V\};$$

$$1U = U;\ (n+1)U = (nU) \circ U \quad (0 < n \in \omega).$$

Recall that every uniformity has a "symmetric base", i.e., a base consisting of sets U where $U = U^{-1} = \{(x,y) \in S \times S : (y,x) \in U\}$ [4, Theorem 6.6]. A uniform space $(S, \mathcal{U})$ is Hausdorff (i.e. the uniform topology on S induced by $\mathcal{U}$ is Hausdorff) if and only if $\cap \mathcal{U} = id_S = \{(x,x) : x \in S\}$, if and only if $\cap \mathcal{B} = id_S$ for any base $\mathcal{B}$ for $\mathcal{U}$.

Where products of uniform spaces arise, it will be assumed that they are endowed with the product uniformity (see [4, pp.180-184]). In particular, if (S,d) is a pseudo-metric space and $0 < m \in \omega$, then S^m will be considered to have been endowed with the so-called "max-" pseudo-metric, also denoted by d, i.e.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$d(x,y) = \max\{d(x(i),y(i)) : 1 \leq i \leq m\} \qquad (x,y) \in S^m).$$

The following notational convention will be adopted: Let $(S, \mathcal{U})$ be a uniform space with $D \in \mathcal{U}$. If $0 < m \in \omega$ and f is an m-ary operation on S, we define

$$f^{-1}(D) = \{(x,y) \in S^m \times S^m : (f(x),f(y)) \in D\}.$$

Let $t = (F; ar)$ be a "type" of algebras, i.e. F is a set of operation symbols and $ar: F \to \omega$ is the arity function. We define

$$F^+ = \{f \in F : ar(f) > 0\}$$

(i.e., F is the set of all non-constant operation symbols in F).
A structure $\mathcal{A} = (A; F; \mathcal{U})$ is called a uniform algebra (resp. a complete uniform algebra; a Hausdorff uniform algebra) of type t if:

(i) $(A;F)$ is a universal algebra of type t,

(ii) $(A;\mathcal{U})$ is a uniform space (resp. a complete uniform space; a Hausdorff uniform space) and

(iii) for each $f \in F$, the mapping $f_{\mathcal{A}} : A^{ar(f)} \to A$ is uniformly continuous in $\mathcal{U}$. (In the absence of any possible confusion we shall denote $f_{\mathcal{A}}$ by f).

In particular, if $\mathcal{U}$ is the (pseudo-) metric uniformity induced by a (pseudo-) metric d on A and $(A; F; \mathcal{U})$ is a uniform algebra of type t then $(A; F; \mathcal{U})$ will be called a (pseudo-) metric algebra of type t. Such algebras and their completions were studied in [6]. □

1. In this paper, we consider uniform algebras and their completions. While the results are quite general, they arose out of a consideration of uniformities on BCK-algebras: a uniform BCK-algebra $(A; \cdot, 0, \mathcal{U})$ will be called a strongly uniform BCK-algebra if it satisfies the following condition: for any Cauchy nets $(x_\gamma; \gamma \in \Gamma)$ and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$(y_\gamma ; \gamma \in \Gamma)$ in A such that the nets $(x_\gamma y_\gamma ; \gamma \in \Gamma)$ and $(y_\gamma x_\gamma ; \gamma \in \Gamma)$ each converge to 0, the net $((x_\gamma, y_\gamma) ; \gamma \in \Gamma)$ in $A \times A$ is eventually in every element of $\mathcal{U}$. Note that such an algebra also satisfies

$$(xy,0),\ (yx,0) \in \cap\, \mathcal{U} \implies (x,y) \in \cap\, \mathcal{U}$$

for all $x,y \in A$. We cite some examples.

1.1. A structure $(A; .,\ 0; n)$ is called a pseudo-normed BCK-algebra if $(A; .,\ 0)$ is a BCK-algebra and n is a pseudo-norm on A, that is, a real valued function on A satisfying $n(0) = 0$ and $n(x) \leqslant$ $\leqslant n(xy) + n(y)$ for all $x,y \in A$. We call n a norm (and $(A; .,\ 0; n)$ a normed BCK-algebra) if in addition, $x = 0$ whenever $n(x) = 0$. (Pseudo-) normed BCK-algebras were introduced and studied in [6]: in particular if A is interpreted as an algebra of sets and $\cdot$ as the operation of set difference then a pseudo-norm on A has some of the properties of a measure; on the other hand if A is an implicational calculus and xy denotes the propositional formula $y \longrightarrow x$ then pseudo-norms on A may be interpreted as "falsity-valuations". A (pseudo-) norm n on a BCK-algebra A defines a (pseudo-) metric d_n on A: we set $d_n(x,y) = n(xy) + n(yx)$, $x,y \in A$. In [6, Lemma 2] it was proved that the BCK-operation $\cdot : A^2 \longrightarrow A$ is a uniformly continuous mapping, so $(A; .,\ 0; d_n)$ is a (pseudo-) metric BCK-algebra. The set

$$\mathcal{B} = \left\{ \left\{ (x,y) \in A \times A : d_n(x,y) < \varepsilon \right\} : 0 < \varepsilon \in R \right\}$$

is a symmetric base for the (pseudo-) metric uniformity $\mathcal{U}$ on A induced by d_n. Now if $(x_\gamma ; \gamma \in \Gamma)$, $(y_\gamma ; \gamma \in \Gamma)$ are Cauchy nets in A and the nets $(x_\gamma y_\gamma ; \gamma \in \Gamma)$ converge to 0 then for $0 < \varepsilon \in R$, and sufficiently large $\gamma \in \Gamma$ we have

$$d_n(x_\gamma y_\gamma, 0),\ d_n(y_\gamma x_\gamma, 0) < \varepsilon/2,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

hence $$n(x_\gamma y_\gamma) + n(y_\gamma x_\gamma) < \varepsilon ,$$

i.e. $$d_n(x_\gamma y_\gamma) < \varepsilon .$$

It follows that $(A; ., 0; \mathcal{U})$ is a strongly uniform BCK-algebra. □

1.2. In [1], Aló and Deeba showed that if $(A; ., 0)$ is a BCK-algebra then $\mathcal{J}_A = \{\theta_I : I \in Id(A)\}$ is a base for a uniformity on A. Actually $\mathcal{J}_A$ is a base for the discrete uniformity on A, since $id_A = \theta_{\{0\}} \in \mathcal{J}_A$. However, nontrivial uniformities on A may be obtained in the same way if we replace $\mathcal{J}_A$ by any non-empty subset $\mathcal{B}$ of $Con(A)$ which is closed under finite intersections. We prove a more general proposition. (First recall that a universal algebra $(A; F)$ (of given type) is said to be subdirectly reducible if $\bigcap Con_F(A) \setminus \{id_A\} = id_A$, where $Con_F(A)$ is the complete lattice of all F-congruences on A, in which case $(A; F)$ may be decomposed nontrivially as a subdirect product of algebras of the same type. It can be shown that a BCK-algebra $(A; ., 0)$ is subdirectly reducible (as an algebra of type (2,0)) if and only if $\bigcap Id(A) \setminus \{\{0\}\} = \{0\}$). □

1.3. Proposition. (i) Let $(A; F)$ be an algebra of type t and let $\mathcal{B}$ be a subset of $Con_F(A)$ which is closed under finite intersections. Then $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A and $(A; F; \mathcal{U})$ is a uniform algebra of type t. If $(A; \mathcal{U})$ is Hausdorff but $\mathcal{U}$ is not discrete then $(A; F)$ is subdirectly reducible.

(ii) Let $(A; ., 0)$ be a BCK-algebra and $\mathcal{B}$ a non-empty subset of $\mathcal{J}_A$ which is closed under finite intersections. Then $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A and $(A; ., 0, \mathcal{U})$ is a strongly uniform BCK-algebra.

Proof. (i) Each element θ of $\mathcal{B}$, being an equivalence relation on A, satisfies $id_A \subseteq \theta^{-1} = \theta = 2\theta$, where $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on A. If $f \in F^+$ with $ar(f) = m$, then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\bar{\theta} = \bigcap \left\{ \pi_i^{-1}(\theta) : i=1,\ldots,m \right\}$$

is an element of the product uniformity on A^m, and since θ is a congruence relation, we have, for $x,y \in A^m$,

$$(x,y) \in \bar{\theta} \Longrightarrow (f(x),f(y)) \in \theta$$

whence f is uniformly continuous. If the uniformity on A is not discrete then $\bigcap \mathrm{Con}_F(A) \setminus \left\{ \mathrm{id}_A \right\} \subseteq \mathscr{B} = \bigcap \mathcal{U}$ and the last assertion follows.

(ii) It follows from (i) that $\mathscr{B}$ is a base for a uniformity $\mathcal{U}$ on A such that $(A; .,\ 0, \mathcal{U})$ is a uniform BCK-algebra. Now let $(x_\gamma; \gamma \in \Gamma)$ and $(y_\gamma; \gamma \in \Gamma)$ be Cauchy nets in A with $(x_\gamma y_\gamma; \gamma \in \Gamma)$ and $(y_\gamma x_\gamma; \gamma \in \Gamma)$ each converging to 0. Let $\theta_I \in \mathscr{B}$. For sufficiently large γ we have $(x_\gamma y_\gamma, 0)$, $(y_\gamma x_\gamma, 0) \in \theta_I$, whence $(x_\gamma, y_\gamma) \in \theta_I$, (since A/θ_I is a BCK-algebra). We deduce that $(A; .,\ 0; \mathcal{U})$ is a strongly uniform BCK-algebra. □

1.4. A complete Hausdorff uniform BCK-algebra $(A; .,\ 0; \mathcal{U})$ is strongly uniform. Indeed if $(x_\gamma; \gamma \in \Gamma)$, $(y_\gamma; \gamma \in \Gamma)$ are Cauchy nets in A, then they converge, say to x and y respectively. Since $\cdot$ is uniformly continuous, the nets $(x_\gamma y_\gamma; \gamma \in \Gamma)$, $(y_\gamma x_\gamma; \gamma \in \Gamma)$ converge to xy and yx respectively. If these latter nets also converge to 0, then since $(A; \mathcal{U})$ is Hausdorff, we have $xy = 0 = yx$, hence $x = y$. Therefore the net $((x_\gamma, y_\gamma); \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}$. □

Another class of strongly uniform BCK-algebras will be identified in Corollary 14. In section 15 we shall present examples which show that neither a complete uniform BCK-algebra nor a Hausdorff uniform BCK-algebra need be strongly uniform, and that a Hausdorff strongly uniform BCK-algebra need not be complete.

UA shall denote the category of all uniform algebras of a given type t, where for $\mathscr{A} = (A; F; \mathcal{U})$ and $\mathscr{B} = (B; F; \mathcal{V})$ in UA, a map $h: \mathscr{A} \to \mathscr{B}$ is a UA-morphism if and only if

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $h: (A; F) \longrightarrow (B; F)$ is a homomorphism of algebras, and

(ii) $h: (A; \mathcal{U}) \longrightarrow (B; \mathcal{V})$ is a uniformly continuous function.

We shall also consider the following full subcategories of UA:

HA, whose objects are the Hausdorff uniform algebras of type t;

CA, whose objects are the complete uniform algebras of type t;

CHA, whose objects are the complete Hausdorff uniform algebras of type t.

2. Lemma. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Then $\cap\mathcal{U}$ is a congruence relation on $(A; F)$. If $A^* = A/\cap\mathcal{U}$, and $\mathcal{U}^* = \{U^*: U \in \mathcal{U}\}$ where $U^* = \{(X,Y) \in A^* \times A^*: (x,y) \in U \text{ for all } x \in X, y \in Y\}$ then the canonical surjection $\lambda = \lambda_{\mathcal{A}} : A \longrightarrow A^*$ is an HA-reflection of $\mathcal{A}$ onto $\mathcal{A}^* = (A^*; F; \mathcal{U}^*)$.

Proof. Let $\mathcal{D}$ be a symmetric base for $\mathcal{U}$. Then $\cap\mathcal{D} = \cap\mathcal{U}$. For each $D \in \mathcal{D}$, we have $id_A \subseteq D = D^{-1}$ so $\cap\mathcal{U}$ if reflexive and symmetric. Transitivity of $\cap\mathcal{U}$ follows from the fact that each $D \in \mathcal{D}$ contains $2E$ for some $E \in \mathcal{D}$, while the uniform continuity of each $f \in F$ ensures that $\cap\mathcal{U}$ is an F-congruence relation on A. There is no difficulty in checking that $\mathcal{A}^*$ is also a uniform algebra of type t and since $\cap\mathcal{U}^* = id_{A^*}$, we have $\mathcal{A}^* \in HA$. Clearly $\lambda \in Hom_{UA}(\mathcal{A}, \mathcal{A}^*)$.

If $\mathcal{B} = (B; F; \mathcal{V}) \in HA$ and $\Phi \in Hom_{UA}(\mathcal{A}, \mathcal{B})$, define $\Phi^*: A^* \longrightarrow B$ by $\Phi^*(X) = \Phi(x)$, where $x \in X \in A^*$. Φ^* is properly defined since Φ is uniformly continuous and $(B, \mathcal{V})$ is Hausdorff. Since Φ is a UA-morphism, it follows that Φ^* is an HA-morphism, unique such that $\Phi^*\lambda = \Phi$. Thus λ is an HA-reflection. □

Note that in case $\mathcal{A}$ is a pseudo-metric algebra, $\mathcal{A}^*$ is the metric identification of $\mathcal{A}$ and the above result specializes to [6, Lemma 3].

The following result is an algebraic variant of the "Metrization Lemma".

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Lemma. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Let $(D_n; n \in \omega)$ be a sequence of elements of $\mathcal{U}$ such that:

(1) $D_0 = A \times A$,

(2) $3D_{n+1} \subseteq D_n$ and

(3) $\cap \left\{ \pi_i^{-1}(D_{n+1}) : i=1,\dots,m \right\} \subseteq f^{-1}(D_n)$

for all $n \in \omega$ and all $f \in F^+$ with $ar(f) = m$. Then there exists a pseudo-metric d on A such that

(i) $D_n \subseteq \left\{ (x,y) \in A \times A : d(x,y) < 2^{-n} \right\} \subseteq D_{n-1}$ for $0 < n \in \omega$

(ii) $(A; F; d)$ is a pseudo-metric algebra of type t.

Proof. The conditions of the Metrization Lemma [4, Lemma 6.12] hold so there exists a pseudo-metric d on A satisfying (i). Let $f \in F^+$ with $ar(f) = m$ and let $0 < \varepsilon \in R$ be given. Choose $n \in \omega$ such that $2^{-n} < \varepsilon$. Let $\delta = 2^{-n-2}$. For $x, y \in A^m$, we have

$$\begin{aligned} d(x,y) < \delta &\Longrightarrow d(x(i),y(i)) < \delta && (i=1,\dots,m) \\ &\Longrightarrow (x(i),y(i)) \in D_{n+1} && (i=1,\dots,m) \\ &\Longrightarrow (f(x),f(y)) \in D_n \\ &\Longrightarrow d(f(x),f(y)) < 2^{-n} < \varepsilon \end{aligned}$$

so f is uniformly continuous in d, and (ii) follows. □

If $(A, \mathcal{U})$ is a uniform space and Q is a family of pseudo-metrics on A, we define $V(d,\varepsilon) \subseteq A \times A$ by

$$V(d,\varepsilon) = \left\{ (x,y) \in A \times A : d(x,y) < \varepsilon \right\}$$

for $d \in Q$ and $0 < \varepsilon \in R$. The set

$$\left\{ V(d,\varepsilon) : d \in Q,\ 0 < \varepsilon \in R \right\}$$

is a subbase for a uniformity $\mathcal{U}(Q)$ on A. If $\mathcal{U} = \mathcal{U}(Q)$, we say that $\mathcal{U}$ is generated by Q.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Corollary. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t and suppose F^+ is finite. Then $\mathcal{U}$ is generated by the family Q of all uniformly continuous pseudo-metrics d such that $f_d: (A^{ar(f)}, d) \longrightarrow (A,d)$ is uniformly continuous for each $f \in F$.

Proof. If $d \in Q$ and $0 < \varepsilon \in R$ then $V(d,\varepsilon)$ contains an element of $\mathcal{U}$ by [4, Theorem 6.11, p.183]. On the other hand if $D \in \mathcal{U}$, define

$$D_0 = A \times A, \quad D_1 = D$$

and choose $E \in \mathcal{U}$ such that $3E \subseteq D$. Noting that for each $f \in F^+$ (with $ar(f) = m$), $f^{-1}(D)$ is a member of the product uniformity on A^m, we may choose $G_{f,1} \ldots G_{f,m} \in \mathcal{U}$ such that

$$\cap \left\{ \pi_i^{-1}(G_{f,i}) : i = 1, \ldots, m \right\} \subseteq f^{-1}(D).$$

Let $D_2 = E \cap (\cap \left\{ G_{f,i} : i = 1, \ldots, ar(f);\ f \in F^+ \right\})$. Then for each $f \in F^+$,

$$\cap \left\{ \pi_i^{-1}(D_2) : i = 1, \ldots, ar(f) \right\} \subseteq f^{-1}(D), \text{ and } 3D_2 \subseteq 3E \subseteq D.$$

Continuing in this fashion, we obtain a sequence $(D_n;\ n \in \omega)$ satisfying the conditions of Lemma 3. It follows from Lemma 3 that there exists $d \in Q$ such that $V(d, \frac{1}{4}) \subseteq D$, hence the result. □

If $\mathcal{A} = (A; F; \mathcal{U})$ is a uniform algebra of a given type and $(B; F)$ is a subalgebra of $(A; F)$, we call $\mathcal{B} = (B; F; \mathcal{U}_B)$ a uniform subalgebra od $\mathcal{A}$, where

$$\mathcal{U}_B = \left\{ U \cap (B \times B) : U \in \mathcal{U} \right\}.$$

5. Theorem. Let $\mathcal{A} = (A; F; \mathcal{U})$ be a uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of pseudo-metric algebras of type t. If $\mathcal{A}$ is Hausdorff and F^+ is finite then $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of metric algebras of type t.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let Q be as in Corollary 4. For each $d \in Q$, set $\mathcal{A}_d = (A; F; d)$ and let $\lambda_d : \mathcal{A}_d \to \mathcal{A}_d^*$ be the metric identification of $\mathcal{A}_d$. Let $\mathcal{A}' = \Pi(\mathcal{A}_d; d \in Q)$, $\mathcal{A}'' = \Pi(\mathcal{A}_d^*; d \in Q)$. Define $\Phi : \mathcal{A} \to \mathcal{A}'$ by

$$\pi_d \phi(x) = x \quad (x \in A, d \in Q).$$

ϕ is clearly a one-to-one F-homomorphism and since $\mathcal{U}$ contains each $V(d, \varepsilon)$, $\mathcal{U}$ is the smallest uniformity on A making each $\pi_d \phi$ (and hence ϕ itself) uniformly continuous. Now suppose $\mathcal{A}$ is Hausdorff and F^+ is finite. Then the map $\psi : \mathcal{A} \to \mathcal{A}''$ defined by

$$\pi_d \psi(x) = \lambda_d(x) \quad (x \in A, d \in Q)$$

is one-to-one (since by Corollary 4, any two distinct points $x, y \in A$ must satisfy $d(x,y) \neq 0$ for some $d \in Q$), and an F-homomorphism. Since each λ_d is uniformly continuous, so is ψ. □

It is not clear whether the restriction on F^+ in the latter assertion of Theorem 5 may be dropped. We consider Theorem 5 to be of interest in its own right: in particular it implies that any uniform BCK-algebra is uniformly BCK-embeddable in a product of pseudo-metric BCK-algebras. As a case of special interest we may consider the strongly uniform BCK-algebras whose uniformities are induced by sets of ideals. Let A, $\mathcal{O}$, $\mathcal{U}$ and $\mathcal{A} = (A; .\,, 0; \mathcal{U})$ be as in Proposition 1.3(ii) and let

$$Q = \left\{ d_n : n \text{ is a pseudo-norm on A and } d_n : A \times A \to R \text{ is uniformly continuous} \right\}.$$

We may show that $\mathcal{U}$ is generated by Q. If $d_n \in Q$ then

$$J = J_n = \left\{ x \in A : n(x) = 0 \right\}$$

is an ideal of A [6, Lemma 1] and it is easy to see that $\theta_J \subseteq V(d_n, \varepsilon)$ for each real $\varepsilon > 0$. Conversely, given $\theta_J \in \mathcal{O}$, define $n : A \to R$ by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$n(x) = \begin{cases} 0 & \text{if } x \in J \\ 1 & \text{otherwise.} \end{cases}$$

By [6, Lemma 1], n is a pseudo-norm on A and

$$d_n(x,y) < 1 \iff n(xy) = n(yx) = 0$$
$$\iff xy,\ yx \in J$$

so $V(d_n,1) = \theta_J$. Hence $d_n \in Q$ and $\mathcal{U}$ is generated by Q. It follows in the spirit of Theorem 5 that $\mathcal{A}$ is uniformly embeddable in $\Pi((A; .,\ 0;\ d_n) : d_n \in Q)$. Furthermore the metric identification $(A_n^*; .,\ 0,\ d_n^*)$ of a pseudo-normed BCK-algebra $(A; .,\ 0,\ d_n)$ is a normed BCK-algebra [6, Theorem 10(ii)] so if $(A; \mathcal{U})$ is Hausdorff (i.e. $\cap \mathcal{O} = id_A$), the method of Theorem 5 may be applied to prove that $\mathcal{A}$ is uniformly embeddable in $\Pi((A_n^*; .,\ 0;\ d_n^*) : d_n \in Q)$. In summary: a uniform (resp. Hausdorff uniform) BCK-algebra whose uniformity is induced by a set of ideals is isomorphic to a uniform subalgebra of a product of pseudo-normed (resp. normed) BCK-algebras.

At the same time the purely topological counterpart of Theorem 5 [4, Theorem 6.16, p.188] plays a key role in the construction of completions of uniform spaces. Indeed, if $(A; \mathcal{U})$ is a uniform space embedded in a product of (pseudo-) metric spaces $(A_i;d_i)$ then the closure of $(A; \mathcal{U})$ in a product of (pseudo-) metric completions of the $(A_i;d_i)$ is a uniform completion of $(A; \mathcal{U})$ [4, pp.196-197]. Naturally we would like to be able to embed Hausdorff uniform algebras $(A; F; \mathcal{U})$ into complete Hausdorff uniform algebras in a similar manner without imposing restrictions on the cardinality of F^+, but Theorem 5 will not lead to such a general result. Fortunately we may still obtain a general embedding theorem if we first consider a (purely topological) completion $(\bar{A}; \bar{\mathcal{U}})$ of the underlying uniform space $(A; \mathcal{U})$ and extend each operation f to a uniformly continuous operation on $\bar{A}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Recall that the closure $\tilde{B}$ of a subset B of a uniform space $(A,\mathcal{U})$ consists of all points $x \in A$ to which some Cauchy net $(x_\gamma : \gamma \in \Gamma)$ in B converges [4, Theorem 6.21]. If $A = \tilde{B}$ then B is called a dense subset of $(A, \mathcal{U})$.

6. Lemma. Let $\mathcal{B} = (B;F;\mathcal{U}_B)$ be a uniform subalgebra $\mathcal{A} = (A;F;\mathcal{U})$. Let $f \in F$ with $ar(f) = m$ and suppose that for $i = 1,\dots,m$ there exists a (Cauchy) net $(x_\alpha; \alpha \in \Gamma_i)$ in B which converges to $x_i \in A$. Then there exists a (Cauchy) net $(z_\gamma; \gamma \in \Gamma)$ in B which converges to $f_{\mathcal{A}}(x_1,\dots,x_m)$.

Proof. Let $\Gamma = \prod(\Gamma_i : i=1,\dots,m)$ and define, for $\gamma, \gamma' \in \Gamma$,

$$\gamma < \gamma' \text{ iff } \gamma(i) < \gamma'(i) \quad \text{for} \quad i=1,\dots,m.$$

Then Γ is a directed set. For $\gamma \in \Gamma$, define

$$x_\gamma = (x_{\gamma(1)},\dots,x_{\gamma(m)}); \quad z_\gamma = f_{\mathcal{B}}(x_\gamma); \quad x = (x_1,\dots,x_m).$$

Let $D \in \mathcal{U}$ and choose $E \in \mathcal{U}$ such that $\bigcap \left\{ \pi_i^{-1}(E) : i=1,\dots,m \right\} \subseteq$ $\subseteq f_{\mathcal{A}}^{-1}(D)$. For $i = 1,\dots,m$, choose $\alpha_i \in \Gamma_i$ such that $\alpha_i < \alpha \in \Gamma_i \Rightarrow$ $\Rightarrow (x_\alpha, x_i) \in E$. Let $\alpha = (\alpha_1,\dots,\alpha_m)$. Then

$$\alpha < \gamma \in \Gamma \Longrightarrow (x_\gamma, x) \in \bigcap \left\{ \pi_i^{-1}(E); \ i=1,\dots,m \right\}$$
$$\Longrightarrow (z_\gamma, f_{\mathcal{A}}(x)) \in D,$$

and it follows that the net $(z_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{A}}(x)$. □

7. Corollary. Let $\mathcal{B} = (B;F;\mathcal{U}_B)$ be a uniform subalgebra of $\mathcal{A} = (A;F;\mathcal{U})$ and let $\mathcal{C} = (C;F;\mathcal{V})$ be a Hausdorff uniform algebra of the same type. Let $\phi : (\tilde{B};\mathcal{U}_{\tilde{B}}) \longrightarrow (C;\mathcal{V})$ be a uniformly continuous function.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $\tilde{B} = (B;F;\mathcal{U}_{\tilde{B}})$ is a uniform subalgebra of $\mathcal{A}$.

(ii) If $\phi|_B$ is an F-homomorphism then so is ϕ.

(iii) If $\mathcal{C}$ is complete then every UA-morphism form $\mathcal{B}$ to $\mathcal{C}$ may be extended uniquely to a UA-morphism from $\tilde{\mathcal{B}}$ to $\mathcal{C}$.

($\tilde{\mathcal{B}}$ will be called the closure of $\mathcal{B}$ in $\mathcal{A}$).

Proof. (i) follows immediately from the previous lemma.

(ii) Let $f \in F$ with $ar(f) = m$ and let $x = (x_1,\ldots,x_m) \in \tilde{\mathcal{B}}^m$. For $i = 1,\ldots,m$, choose a Cauchy net $(x_\alpha; \alpha \in \Gamma_i)$ in B converging to x_i. Since ϕ is uniformly continuous, the net $(\phi(x_\alpha); \alpha \in \Gamma_i)$ converges to $\phi(x_i)$ for each i. Let Γ be the directed set $\prod \Gamma_i$. For $\gamma \in \Gamma$, define x_γ and z_γ as in Lemma 6, and also

$$y_\gamma = (\phi(x_{\gamma(1)}),\ldots,\phi(x_{\gamma(m)})); \quad w_\gamma = f_{\mathcal{C}}(y_\gamma).$$

By the proof of Lemma 6, $(z_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{A}}(x)$ and $(w_\gamma; \gamma \in \Gamma)$ converges to $f_{\mathcal{C}}(\phi(x_1),\ldots,\phi(x_m))$. Since ϕ is uniformly continuous, the net $(\phi(z_\gamma); \gamma \in \Gamma)$ must converge to $\phi(f_{\mathcal{A}}(x))$. Now since $\phi|_B$ is an F-homomorphism, we have

$$\phi(z_\gamma) = \phi(f_{\mathcal{B}}(x_\gamma)) = f_{\mathcal{C}}(y_\gamma) = w_\gamma$$

($\gamma \in \Gamma$). Since $\mathcal{C}$ is Hausdorff, it follows that

$$\phi(f_{\mathcal{A}}(x)) = f_{\mathcal{C}}(\phi(x_1),\ldots,\phi(x_m))$$

and so ϕ is an F-homomorphism.

(iii) Let $\mathcal{C}$ be complete and let $\varrho : \mathcal{B} \longrightarrow \mathcal{C}$ be a UA-morphism. By [7, Theorem 39.10], $\varrho : (B; \mathcal{U}_B) \longrightarrow (C; \Upsilon)$ may be extended uniquely to a uniformly continuous function $\tilde{\varrho} : (\tilde{B}; \mathcal{U}_{\tilde{B}}) \longrightarrow (C; \Upsilon)$ and $\tilde{\varrho}$ is a UA-morphism by (ii). □

8. Theorem. Let $\mathcal{A}$ be a uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a dense subalgebra of a complete uniform algebra $\tilde{\mathcal{A}}$ of type t.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. By Theorem 5, $\mathcal{A}$ is isomorphic to a uniform subalgebra of a product of pseudo-metric algebras of type t, each of which is isomorphic to a uniform subalgebra of a complete pseudo-metric algebra of type t [6, Lemma 4]. The product $\mathcal{C}$ of these complete algebras is a complete uniform algebra of type t and contains the closure $\hat{\mathcal{A}}$ of the image of $\mathcal{A}$ as a complete uniform subalgebra (by Corollary 7(i)). □

9. Theorem. Let $\mathcal{A}$ be a Hausdorff uniform algebra of type t. Then $\mathcal{A}$ is isomorphic to a dense subalgebra of a complete Hausdorff uniform algebra $\bar{\mathcal{A}}$ which is unique in the sense that if $\mathcal{B}$ is any complete Hausdorff uniform algebra containing $\mathcal{A}$ as a dense uniform subalgebra then the identity map on $\mathcal{A}$ extends to a unique isomorphism from $\bar{\mathcal{A}}$ to $\mathcal{B}$. The embedding $\mathcal{A} \rightarrowtail \bar{\mathcal{A}}$ is a simple CHA-reflection of $\mathcal{A}$.

Proof. Let $\mathcal{A} = (A;F;\mathcal{U})$. By [4, Theorem 6.28], $(A;\mathcal{U})$ is uniformly isomorphic to a dense subspace of a complete Hausdorff uniform space $(\bar{A};\bar{\mathcal{U}})$. For each $f \in F$ (with $ar(f) = m$, say), $f_{\mathcal{A}}$ is a uniformly continuous function from A^m to A (with respect to $\mathcal{U}$). Since $(\bar{A}^m;\bar{\mathcal{U}})$ is a complete Hausdorff uniform space containing A^m as a dense subspace, it follows from [7, Theorem 39.10] that $f_{\mathcal{A}}$ may be extended uniquely to a uniformly continuous function $f_{\bar{\mathcal{A}}} : \bar{A}^m \rightarrow \bar{A}$, so we obtain a CHA-object $\bar{\mathcal{A}} = (\bar{A};F;\bar{\mathcal{U}})$ and an HA-embedding $\mathcal{A} \rightarrowtail \bar{\mathcal{A}}$. If $\mathcal{B} \in$ CHA and $\phi : \mathcal{A} \rightarrowtail \mathcal{B}$ is an HA-morphism then by Corollary 7, ϕ may be extended to a unique CHA-morphism $\psi : \bar{\mathcal{A}} \rightarrow \mathcal{B}$, so $\mathcal{A} \rightarrowtail \bar{\mathcal{A}}$ is a simple CHA-reflection. In particular, if $\mathcal{A}$ is a dense subalgebra of $\mathcal{B}$ then id_A extends to CHA-morphism $\psi : \bar{\mathcal{A}} \rightarrow \mathcal{B}$ and $\varrho : \mathcal{B} \rightarrowtail \bar{\mathcal{A}}$. Since $\varrho\psi|_{\mathcal{A}} = id_{\mathcal{A}} = id_{\bar{\mathcal{A}}}|_{\mathcal{A}}$, it follows from Corollary 7 than $\varrho\psi = id_{\bar{\mathcal{A}}}$ and similarly $\psi\varrho = id_{\mathcal{B}}$. Thus ψ is an isomorphism. □

If $\mathcal{M}$ is a class of universal algebras of type t, then a uniform algebra $\mathcal{A} = (A;F;\mathcal{U})$ will be called a uniform $\mathcal{M}$-algebra if $(A;F) \in \mathcal{M}$. Bearing in mind the construction of the (Hausdorff) uniform completion

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

of a (Hausdorff) uniform space described in Theorem 8 and the remarks following Theorem 5, the following theorem is an obvious consequence of Lemma 2 and Theorems 8 and 9. The symbols $\mathcal{A}^*$, $\hat{\mathcal{A}}$ and $\bar{\mathcal{A}}$ are used in the sense of the statements and/or proofs of these three results.

10. Theorem. Let $\mathcal{U}$ be a class of universal algebras of type t and let $\mathcal{A}$ be a uniform $\mathcal{U}$-algebra.

(i) If $\mathcal{U}$ is closed under the formation of homomorphic images then $\mathcal{A}^*$ is a Hausdorff uniform $\mathcal{U}$-algebra.

(ii) If $\mathcal{U}$ is closed under the formation of direct products, isomorphic images and subalgebras then $\hat{\mathcal{A}}$ is a complete uniform $\mathcal{U}$-algebra.

(iii) If $\mathcal{U}$ is closed under the formation of direct products, homomorphic images and subalgebras and $\mathcal{A}$ is Hausdorff then $\bar{\mathcal{A}}$ is a complete Hausdorff uniform $\mathcal{U}$-algebra. □

We shall now consider classes of BCK-algebras and obtain a BCK-analogue of Theorem 10. Having already noted that the $\{\cdot,0\}$-quasi-variety of all BCK-algebras is not a variety, we should not expect the Hausdorff completion of every Hausdorff uniform BCK-algebra to be a BCK-algebra. The situation is better in the case of strongly uniform Hausdorff BCK-algebras.

11. Lemma. Let $\mathcal{B} = (B;\cdot,0;\mathcal{U}_B)$ be a strongly uniform BCK-algebra and a uniform subalgebra of the Hausdorff uniform algebra $\mathcal{A} = (A;\cdot,0;\mathcal{U})$ of type $(2,0)$. Then the closure $\tilde{\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{A}$ satisfies the quasi-identity

$$xy = 0 = yx \Longrightarrow x = y.$$

Proof. Let $x,y \in \tilde{B}$ with $xy = 0 = yx$. Choose nets $(x_\alpha;\ \alpha \in \Gamma_1)$ and $(y_\beta;\ \beta \in \Gamma_2)$ in B, converging to x and y respectively. Let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\Gamma = \Gamma_1 \times \Gamma_2$ and define $(\alpha,\beta) < (\alpha',\beta')$ iff $\alpha < \alpha'$ and $\beta < \beta'$. Define $x_{\alpha,\beta} = x_\alpha$, $y_{\alpha,\beta} = y_\beta$ $(\alpha \in \Gamma_1, \beta \in \Gamma_2)$. Then the nets $(x_\gamma; \gamma \in \Gamma)$ and $(y_\gamma; \gamma \in \Gamma)$ converge to x and y respectively, and by the proof of Lemma 6, the nets $(x_\gamma y_\gamma; \gamma \in \Gamma)$ and $(y_\gamma x_\gamma; \gamma \in \Gamma)$ each converge to 0. Since $\mathcal{B}$ is strongly uniform, the net $((x_\gamma, y_\gamma); \gamma \in \Gamma)$ is eventually in every element of $\mathcal{U}_B$ (hence in every element of $\mathcal{U}_{\tilde{B}}$). For $D \in \mathcal{U}_{\tilde{B}}$ choose $E \in \mathcal{U}_{\tilde{B}}$ such that $3E \subseteq D$, and $\gamma' \in \Gamma$ such that for $\gamma' < \gamma \in \Gamma$,

$$(x,x_\gamma),\ (x_\gamma,y_\gamma),\ (y_\gamma,y) \in E.$$

Then $(x,y) \in 3E \subseteq D$, and so $(x,y) \in \bigcap \mathcal{U}_{\tilde{B}} = \mathrm{id}_{\tilde{B}}$, i.e., $x = y$. □

We omit a routine proof of the following lemma.

12. Lemma. If UA is the category of all uniform algebras of type (2,0) then the class of all strongly uniform BCK-algebras is closed under the formation of UA-direct products, UA-subalgebras and UA-isomorphic images. □

13. Theorem. Let $\mathcal{A} = (A;.,0;\mathcal{U})$ be a uniform BCK-algebra.

(i) $\hat{\mathcal{A}}$ is a complete uniform BCK-algebra. $\hat{\mathcal{A}}$ is strongly uniform if and only if $\mathcal{A}$ is.

(ii) If $\mathcal{A}$ is a strongly uniform then $\mathcal{A}^*$ is a Hausdorff strongly uniform BCK-algebra.

(iii) If $\mathcal{A}$ is Hausdorff then the following conditions are equivalent:

(1) $\mathcal{A}$ is strongly uniform;

(2) $\bar{\mathcal{A}}$ is a uniform BCK-algebra;

(3) $\bar{\mathcal{A}}$ is a complete Hausdorff strongly uniform BCK-algebra.

Proof. (i) The class of all BCK-algebras is closed under direct products, subalgebras and isomorphic images, so the first assertion follows from Theorem 10(ii). The second assertion follows from Lemma 12.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Let $(X_\gamma;\ \gamma\in\Gamma)$ and $(Y_\gamma;\ \gamma\in\Gamma)$ be Cauchy nets in $\mathcal{A}^*$ such that $(X_\gamma Y_\gamma;\ \gamma\in\Gamma)$ and $(Y_\gamma X_\gamma;\ \gamma\in\Gamma)$ each converge to $0_{\mathcal{A}^*}$ in $\mathcal{U}^*$. Choose $x_\gamma\in X_\gamma$ and $y_\gamma\in Y_\gamma$ for each $\gamma\in\Gamma$. Then $(x_\gamma y_\gamma;\ \gamma\in\Gamma)$ and $(y_\gamma x_\gamma;\ \gamma\in\Gamma)$ each converges to 0 in $\mathcal{U}$. Since $\mathcal{A}$ is strongly uniform, the net $((x_\gamma, y_\gamma);\ \gamma\in\Gamma)$ is eventually in every element of $\mathcal{U}$, whence the net $((X_\gamma, Y_\gamma);\ \gamma\in\Gamma)$ is eventually in every element of $\mathcal{U}^*$. In particular, for $X,Y\in\mathcal{A}^*$, we have

$$XY = 0_{\mathcal{A}^*} = YX \Longrightarrow (X,Y)\in\cap\,\mathcal{U}^* \Longrightarrow X = Y.$$

Hence $\mathcal{A}^*$ is a strongly uniform BCK-algebra. $\mathcal{A}^*$ is Hausdorff by Lemma 2.

(iii) (1) $\Longrightarrow$ (2). A product of homomorphic images of BCK-algebras is a universal algebra of type (2,0) satisfying the four defining $\{.,0\}$-identities of a BCK-algebra. $\bar{\mathcal{A}}$ is the closure in such an algebra of an isomorphic image of $\mathcal{A}$ and so, by Lemma 11, satisfies the quasi-identity

$$xy = 0 = yx \Longrightarrow x = y$$

also. It follows that $\bar{\mathcal{A}}$ is a uniform BCK-algebra.
(2) $\Longrightarrow$ (3). Let $\bar{\mathcal{A}}$ be a uniform BCK-algebra. $\bar{\mathcal{A}}$ is also complete and Hausdorff by Theorem 9. Therefore $\bar{\mathcal{A}}$ is strongly uniform by 1.4.
(3) $\Longrightarrow$ (1) follows from Lemma 12. □

14. Corollary. Let $\mathcal{A} = (A;.,0;\mathcal{U})$ be a Hausdorff uniform BCK-algebra and suppose $(A;.,0)$ is a member of some $\{.,0\}$-variety of BCK-algebras. Then $\mathcal{A}$ is a strongly uniform BCK-algebra.

Proof. By Theorem 10(iii) $\bar{\mathcal{A}}$ is a (complete Hausdorff) uniform BCK-algebra. Now by Theorem 13(iii), $\mathcal{A}$ is strongly uniform. □

In the next section we show, inter alia, that we may not drop from the above corollary the condition that $(A;.,0)$ is a member of a BCK-variety.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

15. Examples

15.1. We exhibit a complete uniform BCK-algebra which is not strongly uniform. (Hence, in 1.4, the condition that $(A;.,0;\mathcal{U})$ be Hausdorff may not be dropped). We exploit Wroński's example from [8]. Let B and C be countably infinite sets such that the sets B, C and ω are mutually disjoint. Order B and C into one-to-one sequences $(b_i;\ i\in\omega)$ and $(c_i;\ i\in\omega)$ respectively. For $i,j\in\omega$, define

$$ij = i \dot{-} j = \max\{i \dot{-} j, 0\}$$

$$ib_j = i_{cj} = 0$$

$$b_i j = b_{i+j},\ c_i j = c_{i+j}$$

$$b_i b_j = c_i c_j = j \dot{-} i$$

$$b_i c_j = c_i b_j = (j+1) \dot{-} i.$$

Let $A = B\cup C\cup\omega$. By [8], $(A;\ .,\ 0)$ is a BCK-algebra and $\theta = B^2\cup C^2\cup\omega^2\in \mathrm{Con}(A)$. By Proposition 1.3, $\{\theta\}$ is a base for a uniformity $\mathcal{U}$ on A and $\mathcal{A} = (A;.,0.,\mathcal{U})$ us a uniform BCK-algebra. ($\mathcal{A}$ is not Hausdorff since $\cap\mathcal{U} = \theta$). Now if $(x_\gamma;\ \gamma\in\Gamma)$ is a Cauchy net in $(A;\mathcal{U})$ then there exists $\gamma\in\Gamma$ such that

$$\gamma<\alpha,\beta\in\Gamma \Longrightarrow (x_\alpha,x_\beta)\in\theta .$$

Hence there exists $X\in\{B,C,\omega\}$ such that

$$\gamma<\alpha\in\Gamma\ x_\alpha\in X.$$

This implies that the net $(x_\gamma;\ \gamma\in\Gamma)$ converges to any given element of X. We conclude that $\mathcal{A}$ is complete. However, for any $i,j\in\omega$, we have

(1) $(b_i,b_j),\ (c_i,c_j)\in\theta$,

(2) $(b_ic_i,0) = (c_ib_i,0) = (1,0)\in\theta$,

(3) $(b_i,c_i)\notin\theta$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now (1) implies that $(b_i ; i \in \omega)$ and $(c_i ; i \in \omega)$ are Cauchy sequences in $(A;\mathcal{U})$ while (2) implies that the sequences $(b_i c_i ; i \in \omega)$ and $(c_i b_i ; i \in \omega)$ each converge to 0. However, by (3), the sequence $((b_i,c_i) ; i \in \omega)$ is never in θ, so $\mathcal{A}$ is not a strongly uniform BCK-algebra. □

15.2. We show that a Hausdorff uniform BCK-algebra need not be strongly uniform. Let B, C, A, and θ be exactly as in 15.1. Let $(E,.,0_E)$ be the direct product BCK-algebra $\Pi((A;.,0) : i \in \omega)$. For each $n \in \omega$, define

$$\theta_n = \{(x,y) \in E \times E : x(i) = y(i) \text{ for } 0 \leq i \leq n \text{ and } (x(i),y(i)) \in \theta \text{ for all but finitely many } i \in \omega\}.$$

It is easily checked that $\theta_n \in \mathrm{Con}(E)$ for all $n \in \omega$, that $\mathcal{B} = \{\theta_n : n \in \omega\}$ is closed under finite intersections and that $\cap \mathcal{B} = \mathrm{id}_E$. It follows from Proposition 1.3(i) that $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$ on E and that $\mathcal{E} = (E;.,0;\mathcal{U})$ is a Hausdorff uniform BCK-algebra. Now consider the sequences $(x_i ; i \in \omega)$ and $(y_i ; i \in \omega)$ defined by

$$x_i(k) = b_0 ,$$

$$y_i(k) = \begin{cases} b_0 & \text{for } 0 \leq k \leq i \\ c_0 & \text{for } k > i \end{cases}$$

$(i,k \in \omega)$. The constant sequence $(x_i ; i \in \omega)$ is obviously Cauchy, while for any $n,i,j \in \omega$ with $n \leq i < j$, we have

$$y_i(k) \neq y_j(k) \iff i \leq k \leq j,$$

and hence $(y_i,y_j) \in \theta_n$. It follows that $(y_i ; i \in \omega)$ is also Cauchy. Now if $n, i \in \omega$ with $\leq i$, we have

$$(x_i y_i)(k) = (y_i x_i)(k) = \begin{cases} 0 & \text{for } 0 \leq k \leq i \\ 1 & \text{for } k > i \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$(k \in \omega)$ and since $(0,1) \in \theta$, it follows that $(x_i y_i, 0)$, $(y_i x_i, 0) \in \theta_n$. Thus the sequences $(x_i y_i;\ i \in \omega)$ and $(y_i x_i;\ i \in \omega)$ each converge to 0_E. However $(b_0, c_0) \notin \theta$ and therefore there are no n, $i \in \omega$ such that $(x_i, y_i) \in \theta_n$. This implies that $\mathcal{E}$ is not a strongly uniform BCK-algebra. (Note that by 1.4, $(E, \mathcal{U})$ is not a complete space; more explicitly, $(y_i;\ i \in \omega)$ does not converge). □

15.3. We conclude with an example of Hausdorff strongly uniform BCK-algebra which is not complete. Let $\rho(\omega)$ denote the set of all subsets of ω. Then $(\rho(\omega), \setminus, \phi)$ is a BCK-algebra (and, in fact, a member of every variety of BCK-algebras). For each $n \in \omega$, define

$$I(n) = \{z \in \rho(\omega) : z \text{ is finite and } 0,1,\ldots,n \notin z\},$$
$$\theta_n = \theta_{I(n)} = \{(x,y) \in \rho(\omega) \times \rho(\omega) : x \setminus y,\ y \setminus x, \in I(n)\},$$
$$\mathcal{B} = \{\theta_n : n \in \omega\}.$$

It is easily checked that $I(n)$ is an ideal of $\rho(\omega)$ for each $n \in \omega$, that $\{I(n) : n \in \omega\}$ (and hence $\mathcal{B}$) is closed under finite intersections and that $\cap\{I(n) : n \in \omega\} = \{\phi\}$ (whence $\cap \mathcal{B} = \mathrm{id}_{\rho(\omega)}$). It follows from Proposition 1.3(ii) that $\mathcal{B}$ is a base for a (Hausdorff) uniformity $\mathcal{U}$ on $\rho(\omega)$ and that $\mathcal{A} = (\rho(\omega), \setminus, \phi, \mathcal{U})$ is a Hausdorff strongly uniform BCK-algebra. Consider the sequence $(x_n;\ n \in \omega)$ defined by

$$x_n = \{0,1,\ldots,n\} \quad (n \in \omega).$$

Given $n,m,k \in \omega$ with $n \leq m < k$, we have

$$x_m \setminus x_k = \phi \in I(n);$$
$$x_k \setminus x_m = \{m+1,\ m+2,\ldots,k\} \in I(n),$$

and it follows that $(x_n;\ n \in \omega)$ is Cauchy. Suppose $(x_n;\ n \in \omega)$ converges to some $y \in \rho(\omega)$. For any $m \in \omega$ and for a sufficiently large integer $k > m$, we must have $x_k \setminus y \in I(m)$, whence $m \in y$. Thus y must be ω.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

But for each $m \in \omega$, the set $\omega \setminus x_m$, being infinite, cannot be an element of any $I(n)$. Thus $(x_n; n \in \omega)$ does not converge to ω, and hence does not converge. □

Acknowledgement: The second author was partially supported by South African FRD Grant 883-474-10.

130274

REFERENCES

[1] R.A. Aló, E.Y. Deeba: A note on uniformities of a BCK-algebra, Math. Japon., 30 (1985), 237-240.

[2] W.H. Cornish: On Iséki's BCK-algebras, Lecture Notes in Pure and Appl. Math. Vol. 74, 101-122, Marcel Dekker, New York, 1982.

[3] K. Iséki, S. Tanaka: An introduction to the theory of BCK-algebras, Math. Japon., 23 (1978), 1-26.

[4] J.L. Kelly: General Topology. Van Nostrand, Princeton, 1955.

[5] J.G. Raftery, T. Sturm: On ideal and congruence lattices of BCK-semilattices, Math. Japon., 32 (1987), 465-474.

[6] J.G. Raftery, T. Sturm: On completions of pseudo-normed BCK-algebras and pseudo-metric universal algebras, to appear in Mathematica Japonica, 33 (1988).

[7] S. Willard: General Topology. Addison Wesley, 1968.

[8] A. Wroński: BCK-algebras do not form a variety, Math. Japon., 28 (1983), 211-213.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF NATAL, SCOTTSVILLE, PIETERMARITZBURG 3201, REPUBLIC OF SOUTH AFRICA;

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF NATAL, DURBAN 4001, REPUBLIC OF SOUTH AFRICA

Received December 13, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

H.W. Pu, H.H. Pu

SELECTIVE DERIVATIVES FOR MONOTONE FUNCTIONS

1. The notion of the selective derivative was introduced by R.J. O'Malley who showed that the finite selective derivative is closely related to the approximate derivative ([3], Theorems 11 and 16). In the same paper, he also constructed a function whose finite selective derivative is not of Baire 1 and pointed out that the selective derivative of a monotone function is not the derivative. These two properties are possessed by every approximate derivative. In [2], M. Laczkovich proved that every finite selective derivative is of Baire 2 and later O'Malley proved that every selective derivative with respect to a balanced selection is of Baire 1 [4]. In the present paper, one more interesting property enjoyed by every approximate derivative is found and a sufficient condition, under which the selective derivative of a monotone function is the derivative, is obtained.

In the sequal, all functions are real valued, defined on the real line R unless otherwise is stated. Here we give some basic definitions found in [3] and [4].

Definition 1. A selection s is an interval function defined on the class $\mathcal{A}$ of all nondegenerate closed intervals $[a,b]$ in R such that $a < s[a,b] < b$.

Definition 2. For $[a,b] \in \mathcal{A}$ and $\alpha \in (0,1)$, $\alpha([a,b])$ denotes

$$\left[\frac{a+b}{2} - \frac{\alpha(b-a)}{2}, \frac{a+b}{2} + \frac{\alpha(b-a)}{2}\right],$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the interval concentric with $[a, b]$ and having the length $\alpha(b-a)$. Then a selection s is said to be balanced (or, more specifically, α-balanced) if $s[a,b] \in \alpha([a,b])$ for every $[a,b] \in \mathcal{A}$.

Definition 3. The selective derivative $sf'(x)$ of a function f, with respective to a selection s, at a point x is defined by

$$sf'(x) = \lim_{h \to 0} \frac{f(s[x, x+h]) - f(x)}{s[x, x+h] - x}$$

if the limit exists (for $h<0$, $[x, x+h]$ denotes the interval $[x+h, x]$).

Definition 4. The approximate derivative $apf'(x)$ of a measurable function f at x is defined by

$$apf'(x) = \lim_{\substack{y \to x \\ y \in E_x}} \frac{f(y) - f(x)}{y - x}$$

where E_x is some measurable set having x as a point of density, if such a set and the limit exist.

It should be noted that this definition is formally different from, but actually equal to the definition used in [3]. (See [5], p.219).

2. Now we proceed to the study of the selective derivative of a monotone function.

Theorem 1. There exists a monotone continuous function f and as balanced selection s such that the selective derivative $sf'(0)$ exists and is finite, but f is not differentiable at 0.

Proof. Consider the function f defined by

$f(x) = x$ if $x \leq 0$ or $x > 1$,

$= \left(\frac{2}{3}\right)^n$ if $\frac{3}{4}\left(\frac{2}{3}\right)^n \leq x \leq \left(\frac{2}{3}\right)^n$ for $n = 0,1,2,\ldots$,

linear on $\left(\frac{2}{3}\right)^{n+1} \leq x \leq \frac{3}{4}\left(\frac{2}{3}\right)^n$ for $n = 0,1,2,\ldots$.

It is clear that f is continuous and non-decreasing.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We define the selection s as follows: For given $a<b$, let $s[a,b] = \frac{1}{2}(a+b)$ if $a \neq 0$ or $b>1$. If $a = 0<b\leq 1$, then $s[a,b] = (2/3)^{n+2}$, where n is the unique integer such that $(2/3)^{n+1} < b \leq (2/3)^n$. It is easy to show that $s[a,b] \in (1/3)([a,b])$ for all $[a,b] \in \mathcal{A}$ and $sf'(0) = 1$. However, f is not differentiable at 0.

Theorem 2. Let f be an approximately differentiable function. Then for every $\alpha \in (0,1)$, there is an α-balanced selection s_α such that $s_\alpha f'(x) = apf'(x)$ for all x.

Proof. By the definition of the approximate derivative, each point x is associated with a measurable set E_x whose density at x is 1. Consequently, for each x, there is a positive number $\delta(x)$ such that $\mu(E_x \cap I) > (1 - \frac{\alpha}{2})\mu(I)$ whenever I is an interval containing x and $\mu(I) \leq \delta(x)$, where μ denotes the Lebesgue measure. We can easily define s_α so that $s_\alpha[a,b] \in E_e \cap \alpha([a,b])$ if $b-a<\delta(e)$, where e denotes an endpoint of $[a,b]$. (Note that $s_\alpha[a,b] \in E_a \cap E_b \cap \alpha([a,b])$ if $b-a<\min\{\delta(a), \delta(b)\}$). If $b-a \geq \max\{\delta(a), \delta(b)\}$, simply let $s_\alpha[a,b] = \frac{1}{2}(a+b)$. Then s_α is clearly α-balanced and $s_\alpha f'(x) = apf'(x)$ for all x.

Theorem 3. Let f be non-decreasing and continuous. Suppose that there is, in $(0,1)$, a decreasing sequence $\{\alpha_n\}$ converging to 0, and x_o and r satisfy the following:

For each n, there is an α_n-balanced selection s_n such that $s_n f'(x_o) = r$. Then f is differentiable at x_o and $f'(x_o) = r$.

Proof. Without loss of generality, we assume that $x_o = 0$ and $f(0) = 0$. It suffices to prove that $f'_+(0) = r$ and $f'_-(0) = r$. Since the proofs for these two equalities are similar, we only give the proof for $f'_+(0) = r$ here.

Let $0<\varepsilon<2$ be given. We want to show that there exists $\delta>0$ such that

(1)
$$\left|\frac{f(h)}{h} - r\right| < \varepsilon \quad \text{if} \quad 0<h<\delta.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since f is nondecreasing, $1+r>0$. There exists n such that

$$\frac{2\alpha_n}{1-\alpha_n} < \frac{\varepsilon/2}{1+r} . \tag{2}$$

For such an n, since $s_n f'(0) = r$, there exists $\delta_n > 0$ such that

$$\left| \frac{f(s_n[0,h])}{s_n[0,h]} - r \right| < \frac{\varepsilon}{2} \text{ whenever } 0<|h|\leq\delta_n. \tag{3}$$

Let $x_1 = s_n[0, \delta_n]$. Then $x_1>0$ and $r - \frac{\varepsilon}{2} < \frac{f(x_1)}{x_1} < r + \frac{\varepsilon}{2}$. Continuity of f at x_1 yields the existence of an interval (y,z) containing x_1 such that

$$r - \frac{\varepsilon}{2} < \frac{f(x)}{x} < r + \frac{\varepsilon}{2} \text{ for every } x \in (y,z). \tag{4}$$

Let $A = \left\{ t \in [0,z) : r - \varepsilon < \frac{f(x)}{x} < r + \varepsilon \text{ for all } x \in (t,z) \right\}$ and $a = \inf A$ (note that $A \neq \emptyset$). Clearly $a \in A$. We shall show that $a = 0$.

Suppose $a>0$. Since f is continuous at $a = \inf A$, we have

$$r - \varepsilon \leq \frac{f(a)}{a} \leq r + \varepsilon . \tag{5}$$

In fact, one of the equal signs in (5) must hold. Otherwise, a would not be inf A.

Case 1. $f(a)/a = r - \varepsilon$. Let $h = 2a/(1+\alpha_n)$ and $x_2 = x_n[0,h]$. Then $x_2 \in \alpha_n([0,h]) = \left[\frac{h}{2} - \frac{\alpha_n}{2}h, \frac{h}{2} + \frac{\alpha_n}{2}h\right] = \left[\frac{h}{2}(1-\alpha_n), \frac{h}{2}(1+\alpha_n)\right]$. Noting that $\frac{h}{2}(1+\alpha_n) = a$, we have $\frac{h}{2}(1-\alpha_n) \leq x_2 \leq a$ and $0 \leq f(x_2) \leq f(a)$. Hence

$$\begin{aligned} \frac{f(x_2)}{x_2} &\leq \frac{f(a)}{a}\frac{a}{x_2} = \frac{f(a)}{a} \cdot \left(\frac{h}{2}(1+\alpha_n)/x_2 \right) \leq \\ &\leq \frac{f(a)}{a}\left(\frac{h}{2}(1+\alpha_n) \Big/ \frac{h}{2}(1-\alpha_n) \right) = \frac{f(a)}{a}\frac{1+\alpha_n}{1-\alpha_n} = \\ &= (r-\varepsilon)\left(1 + \frac{2\alpha_n}{1-\alpha_n}\right) < (r-\varepsilon)\left(1 + \frac{\varepsilon/2}{1+r}\right) < r - \varepsilon + \frac{\varepsilon}{2} = r - \frac{\varepsilon}{2}. \end{aligned} \tag{6}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

On the other hand, we see from (4) that $y \in A$ and hence $a \leq y < x_1$. It follows that $0 < h = \frac{2a}{1+\alpha_n} < \frac{2}{1+\alpha_n} x_1 < \frac{2}{1+\alpha_n}\left(\frac{1}{2}\delta_n + \frac{\alpha_n}{2}\delta_n\right) = \delta_n$ and, by (3),

$$r - \frac{\varepsilon}{2} < \frac{f(x_2)}{x_2} < r + \frac{\varepsilon}{2}.$$

This is a contradiction to (6).

Case 2. $f(a)/a = r + \varepsilon$. Let $B = \left\{ t > a: \frac{f(x)}{x} > r + \frac{\varepsilon}{2} \text{ for all } x \in (a,t) \right\}$ and $b = \sup B$. Then $b \in B$ and $f(b)/b = r + \frac{\varepsilon}{2}$. Let $h' = 2b/(1+\alpha_n)$ and $x_3 = s_n[0,h']$. By (4), $b \leq y$. Just as in Case 1, we can show that $\frac{h'}{2}(1-\alpha_n) \leq x_3 \leq \frac{h'}{2}(1+\alpha_n) = b$ and $h' < \delta_n$. Therefore

$$r - \frac{\varepsilon}{2} < \frac{f(x_3)}{x_3} < r + \frac{\varepsilon}{2}$$

and, owing to the fact $b \in B$, $x_3 \leq a$. Moreover,

$$\frac{f(a)}{a} \leq \frac{f(b)}{x_3} = \frac{f(b)}{b}\frac{b}{x_3} \leq \left(r + \frac{\varepsilon}{2}\right)\left(\frac{1+\alpha_n}{1-\alpha_n}\right) <$$

$$< \left(r + \frac{\varepsilon}{2}\right)\left(1 + \frac{\varepsilon/2}{1+r}\right) = r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{r + \varepsilon/2}{r+1} < r + \varepsilon.$$

This is a contradiction to the assumption $f(a)/a = r + \varepsilon$.

We have shown that $a = 0$ and clearly (1) follows if we take $\delta = z$. The theorem is proved.

Remark. To prove Theorem 3, we could define a local system S (see [6], p.3) in the following manner: $S \in S(x)$ if there exists a sequence of positive numbers $\{\varepsilon_n\}$ such that

$$\bigcup_{n=1}^{\infty} \left\{ s_n[x, x+h]: 0 < |h| < \varepsilon_n \right\} \cup \{x\} \subset S.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then we could prove that every $S \in S(x)$ is nonporous at x and use (65.1) on p. 155 in [6]. However, this method is too involved. We choose to present the above proof which is straightforward.

Definition. Let functions f and g be given. g is said to be a super-balanced selective derivative of f is the following condition is fulfilled:

There exists, in $(0,1)$, a decreasing sequence $\{\alpha_n\}$ converging to 0 and, for each n, an α_n-balanced selection s_n such that $s_n f'(x) = g(x)$ for every x.

Using this term, we can say that every approximate derivative is a super-balanced selective derivative by Theorem 2. Also, we obtain the following corollaries to Theorem 3:

Corollary 1. A super-balanced selective derivative g of a monotone function f is the derivative f'.

Proof. Since f has a selective derivative g, f is Darboux by Theorem 11 in [3]. A Darboux monotone function must be continuous and the corollary follows from Theorem 3 immediately.

Corollary 2. If g is a super-balanced selective derivative of f and φ is a derivative such that $g \leq \varphi$ or $g \geq \varphi$, then $g = f'$.

Proof. Let $\varphi = \psi'$. Clearly $g - \varphi$ is a super-balanced selective derivative of $h = f - \psi$. By Theorem 2 in [3], h is monotone. Corollary 1 tells that $g - \varphi = h'$. Hence $g = \varphi + h' = \psi' + h' = f'$.

Corollary 3. Let g be a super-balanced selective derivative of f. Then there is a dense open set V on which f is differentiable.

Proof. The proof is standard (see, for example, Theorem 7.6 of [1]). By [4], g is Baire 1 and hence there exists a dense open set V such that g is bounded on some neighborhood of each point of V. By Corollary 2, $g = f'$ on V.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] A.M. Bruckner, R.J. O'Malley, B.S. Thomson: Path derivatives: a unified view of certain generalized derivatives, Trans. Amer. Math. Soc. 283 (1984), 97-125.

[2] M. Laczkovich: On the Baire class of selective derivatives, Acta Math. Acad. Sci. Hungar., 29 (1977), 99-105.

[3] R.J. O'Malley: Selective derivatives, Acta Math. Acad. Sci. Hungar., 29 (1977), 77-97.

[4] R.J. O'Malley: Balanced selections, Real Analysis Exchange, 8 (1982-83), 504-508.

[5] S. Saks: Theory of the integral. Monografie Matematyczne, vol. 7, Warsaw-Lwów, 1937.

[6] B.S. Thomson: Real functions, Lecture Notes in Mathematics, No. 1170, Springer, 1985.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY,
COLLEGE STATION, TEXAS 77843 U.S.A.

Received December 16, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Maciej Sablik

SOME REMARKS ON CAUCHY EQUATION ON A CURVE

1. In recent years a still increasing number of papers have been dealing with so called conditional Cauchy equations. The problem can be briefly stated as follows. Let X and E be some nonempty sets and let $\circ : X \times X \to X$ and $* : E \times E \to E$ be some binary operations. Further, let $Z \subset X \times X$ be a nonempty set. Consider the following functional equation for $f : X \to E$ (usually called Cauchy functional equation)

$$f(x \circ y) = f(x) * f(y). \tag{1}$$

The problem is to find all functions f satisfying (1) for all $x, y \in Z$ or rather to compare the set of these functions with the set of functions satisfying (1) for all $(x,y) \in X \times X$. As stated here, the question is very general and no wonder that usually some additional assumptions are made about X, E, $\circ$, $*$, Z and even the class of functions in which we are looking for solutions of (1). The reader is referred to J. Dhombres [1] and R. Ger [3] and [4] for more details.

One of the most studied situations is the case where $X = (0, +\infty)$, $E = \mathbb{R}$, $\circ$ and $*$ are both usual addition. In the present paper we shall restrict ourselves to this case. Moreover, Z will be a curve in $X \times X$. In other words we are going to deal with the equation

$$f(x(t) + y(t)) = f(x(t)) + f(y(t)) \tag{2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $t \in (0,+\infty)$, where $x,y:(0,+\infty) \longrightarrow (0,+\infty)$ are some functions. Let us mention here J. Dhombres [1], G.L. Forti [2], W. Jarczyk [7], M. Laczkovich [10] and M.C. Zdun [12] among those who obtained numerous results in this particular area.

Typical results read as follows. Under some assumptions on functions x and y every solution f of (2) which belongs to a prescribed class (for instance f is continuous, measurable or $\lim_{t\to 0} f(t)/t$ exists) has to be additive (or sometimes equal to an additive function almost everywhere in the sense of Lebesgue measure), i.e. f fulfils

$$f(s+t) = f(s) + f(t) \tag{3}$$

for all $s,t \in (0,+\infty)$.

In what follows we shall consider functions $x,y:(0,+\infty) \longrightarrow (0,+\infty)$ which are continuous and such that $x+y$ maps $(0,+\infty)$ homeomorphically onto itself. If we adopt these assumptions then (2) may be equivalently written as

$$f(t) = f(r_1(t)) + f(r_2(t)) \tag{4}$$

for all $t \in (0,+\infty)$, where $r_1 = x \circ (x+y)^{-1}$ and $r_2 = y \circ (x+y)^{-1}$ and hence r_i are continuous, $0 < r_i(t) < t$ for $t > 0$ and $r_1 + r_2 = \mathrm{id}$. Under these assumptions one can prove (cf. for instance Dhombres [1], Forti [2], Zdun [12] and M. Sablik [11]) that every solution f of (4) such that $\lim_{t\to 0} f(t)/t$ exists is linear, i.e. there is a constant $c \in \mathbb{R}$ such that $f(t) = ct$, $t > 0$. We can say that (3) and (4) are equivalent in the class

$$\underset{\sim}{D} = \left\{ f:(0,+\infty) \longrightarrow \mathbb{R} : \lim_{t\to 0} f(t)/t \text{ exists and is finite} \right\}.$$

(it is a well known fact that any solution of (3) belonging to $\underset{\sim}{D}$ has to be linear). Obviously this equivalence holds also in the class

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\underset{\sim}{A} = \underset{\sim}{D} + \underset{\sim}{Ad}$, where $\underset{\sim}{Ad}$ denotes the class of all solutions of (3). One cannot expect equivalence of (3) and (4) in the class of all functions because if we take $r_i = (1/2)\mathrm{id}$ then there exist nonadditive solutions of (4) (even very regular ones, cf. M. Kuczma [8]).

For every $f:(0,+\infty) \longrightarrow \mathbb{R}$ we call the Cauchy difference of f the function $C_f:(0,+\infty)\times(0,+\infty) \longrightarrow \mathbb{R}$ defined by

$$C_f(s,t) = f(s+t) - f(s) - f(t).$$

Consider the following class of functions

$$\underset{\sim}{B} = \left\{ f:(0,+\infty) \longrightarrow \mathbb{R} : C_f(s,t) = o(s+t), (s,t) \longrightarrow (0,0) \right\}^{1)}.$$

Let $f \in \underset{\sim}{A}$ and write $f = g + a$, where $g \in \underset{\sim}{D}$ and $a \in \underset{\sim}{Ad}$. Denote $L = \lim_{t\to 0} g(t)/t$. Then we have for $s,t>0$

$$C_f(s,t)/(s+t) = [(g(s+t)/(s+t))-L] - (s/(s+t))[(g(s)/s)-L] - (t/(s+t))[(g(t)/t)-L],$$

whence it easily follows that $\lim_{(s,t)\to(0,0)} C_f(s,t)/(s+t) = 0$. We have proved therefore that $\underset{\sim}{A} \subset \underset{\sim}{B}$. The reverse inclusion does not hold as is shown by the following example which we owe to Z. Gajda.

Example. Let $h:(0,+\infty) \longrightarrow \mathbb{R}$ be given by $h(t) = t - [t]$ and put

$$f(t) = \sum_{n=1}^{\infty} h(2^n t)/n2^n$$

for every $t>0$. It can be checked that f is continuous, $\lim_{t\to 0} f(t)/t = +\infty$ and $f \in \underset{\sim}{B}$. If we had $f = g + a$ with $g \in \underset{\sim}{D}$ and an additive a then,

1) Here and in the sequel we use the Landau notation in what concerns functions with limits equal to zero.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

in view of continuity of f, a would be linear and hence differentiable at 0. Thus f would be in $\underset{\sim}{D}$, a contradiction.

It is still an open question whether (3) and (4) are equivalent in the class $\underset{\sim}{B}$ for any suitable pair of r_i's. Below we prove a result showing that such an equivalence actually holds if we consider some special curves.

Theorem 1. Let $\underset{\sim}{C}$ be the class of functions defined by

$$\underset{\sim}{C} = \left\{ f:(0,+\infty) \to \mathbb{R} : C_f(t\cos\alpha, t\sin\alpha) = o(t),\ t \to 0,\ \text{for all } \alpha \in (0,\pi/2) \right\}.$$

Further, assume that

(H) $r_i:(0,+\infty) \to (0,+\infty)$, $i = 1,2$, are continuous and $r_1 + r_2 = \mathrm{id}$.

Then every function $f \in \underset{\sim}{C}$ satisfying (4) and

$$f(2t) = f(2r_1(t)) + f(2r_2(t)) \tag{5}$$

for $t > 0$, is additive.

Proof. Define $F:(0,+\infty) \to \mathbb{R}$ by $F(t) = C_f(t,t)$. By (4) and (5) we have for every $t > 0$

$$F(t) = f(2t) - 2f(t) = f(2r_1(t)) - 2f(r_1(t)) + f(2r_2(t)) - 2f(r_2(t)) =$$

$$= F(r_1(t)) + F(r_2(t)).$$

Moreover, we have for every $s \in (0,+\infty)$

$$F(s)/s = C_f(s,s)/s = \sqrt{2}\, C_f(\sqrt{2}s/\sqrt{2}, \sqrt{2}s/\sqrt{2})/\sqrt{2}s =$$

$$= \sqrt{2} C_f(\sqrt{2}s \cos(\pi/4),\ \sqrt{2}s \sin(\pi/4)),$$

whence

$$F(s) = o(s), \quad s \to 0+, \tag{6}$$

because $f \in \underset{\sim}{C}$. Thus F fulfils (4) and (6) which means that $F = 0$ (cf. e.g. Dhombres [1]). Fix now $s,t > 0$. We get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$0 = F(s+t) = f(2s+2t) - 2f(s+t) = C_f(2s,2t) + C_f(s,s) + C_f(t,t) - 2C_f(s,t) = C_f(2s,2t) - 2C_f(s,t)$$

whence

$$C_f(2s,2t) = 2C_f(s,t)$$

for every $s,t>0$. An easy induction yields

$$C_f(s,t) = 2^n C_f(s/2^n, t/2^n) \tag{7}$$

for every s,t 0 and $n \in \mathbf{N}$. Fix $s,t>0$ and choose $u>0$ and $\alpha \in (0,\pi/2)$ so that $s = u\cos\alpha$ and $t = u\sin\alpha$. From (7) and our assumptions we infer

$$C_f(s,t) = \lim_{n\to\infty} u[C_f((u/2^n)\cos\alpha, (u/2^n)\sin\alpha)/(u/2^n)] = 0$$

which proves additivity of f.

Corollary 1. Let r_i, $i = 1,2$, satisfy (H) and assume moreover that $r_1(2t) = 2r_1(t)$ for $t>0$. Then every solution of (4) which belongs to $\underset{\sim}{B}$ is additive.

Proof. We get immediately $r_2(2t) = 2r_2(t)$ for $t>0$. This means that every solution of (4) satisfies (5) as well. Since obviously $\underset{\sim}{B} \subset \underset{\sim}{C}$ we get our assertion from Theorem 1.

Observe that assumptions of Corollary 1 are satisfied in particular when $r_1(t) = ct$ for a $c \in (0,1)$.

2. M. Laczkovich in [9] introduced the notion of double difference property which in present circumstances can be formulated as follows. Let $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ be classes of real functions defined in $(0,+\infty)$ and $(0,+\infty)\times(0,+\infty)$, respectively. We say that $(\underset{\sim}{M}_1, \underset{\sim}{M}_2)$ has the double difference property (which we abbreviate to d.d.p. in the sequel) if for every function $f:(0,+\infty) \longrightarrow \mathbf{R}$ we have $C_f \in \underset{\sim}{M}_2$ implies $f = a + g$ where a is additive and $g \in \underset{\sim}{M}_1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Laczkovich proved that $(\underset{\sim}{L}^1, \underset{\sim}{L}^2)$ where $\underset{\sim}{L}^i$ denotes Lebesgue measurable functions of i variables, $i = 1,2$, has the d.d.p. On the other hand, if $\underset{\sim}{M}_i$, $i = 1,2$, is the class of bounded functions then $(\underset{\sim}{M}_1, \underset{\sim}{M}_2)$ has the d.d.p. This last sentence is known under the name of Hyers theorem on the stability of Cauchy functional equation (cf. [5]).

Let us introduce the following classes of functions for every $p \geq 1$:

$$\underset{\sim}{D}^1_p = \left\{ f:(0,+\infty) \to \mathbf{R} : f(t) = o(t^p),\ t \to 0 \right\},$$

and

$$\underset{\sim}{D}^2_p = \left\{ G:(0,+\infty)\times(0,+\infty) \to \mathbf{R} : G(s,t) = o((s+t)^p),(s,t) \to (0,0) \right\}.$$

Gajda's example quoted in § 1 shows that $(\underset{\sim}{D}^1_1, \underset{\sim}{D}^2_1)$ has not the d.d.p. We will show below that $(\underset{\sim}{D}^1_p, \underset{\sim}{D}^2_p)$ has the d.d.p. for every $p > 1$.

We start with a result which could be called d.d.p. on a curve.

Lemma 1. Let r_i, $i = 1,2$, satisfy (H). Further, let $f:(0,+\infty) \to \mathbf{R}$ be a function and define functions $F_n:(0,+\infty) \to \mathbf{R}$, $n \in \mathbf{N}$, by

$$F_1(t) = C_f(r_1(t), r_2(t)) \quad \text{and} \quad F_{n+1}(t) = F_n(r_1(t)) + F_n(r_2(t)).$$

Assume that for some $p \geq 1$ and $t_o > 0$

$$F_1 \in \underset{\sim}{D}^1_p \quad \text{and} \quad \sum_{n=1}^{\infty} F_n(t)/t^p \quad \text{is uniformly convergent in } (0,t_o].$$

Then $f = a + g$ where a fulfils (4) and $g \in \underset{\sim}{D}^1_p$.

Proof. An easy induction shows that for every $n \in \mathbf{N}$ and $t > 0$

$$f(t) = \sum_{i_1,\dots,i_n \in \{1,2\}} f(r_{i_1} \circ \dots \circ r_{i_n}(t)) + \sum_{k=1}^{n} F_k(t). \tag{8}$$

Let $h(t) = \sum_{n=1} F_n(t)/t^p$ for $t \in (0,t_o]$ where $p \geq 1$ and $t_o > 0$ are as in our assumptions. Put $g_o(t) = t^p h(t)$ for $t \in (0,t_o]$. From (8) we infer

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$a_o(t) = \lim_{n\to\infty} \sum_{i_1,\dots,i_n \in \{1,2\}} f(r_{i_1} \circ \dots \circ r_{i_n}(t)) = f(t) - g_o(t)$$

exists for every $t \in (0,t_o]$. Moreover, since

$$\sum_{i_1,\dots,i_{n+1} \in \{1,2\}} f(r_{i_1} \circ \dots \circ r_{i_n}(t)) =$$

$$= \sum_{i=1}^{2} \sum_{i_1,\dots,i_n \in \{1,2\}} f(r_{i_1} \circ \dots \circ r_{i_n} \circ r_i(t))$$

for $n \in \mathbb{N}$ and $t > 0$ we obtain

$$a_o(t) = a_o(r_1(t)) + a_o(r_2(t)) \tag{9}$$

for $t \in (0,t_o]$.

Now, let $n \in \mathbb{N}$ be such that $r_{i_1} \circ \dots \circ r_{i_n}(t) \in (0,t_o]$ for all $i_1,\dots,i_n \in \{1,2\}$ and put

$$a(t) = \sum_{i_1,\dots,i_n \in \{1,2\}} a_o(r_{i_1} \circ \dots \circ r_{i_n}(t)). \tag{10}$$

We show first that the definition (10) is not depending on n. Indeed, let n and m be such that $r_{i_1} \circ \dots \circ r_{i_q}(t) \in (0,t_o]$ for every $i_1,\dots,i_q \in \{1,2\}$ and $q \in \{n,m\}$. Suppose that $n < m$. Since $0 < r_i < \mathrm{id}$ and r_i are continuous, $i = 1,2$, we have $r_{i_1} \circ \dots \circ r_{i_{n+k}}(t) \in (0,t_o]$ for every $k \in \mathbb{N}$ and $i_1,\dots,i_{n+k} \in \{1,2\}$. In particular we get by (9)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sum_{i_1,\dots,i_m\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_2,\dots,i_m\in\{1,2\}} a_o(r_{i_2}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_1,\dots,i_{m-1}\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_{m-1}}(t)).$$

An easy induction leads to

$$\sum_{i_1,\dots,i_m\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_m}(t)) =$$

$$= \sum_{i_1,\dots,i_n\in\{1,2\}} a_o(r_{i_1}\circ \dots \circ r_{i_n}(t))$$

which proves that the definition is correct. A very similar argument shows that the function a defined by (10) for $t>t_o$ and equal to a_o in $(0,t_o]$ fulfils (4).

Define $g:(0,+\infty)\to R$ by $g(t) = f(t) - a(t)$. It remains to prove that $g(t) = o(t^p)$, $t\to 0$. We have $g|(0,t_o] = g_o$. Moreover,

(11) $$F_n(t) = o(t^p), \quad t\to 0,$$

for every $n\in N$. Indeed, we assumed it for $n = 1$ and if (11) holds for an $n\in N$ then from the equality

$$F_{n+1}(t)/t^p = \sum_{i=1}^{2} (r_i(t)/t)^p (F_n(r_i(t))/t^p),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

which is derived from the definition of F_{n+1}, we easily get the validity of (11) for n+1 as well. Combining (11) with the uniform convergence of $\sum_{n=1}^{\infty} F_n(t)/t^p$ we see that $g \in \underset{\sim}{D}_p^1$ which ends the proof.

Taking into account Corollary 1 we get from the above Lemma

Corollary 2. Under the assumptions of Lemma 1, if moreover $C_f \in \underset{\sim}{D}_1^2$ and $r_1(2t) = 2r_1(t)$ for every $t>0$ then $f = a + g$ where a is additive and $g \in \underset{\sim}{D}_1^1$.

Proof. By Lemma 1 the function a fulfils (4). Moreover, $g \in \underset{\sim}{D}_1^1 \subset \underset{\sim}{D} \subset \underset{\sim}{A} \subset \underset{\sim}{B}$. We also have $f \in \underset{\sim}{B}$ and hence obviously $a = f - g \in \underset{\sim}{B}$. Thus a is additive by Corollary 1.

Let us prove now that $(\underset{\sim}{D}_p^1, \underset{\sim}{D}_p^2)$ has the d.d.p. for $p>1$.

Theorem 2. Let $f:(0,+\infty) \longrightarrow R$ be a function such that $C_f \in \underset{\sim}{D}_p^2$ for some $p>1$. Then $f = a + g$ where a is additive and $g \in \underset{\sim}{D}_p^1$.

Proof. Define $r_i:(0,+\infty) \longrightarrow R$ by $r_i(t) = t/2$ for $i = 1,2$. Using the notation of Lemma 1 we get by an easy induction

$$F_1(t) = C_f(t/2,t/2) \text{ and } F_{n+1}(t) = 2^n F_1(t/2^n) \text{ for } n \geq 1. \tag{12}$$

Obviously we have $F_1(t) = o(t^p)$, $t \longrightarrow 0+$, because $C_f \in \underset{\sim}{D}_p^2$. It follows from (12) that

$$F_{n+1}(t)/t^p = (2^n)^{(1-p)}[F_1(t/2^n)/(t/2^n)^p], \quad n \geq 1. \tag{13}$$

Thus $F_{n+1}(t) = o(t^p)$, $t \longrightarrow 0$, for every $n \geq 1$. Choose $t_o > 0$ so that

$$|F_1(t)/t^p| \leq M$$

for a constant $M \geq 0$ and all $t \in (0,t_o]$. Then in view of (13) we have

$$|F_{n+1}(t)/t^p| \leq M(2^{1-p})^n$$

for every $t \in (0,t_o]$ and $n \geq 1$. This proves that the series $\sum_{n=1}^{\infty} F_n(t)/t^p$ is uniformly convergent in $(0,t_o]$. By Lemma 1 we get the decomposition $f = a + g$, where $g(t) = o(t^p)$, $t \longrightarrow 0$, and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$a(t) = 2a(t/2)$$

for every $t>0$. Now, if $C_f \in \underset{\sim}{D}_p^2$ for a $p>1$ then $C_f \in \underset{\sim}{D}_1^2$ and hence $f \in \underset{\sim}{B}$. Now the assertion follows from Corollary 2.

3. Theory of stability of functional equations was born in 1940s with papers of D.H. Hyers and S. Ulam (cf. [5] and [6] for instance). It has much developed since and even a short presentation of its achievements would go far beyond the frame of the present paper. As we have mentioned in the previous section, the stability of Cauchy equation in the sense of Hyers can be looked at as d.d.p. problem. On the other hand those who prefer stability language would rather say that d.d.p problems belong in fact to stability theory - one just replaces boundedness of Cauchy difference by some other regularity property. From this point of view the results of the previous section are also of "stability type" and Lemma 1 cound be called a result on stability on a curve.

In Theorem 2, to prove additivity of a we have used its additivity on the diagonal and the fact that a has some regularity at 0. We could not expect that a similar method would work if we assumed only the boundedness of a. In fact the equation

$$a(t) = 2a(t/2)$$

has many solutions which are bounded and continuous in $(0,+\infty)$, still not additive. Thus one cannot hope that the assertion of Hyers theorem will hold if we assume that C_f is bounded on curve only. To get such an assertion one has to impose some other conditions on f. Below we present a possible approach to this problem.

Let us prove first

Theorem 3. Let $f:(0,+\infty) \longrightarrow [0,+\infty)$ be Lebesgue measurable and such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$|f(t) - 2f(t/2)| \leqslant \varepsilon , \quad t > 0 \tag{14}$$

and

(15) function $t \longrightarrow f(t) - f(ct) - f(dt)$ is bounded in $(0,\infty)$

where $\varepsilon \geqslant 0$ is a constant, $c \in (0,1)$, $d = 1 - c$ and $\ln c/\ln d \notin \mathbb{Q}$. Then $f = a + g$ where a is equal to an additive function almost everywhere, g is measurable and $|g| \leqslant \varepsilon$.

Proof. Observe that (14) is equivalent to $|f(2t) - 2f(t)| \leqslant \varepsilon$ which we write as

$$|C_f(t,t)| \leqslant \varepsilon \tag{16}$$

for every $t>0$. An easy induction shows that

$$f(t) = (1/2)f(2t) - (1/2)C_f(t,t) = \ldots \tag{17}$$

$$= (1/2^n)f(2^n t) - \sum_{k=1}^{n} (1/2^k)C_f(2^k t, 2^k t).$$

From (16) we infer that the series $\sum_{n=1}^{\infty} (1/2^n)C_f(2^n t, 2^n t)$ is convergent. Define $g:(0,+\infty) \longrightarrow \mathbb{R}$ by

$$g(t) = -\sum_{n=1}^{\infty} (1/2^n)C_f(2^n t, 2^n t).$$

Then g is measurable because so is $t \longmapsto C_f(t,t)$, and (cf. (16)) $|g(t)| \leqslant \varepsilon$ for every $t>0$. From (15) we infer that

$$a(t) = \lim_{n\to\infty} (1/2^n)f(2^n t)$$

exists for every $t>0$. It is easy to observe that a is nonnegative, measurable and $a(2t) = 2a(t)$. Furthermore we have by (15) for every $n \in \mathbb{N}$ and $t>0$ and some $K>0$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(1/2^n)[f(2^n t) - f(2^n ct) - f(2^n dt)] \in (1/2^n)[-K, K]$$

whence, letting $n \to \infty$ we obtain

$$a(t) = a(ct) + a(dt).$$

Taking into account Laczkovich's result from [10] we see that a is a.e. equal to an additive function, which ends the proof.

The last result of the paper reads as follows.

Theorem 4. Let $f:(0,+\infty) \to \mathbb{R}$ be a function fulfilling (14) with an $\varepsilon \geq 0$ and suppose that an $n \in \mathbb{N}$ exists such that function

$$t \to |f(t) - f(t/2^N) - f((2^N-1)t/2^N)| \tag{18}$$

or

$$t \to |f(t) - f(t/(2^N+1)) - f(2^N t/(2^N+1))| \tag{19}$$

is bounded in $(0,+\infty)$. If f is measurable then the assertion of Theorem 3 is valid.

Proof. Similarly as in the proof of Theorem 3 we get $f = a + g$ where a and g are measurable, $|g| \leq \varepsilon$, $a(t) = 2a(t/2)$ for $t>0$ and

$$a(t) = a(t/2^N) + a((2^N-1)t/2^N) \tag{20}$$

or

$$a(t) = a(t/(2^N+1)) + a(2^N t/(2^N+1)) \tag{21}$$

depending on whether (18) or (19) is assumed. We easily check that

$$a(2^k t) = 2^k a(t) \quad \text{for all } k \in \mathbb{Z} \text{ and } t>0 \tag{22}$$

and hence also

$$a((2^N-1)^m t) = (2^N-1)^m a(t) \quad \text{for all } m \in \mathbb{Z} \text{ and } t>0, \tag{23}$$

if (20) holds, or

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(24) $a((2^N+1)^m t) = (2^N+1)^m a(t)$ for all $m \in Z$ and $t>0$,

if (21) is valid. In both cases we obtain

(25) $$a(rt) = ra(t)$$

for every $t>0$ and $r \in G$ where $G = \{2^k(2^N-1)^m : n,m \in Z\}$ in the case (22), (23), and $G = \{2^k(2^N+1)^m : n,m \in Z\}$ in the case (22), (24). Since neither $\ln 2/\ln(2^N-1)$ nor $\ln 2/\ln(2^N+1)$ is a rational number, the set G is dense in $(0,+\infty)$. This implies that $\ln G$ is dense in R whence we infer that the function $b: R \to R$ given by

$$b(v) = a(\exp v)/\exp v$$

is microperiodic. Since b is measurable, it has to be constant almost everywhere. Thus there is a $c \in R$ such that

$$a(\exp v) = c\exp v$$

for almost all $v \in R$. Absolute continuity of exp implies that $a(t) = ct$ a.e. in $(0,+\infty)$ which ends the proof.

REFERENCES

[1] J. Dhombres: Some aspects of functional equations. Chulalongkorn University Press, Bangkok 1970.

[2] G.L. Forti: On some conditional Cauchy equations on thin sets, Boll. Un. Mat. Ital. B(6)2(1983), 391-402.

[3] R. Ger: On some functional equations with a restricted domain, Fundamenta Math. 89 (1975), 131-149.

[4] R. Ger: On some functional equations with a restricted domain, II, Fundamenta Math. 98 (1978), 249-272.

[5] D.H. Hyers: On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222-224.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[6] D.H. Hyers, S. Ulam: Approximate isometries of the space of continuous functions, Ann. Math. 48 (1947), 285-289.

[7] W. Jarczyk: On continuous functions which are additive on their graphs, Berichte der mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum Graz, Bericht Nr 292 (1988).

[8] M. Kuczma: Functional equations in a single variable, Monografie Mat. 46, PWN - Polish Scientific Publishers, Warszawa 1968.

[9] M. Laczkovich: Functions with measurable differences, Acta Math. Acad. Sci. Hungar. 35 (1980), 217-235.

[10] M. Laczkovich: Non-negative measurable solutions of a difference equation, J. London Math. Soc. 2 (34) (1986), 139-147.

[11] M. Sablik: Note on Cauchy conditional equation, Radovi Mat. 1 (1985), 241-245.

[12] M.C. Zdun: On the uniqueness of solutions of the functional equation $\varphi(x+f(x)) = \varphi(x) + \varphi(f(x))$, Aequationes Math. 8 (1972), 229-232.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY IN KATOWICE, 40-007 KATOWICE, POLAND

Received December 16, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Antoni Sadowski

A GOURSTAT PROBLEM FOR MANGERON POLYVIBRATING EQUATION IN R^n, $n \geq 3$

Introduction

Goursat problems for partial differential equations in R^n-space, where $n \geq 3$, have been extensively examined in the special case when the boundary conditions are given on the planes of coordinates (see [7], [8], [10], [13], [14], [15] and the references in [19]). The papers devoted to the other cases were less numerous and in almost all of them the boundary conditions were given either on one surface or on a set of characteristics (see [1], [9], [11] and [17]). The much harder case of a greater number of non-characteristic surfaces was examined in papers [3]-[6].

In the present paper we extend the results of [3] to the case of R^n, $n \geq 3$.

1. Let $\Omega = \{x(x_1,\ldots,x_n) \in R^n: 0 \leq x_i \leq A_i < \infty;\ 1 \leq i \leq n\}$ be a parallelepiped in R^n-space ($n \in N^{*)}$, $n \geq 3$), m an element of the set N, $(Y, \|\cdot\|)$ a Banach space and $c : \Omega \to Y$ a given function.

We consider the following partial differential equation

(1) $$L^m u(x) = c(x), \qquad x \in \Omega,$$

*) The symbol N denotes the set of all positive integers.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $L = D_1 \dots D_n$, with $D_i = \frac{\partial}{\partial x_i}$, $i = 1,2,\dots,n$; $L^k = L(L^{k-1})$ for $1 \leq k \leq m$; $L^0 u = u$. Equation (1) is called Mangeron's polyvibrating equation of order nm with n independent variables.

By a solution of equation (1) in Ω we mean a function $u : \Omega \to Y$ such that $D_{i_1} \dots D_{i_l} L^k u \in C^{m-k-1}$ for $k = 0,1,\dots,m-1$; $1 \leq i_1 < \dots < i_l \leq n$; $l = 1,2,\dots,n$, satisfying equation (1) at each point $x \in \Omega$.

Admit the following notation

$$I_i(j) = \begin{cases} j & \text{for } 1 \leq j \leq i-1,\ 2 \leq i \leq n, \\ j+1 & \text{for } 1 \leq j \leq n-1,\ 1 \leq i \leq n-1, \end{cases}$$

$\Omega_i = \bigtimes_{j=1}^{n-1} <0, A_{I_i(j)}>$ for $i = 1,2,\dots,n$, and introduce the system of surfaces $S_1, S_2, \dots, S_n$ given by the equations

$$x_i = f_i(x^{(i)})$$

respectively, where $x^{(i)} = (x^i_1, x^i_2, \dots, x^i_{n-1}) \in \Omega_i$; $x^i_j = x_{I_i(j)}$, $i = 1,2,\dots,n$; $j = 1,2,\dots,n-1$.

In this paper we examine a linear Goursat-like problem that consists in finding a solution u of equation (1) in Ω, satisfying on S_i, $i = 1,2,\dots,n$, the boundary conditions of the form

$$L^r u(x) = N_{i,r}(x^{(i)}) \tag{2}$$

$(x \in S_i;\ x^{(i)} \in \Omega_i)$, where $N_{i,r} : \Omega_i \to Y$ $(i = 1,2,\dots,n;\ r = 0,1,\dots,m-1)$ are given functions.

We make the following assumptions

I. The functions $f_i : \Omega_i \to <0, A_i>$ are of class C^{m-n+2} and satisfy the conditions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$D^{l}_{l_i(j)} f_i(x^{(i)}) = 0 \tag{3}$$

$(x^{(i)} \in \Omega_i;\ i = 1,2,\dots,n;$ when $x^i_j = 0$ for some $j \in \mathbb{N};\ j \leq n-1;$ $l = 0,1,\dots,n-2)$, where $D^{l}_{l_i(j)} = D_{l_i(j)} D^{l-1}_{l_i(j)}$ for $1 \leq l \leq n-2$; $D^{0}_{l_i(j)} f_i = f_i$.

Moreover, the surfaces $S_1, S_2, \dots, S_n$ satisfy the relation $S_p \cap S_q = \{x \in \Omega;\ x_p = x_q = 0\}$, $p,q = 1,2,\dots,n;\ p \neq q$.

II. The functions $N_{i,r} : \Omega_i \longrightarrow Y$ of class $C^{m-r+n-2}$, satisfy the conditions *)

$$N^{(k)}_{i,r}(x^{(i)}) = 0 \quad \text{for} \quad x^{(i)} \in \Omega_i;\ \prod_{j=1}^{n-1} x^i_j = 0, \tag{4}$$

$k = 0,1,\dots,m-r+n-3$, $i = 1,2,\dots,n$; $r = 0,1,\dots,m-1$,

$$\|N^{(k)}_{i,r}(x^{(i)})\|_k \leq K_1 \Big[\min_{1 \leq j \leq n-1} x^i_j\Big]^{\theta} \quad \text{for} \quad x^{(i)} \in \Omega_i \tag{5}$$

$i = 1,2,\dots,n$; $k = m-r+n-2$; $r = 0,1,\dots,m-1$, where θ is a positive constant.

III. $c : \Omega \longrightarrow Y$ is a function of class C^{n-2}.

In what follows we will use the notation

$$M_j = \max_{1 \leq i \leq n} \max_{|\alpha| = j} \sup_{\Omega_i} |D^{\alpha} f_i(x^{(i)})|, \tag{6}$$

$j = 1,2,\dots,m+n-2$, and α is the multi-index with $n-1$ components concerning the differentiation with respect to the coordinates of the point $x^{(i)}$ and let

*) Here and in the sequel, $\|\cdot\|_k$, $k \geq 1$, denotes the norm in the space $L_k(B_1,B_2)$ of k-linear continuous functions, where B_1 and B_2 are Banach spaces. The symbol $\|\cdot\|_0$ is meant as the norm in B_2.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$M_k^* = \max\Big(1, \max_{1\leq j\leq k} M_j\Big), \quad k = 1,2,\ldots,m+n-2. \tag{7}$$

2. Auxiliary theorems

Lemma 1. (cf. Lemma 1 in papers [2] and [3]). Let a function $u : \Omega \to Y$ be given by

$$u(x) = R_m(x) + \sum_{j=1}^{m}\sum_{k=1}^{n} (x_k)^{m-j}\psi_{k,m-j}(x^{(k)}), \tag{8}$$

$x \in \Omega$; $x^{(k)} \in \Omega_k$; $k = 1,2,\ldots,n$, where

$$R_m(x) = [(m-1)!]^{-n} \int_0^{x_1}\cdots\int_0^{x_n} \prod_{k=1}^{n} (x_k-\varrho_k)^{m-1}\, c(\varrho_1,\ldots,\varrho_n)\,d\varrho_1\ldots d\varrho_n, \tag{9}$$

$$\psi_{k,m-j}(x^{(k)}) = \Big\{(m-j)!\,[(m-j-1)!]^{n-1}\Big\}^{-1} \times \int_0^{x_1^k}\cdots\int_0^{x_{n-1}^k} \prod_{l=1}^{n-1} (x_l^k-\varrho_l^k)^{m-j-1}\,\varphi_{k,m-j}(\varrho_1^k,\ldots,\varrho_{n-1}^k)\,d\varrho_1^k\ldots d\varrho_{n-1}^k \tag{10}$$

for $j = 1,2,\ldots,m-1$; $\psi_{k,0} = \varphi_{k,0}$, where the functions $\varphi_{k,m-j} : \Omega \to Y$, $k = 1,2,\ldots,n$; $j = 1,2,\ldots,m$, possess continuous derivatives $D_{l_k(i_1)}\cdots D_{l_k(i_1)}\varphi_{k,m-j}$, $1 \leq i_1 < \ldots < i_l \leq n-1$; $l = 1,2,\ldots,n-1$, of class C^{j-1}, respectively. Under these conditions u is a solution in Ω of equation (1) with $c \in C(\Omega)$.

Conversely, if u is a solution of (1) in Ω, then there are functions $\varphi_{k,m-j} : \Omega_k \to Y$, $k = 1,2,\ldots,n$; $j = 1,2,\ldots,m$, possessing the afore-said derivatives of class C^{j-1} such that (8)-(10) are satisfied at each point $x \in \Omega$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 2. The following inequality

$$|D^{\alpha} f_i(x^{(i)})| \leq M_{n-1}^{*} \left(\min_{1\leq k\leq n-1} x_k^i \right)^{n-1-|\alpha|}, \quad x^{(i)} \in \Omega_i, \tag{11}$$

$i = 1,2,\ldots,n$; $|\alpha| \leq n-2$, is valid.

Lemma 3. For each number λ_f satisfying the condition

$$0 < \lambda_f < \left[(m-1)((2n-3)M_1^{*})^{m+n-3+\theta} \right]^{-1/\theta} \tag{12}$$

there is a number $\delta = \delta(\lambda_f)$, $0 < \delta < \min(1, \min_{1\leq i\leq n} A_i)$, such that

$$|D^{\alpha} f_i(x^{(i)})| \leq \lambda_f \left(\min_{1\leq j\leq n-1} x_j^i \right)^{n-2-|\alpha|}, \quad |\alpha| \leq n-3 \tag{13}$$

holds good for $x^{(i)} \in \Omega_i$; $\min_{1\leq j\leq n-1} x_j^i < \delta$; $i = 1,2,\ldots,n$.

Lemma 3 is a consequence of Lemma 2.

Lemma 4. The following inequality

$$\|N_{i,r}^{(k)}(x^{(i)})\|_k \leq K_1 [(m+n-2-r-k)!]^{-1} \left(\min_{1\leq j\leq n-1} x_j^i \right)^{m+n-2-r-k+\theta}, \tag{14}$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $k = 0,1,\ldots,m+n-2-r$; $r = 0,1,\ldots,m-1$, holds good.

We prove Lemma 4 by using relations (4) and (5) and on applying the Taylor formula.

Now let us introduce the functions $a_{r,j}^{i} : \Omega_i \longrightarrow R$ given by

$$a_{r,j}^{i}(x^{(i)}) = \begin{cases} x_r^j & \text{for } r \neq i, \\ f_i(x^{(i)}) & \text{for } r = i, \end{cases} \tag{15}$$

when $i<j$, $r = 1,2,\ldots,n-1$; $x^{(i)} \in \Omega_i$; $1 \leq i,j \leq n$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(16)\qquad a^{i}_{r,j}(x^{(i)}) = \begin{cases} x^{j}_{r} & \text{for } r \neq i-1, \\ f_i(x^{(i)}) & \text{for } r = i-1, \end{cases}$$

when $j<i$, $r = 1,2,\dots,n-1$; $x^{(i)} \in \Omega_i$; $1 \leq i,j \leq n$.

We consider the function sequences $\left\{ z^{\nu}_{\vec{k}(\alpha)} \right\}$ defined by

$$(17)\qquad z^{\nu}_{\vec{k}(\alpha)}(x^{(\nu)}) = (z^{\nu}_{1,\vec{k}(\alpha)}(x^{(\nu)}),\dots,z^{\nu}_{n-1,\vec{k}(\alpha)}(x^{(\nu)}))$$

where

$$(18)\qquad z^{\nu}_{r,\vec{k}(\alpha)}(x^{(\nu)}) = a^{k_{\alpha-1}}_{r,k_\alpha}(z^{\nu}_{\vec{k}(\alpha-1)}(x^{(\nu)}))$$

for $\alpha = 2,3,\dots$; $r = 1,2,\dots,n-1$, and

$$(19)\qquad z^{\nu}_{r,\vec{k}(1)}(x^{(\nu)}) = a^{\nu}_{r,k_1}(x^{(\nu)})$$

for $r = 1,2,\dots,n-1$ ($\vec{k}(\alpha) = (k_1,\dots,k_\alpha)$; $1 \leq k_1 \leq n$; $k_l \neq k_{l-1}$; $l = 1,2,\dots,\alpha$; $k_0 = \nu$; $x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$.

Lemma 5. There is a number $\alpha_0 \in \mathbb{N}$ such that if $\mathbb{N} \ni \alpha > \alpha_0$ or $\min\limits_{1\leq r\leq n-1} x^{\nu}_{r} \leq \delta$, then [1]

$$(20)\qquad \max_{\vec{k}(\alpha)} \min_{1\leq r\leq n-1} z^{\nu}_{r,\vec{k}(\alpha)}(x^{(\nu)}) \leq \text{const}(\lambda_f)^{\alpha}\Big(\min_{1\leq r\leq n-1} x^{\nu}_{r}\Big)^{n-2}.$$

The validity of Lemma 5 results from Assumption I, Corollary 1 in [18] and Lemma 3.

Using formula (1.43) in [17] one can prove the following lemma (cf. Corollary 1.3 in [17])

---

[1] Here and in the sequel const denotes a positive constant.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 6. The inequality

$$\|z^{\nu}_{k(\alpha)}{}^{(l)}(x^{(\nu)})\|_1 \leq l!\,[M_1^*(2n-3)]^{\alpha l}\,[(n-1)^l M_l^*]^{l-1}, \tag{21}$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $l = 1,2,\dots,m+n-2$; $\alpha = 1,2,\dots$, holds good.

Let $R_{\nu,k}(x^{(\nu)}) = [R_k(x)]_{x=f_\nu(x^{(\nu)})}$, $x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $k = 1,2,\dots,m$, with R_k defined by (9) with the replacement of m by k.

Lemma 7. The inequality

$$\|R^{(r)}_{\nu,k}(x^{(\nu)})\|_r \leq \text{const}\left(\min_{1\leq j\leq n-1} x_j^{\nu}\right)^{k+n-1-r}, \tag{22}$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\dots,n$; $r = 0,1,\dots,k+n-2$; $k = 1,2,\dots,m$, is valid.

We omit an inductive proof of this Lemma.

Let

$$\mathcal{K}_{k,m-j} = \left\{ \varphi : \Omega_k \to Y ;\ \|\varphi^{(r)}(x^{(k)})\|_r \leq \right. \tag{23}$$

$$\left. \leq C(\varphi) \min_{1\leq l\leq n-1} x_l^{k\ j+n-2-r+\theta} \right\}.$$

for $x^{(k)} \in \Omega_k$; $k = 1,2,\dots,n$; $r = 0,1,\dots,j+n-2$; $j = 1,2,\dots,m$, where $C(\varphi)$ is a positive constant depending in general on the function φ.

For a fixed positive integer s such that $2 \leq s \leq m$ and fixed function $\varphi_{k,m-j} \in \mathcal{K}_{k,m-j}$, $k = 1,2,\dots,n$; $j = 1,2,\dots,s-1$, we consider the functions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(24)\qquad Q_{s,j}(x) = \sum_{k=1}^{n} (x_k)^{s-j} \int_0^{x_1^k} \cdots \int_0^{x_{n-1}^k} \prod_{l=1}^{n-1} (x_r^k - \varrho_r^k)^{s-j-1} \times$$

$$\times \varphi_{k,m-j}(\varrho_1^k,\ldots,\varrho_{n-1}^k)\,d\varrho_1^k \ldots d\varrho_{n-1}^k, \qquad x \in \Omega;$$

$$(25)\qquad Q_{\nu,s,j}(x^{(\nu)}) = \left[Q_{s,j}(x)\right]_{x_\nu = f_\nu(x^{(\nu)})},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$; $j = 1,2,\ldots,s-1$; $s = 2,3,\ldots,m$.

We have the following lemma whose inductive proof will be also omitted.

Lemma 8. If $\varphi_{k,m-j} \in \mathcal{K}_{k,m-j}$, then

$$(26)\qquad \|Q_{\nu,s,j}^{(r)}(x^{(\nu)})\|_r \leqslant$$

$$\leqslant \text{const} \max_{1\leqslant k\leqslant n} C(\varphi_{k,m-j}) \left(\min_{1\leqslant l\leqslant n-1} x_l^\nu\right)^{s+n-2-r+\Theta},$$

$x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$; $j = 1,2,\ldots,s-1$; $r = 0,1,\ldots,s+n-2$; $s = 2,3,\ldots,m$.

3. Solution of the problem (1), (2)

Let us observe that if u is a solution of (1) in Ω, then by Lemma 1, we have

$$(27)\qquad L^{m-s}u(x) = R_s^*(x) + \sum_{k=1}^{n} \varphi_{k,m-s}(x^{(k)}), \qquad x \in \Omega,$$

$x^{(k)} \in \Omega_k$; $k = 1,2,\ldots,n$; $s = 1,2,\ldots,m$, where[2]

---

2) As usual, we set $\sum_{k=\alpha}^{\beta} a_k = 0$ for $\beta < \alpha$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(28)\quad R_s^*(x) = R_s(x) + \sum_{r=1}^{s-1} \left\{ (s-r)!\,[(s-r-1)!]^{n-1} \right\}^{-1} \sum_{k=1}^{n} (x_k)^{s-r} \times$$

$$\times \int_0^{x_1^k} \cdots \int_0^{x_{n-1}^k} \prod_{j=1}^{n-1} (x_j^k - \varrho_j^k)^{m-j-1} \varphi_{k,m-r}(\varrho_1^k, \ldots, \varrho_{n-1}^k)\, d\varrho_1^k \ldots d\varrho_{n-1}^k .$$

Imposing on the function u (see (8)) the boundary conditions (2), we obtain a system of linear functional equations, with the unknown functions $\varphi_{k,m-s}$, $k = 1,2,\ldots,n$; $s = 1,\ldots,m$, of the form

$$(29)\quad \varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}(x^{(i)}) - \sum_{\substack{k_1=1 \\ k_1 \neq i}}^{n} \varphi_{k_1,m-s}(z^i_{\vec{k}(1)}(x^{(i)})),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,m$, where

$$(30)\quad W_{i,m-s}(x^{(i)}) = N_{i,m-s}(x^{(i)}) - R^*_{i,s}(x^{(i)}),$$

$$(31)\quad R^*_{i,s}(x^{(i)}) = \left[R^*_s(x)\right]_{x_i = f_i(x^{(i)})}$$

and $z^i_{\vec{k}(1)}(x^{(i)}) = (z^i_{1,\vec{k}(1)}(x^{(i)}), \ldots, z^i_{n-1,\vec{k}(1)}(x^{(i)}))$ are given by formula (19). Let us denote

$$(32)\quad V^{\alpha}_{i,m-s}(x^{(i)}) = (-1)^{\alpha} \sum_{\vec{k}(\alpha)} W_{k_\alpha,m-s}(z^i_{\vec{k}(\alpha)}(x^{(i)})),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,m$; $\vec{k}(\alpha) = (k_1,\ldots,k_\alpha)$; $1 \leq k_l \leq n$; $k_l \neq k_{l-1}$; $l = 1,2,\ldots,\alpha$; $k_0 = i$; $\alpha = 1,2,\ldots$.

We will solve (29) by using the iteration method (see [12]).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1 (cf. Theorem 1 in [3]). If Assumptions I-III are satisfied then for each $1 \leq l \leq m$ the system of functions $\{\varphi_{i,m-s}\}$, $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$, given by

$$\varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}(x^{(i)}) + \sum_{\alpha=1}^{\infty} V^{\alpha}_{i,m-s}(x^{(i)}) \tag{33}$$

is a solution of the system of functional equations

$$\varphi_{i,m-s}(x^{(i)}) = W_{i,m-s}(x^{(i)}) - \sum_{\substack{k_1=1 \\ k_1 \neq i}}^{n} \varphi_{k_1,m-s}\, z^{i}_{k(1)}(x^{(i)}), \tag{34}$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$, in the set Ω. This is the unique solution of system (34) in the class $\mathcal{K}_l$ of all systems of functions $\{\varphi_{i,m-s}\}$, such that $\varphi_{i,m-s} \in \mathcal{K}_{i,m-s}$ for $i = 1,2,\ldots,n$; $s = 1,2,\ldots,l$ (see (23)).

Proof. Apply mathematical induction with respect to l. Let $l = 1$. We will prove the uniform convergence of the series $\sum_{\alpha=1}^{\infty} V^{\alpha(r)}_{i,m-1}(x^{(i)})$, (where $i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$) in the set Ω_i. In virtue of formulas (30), (31) and Lemmas 4 and 7, we have

$$\|W^{(r)}_{i,m-1}(x^{(i)})\|_r \leq \text{const}\left(\min_{1\leq i\leq n-1} x^{i}_{j}\right)^{n-r-1+\theta}, \tag{35}$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$.

If $\alpha > \alpha_0$ or $\min_{1\leq j\leq n-1} x^{i}_{j} < \delta$, then the relations (20), (21) and (35) imply the inequality

$$\|V^{\alpha(r)}_{i,m-1}(x^{(i)})\|_r \leq \text{const}\left[(n-1)(\lambda_f)^{n-r-1+\theta}((2n-3)M_1^*)^r\right]^{\alpha} \times \left(\min_{1\leq j\leq n-1} x^{i}_{j}\right)^{n-r-1+\theta}, \tag{36}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$. If $\alpha \leqslant \alpha_0$ and $\min\limits_{1\leqslant j\leqslant n-1} x_j^i > \delta$, then

$$\|V_{i,m-1}^{\alpha(r)}(x^{(i)})\|_r \leqslant \text{const}\left(\min_{1\leqslant j\leqslant n-1} x_j^i\right)^{n-r-1+\theta}, \tag{37}$$

$i = 1,2,\ldots,n$; $r = 0,1,\ldots,n-1$.

As $0<(n-1)(\lambda_f)^{n-r-1+\theta}((2n-3)M_1^*)^r<1$ (cf. (12)) for $r = 0,\ldots,n-1$, the series appearing in formula (33) with $s = 1$, as well as the series obtained by its differentiation of order $r=1,2,\ldots,n-1$, are uniformly convergent in Ω_i, whence and by (33), (36) and (37) the functions $\varphi_{i,m-1}$, $i = 1,2,\ldots,n$, given by (33) for $s = 1$ belong to the classes $\mathcal{K}_{i,m-1}$ respectively. We will show that $\{\varphi_{i,m-1}\}$, $i = 1,\ldots,n$, satisfies the system of functional equations

$$\varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\k_1\neq i}}^{n} \varphi_{k_1,m-1}(z^i_{\vec{k}(1)}(x^{(i)})). \tag{38}$$

Setting $k_\beta = l_{\beta-1}$ and $\alpha-1 = \mu$ for $\alpha \geqslant 2$ in (32), we get (cf. (18), (19))

$$z^i_{\vec{k}(\alpha)}(x^{(i)}) = z^{k_1}_{\vec{l}(\mu)}\left(z^i_{\vec{k}(1)}(x^{(i)})\right),$$

where $\vec{l}(\mu) = (l_1,\ldots,l_\mu)$; $l_j \neq l_{j-1}$; $j = 1,2,\ldots,\mu$; $l_0 = k_1$.

Basing on this relation and using (32), (33), we obtain the following sequence of equalities

$$\varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\k_1\neq i}}^{n} \left\{ W_{k_1,m-1}\left(z^i_{\vec{k}(1)}(x^{(i)})\right) + \right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\sum_{\mu=1}^{\infty}(-1)^{\mu}\sum_{\vec{l}(\mu)}W_{l_\mu,m-1}\left(z^{k_1}_{\vec{l}(\mu)}\left(z^{i}_{\vec{k}(1)}(x^{(i)})\right)\right)\Bigg\} =$$

$$= W_{i,m-1}(x^{(i)}) - \sum_{\substack{k_1=1\\ k_1\neq i}}^{n}\varphi_{k_1,m-1}\left(z^{i}_{\vec{k}(1)}(x^{(i)})\right),$$

which imply that the system of functions $\left\{\varphi_{i,m-1}\right\}$, $i = 1,2,\ldots,n$, given by (33) satisfies (38) in the set Ω, as required.

Now we shall prove that (38) has exactly one solution such that $\varphi_{i,m-1} \in \varkappa_{i,m-1}$ $(i = 1,2,\ldots,n)$. To this end let us note that if $\left\{\varphi_{i,m-1}\right\}$, $i = 1,2,\ldots,n$, is a solution of (38), then

$$\varphi_{i,m-1}(x^{(i)}) = W_{i,m-1}(x^{(i)}) + \sum_{\alpha=1}^{s-1}V^{\alpha}_{i,m-1}(x^{(i)}) +$$

$$+ (-1)^{s}\sum_{\vec{k}(s)}\varphi_{k_s,m-1}\left(z^{i}_{\vec{k}(s)}(x^{(i)})\right), \qquad s \geqslant 1.$$

As, by assumption, $\varphi_{i,m-1} \in \varkappa_{i,m-1}$, Lemma 5 implies that $\sum_{\vec{k}(\alpha)}\varphi_{k_s,m-1}\circ z^{i}_{\vec{k}(s)} \rightrightarrows 0$, when $s \longrightarrow \infty$.

Thus, the functions $\varphi_{i,m-1}$, $i = 1,2,\ldots,n$, are expressed by formula (33) for $s = 1$, which ends the proof of Theorem 1 for $l = 1$, $m = 1$.

Let us assume that $m \geqslant 2$ and that Theorem 1 is valid for $1 \leqslant l \leqslant l_0$, where $1 \leqslant l_0 < m$. We will prove that it is also valid for $l = l_0+1$, i.e. that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\varphi_{i,m-(l_0+1)}(x^{(i)}) = W_{i,m-(l_0+1)}(x^{(i)}) + \tag{39}$$

$$- \sum_{\substack{k_1=1 \\ k_1 \neq i}}^{n} \varphi_{k_1,m-(l_0+1)}\left(z^{i}_{k_1 k(1)}(x^{(i)})\right),$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$, possesses exactly one solution $\left\{\varphi_{i,m-(l_0+1)}\right\}$, $i = 1,2,\ldots,n$, such that $\varphi_{i,m-(l_0+1)} \in \mathcal{K}_{i,m-(l_0+1)}$, with the functions $\varphi_{i,m-(l_0+1)}$ given by formula (33) for $s = l_0+1$.

By the inductive assumption, definitions (30), (31) and Lemmas 4, 7, 8, the inequality

$$\|W^{(r)}_{i,m-(l_0+1)}(x^{(i)})\|_r \leq \text{const}\left(\min_{1\leq j\leq n-1} x^i_j\right)^{l_0+n-1-r+\theta}, \tag{40}$$

$x^{(i)} \in \Omega_i$; $i = 1,2,\ldots,n$; $r = 0,1,\ldots,l_0+n-1$, holds good.

If $\alpha > \alpha_0$ or $\min_{1\leq j\leq n-1} x^i_j < \delta$, then inequalities (20), (21), (40) imply

$$\|V^{(r)}_{i,m-(l_0+1)}(x^{(i)})\|_r \leq \text{cons}\left[(n-1)(\lambda_f)^{l_0+n-1-r+\theta}((2n-3)M_1^*)^r\right]^{\alpha} \times \left(\min_{1\leq j\leq n-1} x^i_j\right)^{l_0+n-1-r+\theta}, \tag{41}$$

$i = 1,2,\ldots,n$; $r = 0,1,\ldots,l_0+n-1$. And if $\alpha \leq \alpha_0$ and $\min_{1\leq j\leq n-1} x^i_j \geq \delta$, $i = 1,2,\ldots,n$; $r = 0,1,\ldots,l_0+n-1$, holds true.

Since $0 < (n-1)(\lambda_f)^{l_0+n-1-r+\theta}((2n-3)M_1^*)^r < 1$ (cf. (12)) for $r = 0,\ldots,l_0+n-1$, we can conclude, analogously as in the case $s = 1$, that the functions $\varphi_{i,m-(l_0+1)}$ $(i = 1,\ldots,n)$ defined by formula (33) (for $s = l_0+1$) belong to the class $\mathcal{K}_{i,m-(l_0+1)}$, respectively.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Using an argument analogous to that for $l = 1$ one can prove that the system of the said functions is the only solution of (39) such that $\varphi_{i,m-(l_0+1)} \in \mathcal{X}_{i,m-(l_0+1)}$. Hence and from the inductive assumption we can conclude that Theorem 1 is valid for $l = l_0+1$. Using the induction principle we end the proof of Theorem 1.

Lemma 1 and Theorem 1 imply the following theorem.

Theorem 2. If Assumptions I-III are satisfied then the Goursat-like problem (1), (2) possesses a solution (8), where the functions $\psi_{k,m-j}$ are defined by (10) with $\varphi_{k,m-j}$ given by (33) and satisfying the system of functional equations (29). The solution (8) is unique in the set of all solutions u of equation (1) such that the system of functions $\{\varphi_{k,m-j}\}$, $k = 1,2,\ldots,n$; $j = 1,2,\ldots,m$, appearing in (10) belongs to the class $\mathcal{X}_m$.

REFERENCES

[1] S.A. Aldashev: On a nonlocal boundary value problem and the Goursat problem for a higher-order equation (in Russian), Izv. Akad. Nauk Kazakh. SSR Ser. Fiz-Mat. 3 (1982), 51-54.

[2] A. Borzymowski: A Goursat problem for some partial differential equation of order 2p, Bull. Acad. Pol. Sci.: Math. 32 (1984), 577-580.

[3] A. Borzymowski, A. Sadowski: A Goursat problem for Mangeron's polyvibrating equation in R^3, Applicable Anal. 23 (1987), 319-332.

[4] A. Borzymowski, A. Sadowski: A boundary value problem for the Mangeron polyvibrating equation, Rend. Mat. 6 (1986), 565-585.

[5] S.C. Chu: On a mixed boundary value problem for linear third order hyperbolic partial differential equations, Rend. Circ. Mat. Palermo 13 (1964), 199-208.

[6] S.C. Chu and J.B. Diaz: The Goursat problem for the partial differential equation $u_{xyz} = f(x,y,z,u,u_x,\ldots,u_{yz})$, A Mirage, J. Math. Mech. (16) (1967), 709-713.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[7] J. Conlan: An existence theorem for the equation $u_{xyz} = f$, Arch. Rational Mech. Anal. 9 (1962), 64-76.

[8] J. Conlan, J.B. Diaz: Existence of solutions of n-th order hyperbolic partial differential equations, Contrib. Diff. Equations 2 (1963), 277-289.

[9] Z. Denkowski, A. Pelczar: On the existence and uniqueness of solutions of some partial differential functional equations, Ann. Polon. Math. 35 (1978), 261-304.

[10] I.I. Glick: On an analog of the Euler-Cauchy polygon method for the partial differential equation $u_{x_1 \dots x_n} = f(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_2 \dots x_n})$, Contrib. Diff. Equations 2 (1963), 1-59.

[11] M.G. Gonzales: Resolucion del problema de Cauchy para ecuaciones en derivades parciales totalmente hiperbolicas, Collect. Math. 21 (1970), 199-283.

[12] M. Kuczma: Functional equations in a single variable, PWN, Warszawa, 1968.

[13] M. Kwapisz, B. Palczewski, W. Pawelski: Sur l'existence et l'unicité des solutions du certaines équations differentielles du type $u_{xyz} = f(x,y,z,u,u_x,\dots,u_{yz})$, Ann. Polon. Math. 11 (1961), 75-106.

[14] L. Nappi: Un teorema di esistenza e di unicita per il problema di Darboux in n variabili, Matematiche 23 (1968), 224-241.

[15] J. Persson: Global linear Goursat problems for functions of Gevrey-Lednev type, Ann. Scuola Norm. Sup. Pisa, 23 (1969), 387-412.

[16] J. Persson: Non-characteristic Cauchy problems and generalized Goursat problems in R^n. J. Math. Mech. 18 (1969), 1087-1094.

[17] A. Sadowski: Goursat-like boundary value problems for some partial differential equations with the number of independent variables greater than two (in Polish), Doctoral Thesis, Warsaw University of Technology, Warszaw 1986.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[18] A. Sadowski: Concerning some sequences of iterates in R^n-space, Demonstratio Math. 23 (4) (1990).

[19] W. Walter: Differential and integral inequalities, Springer.-Verlag, Berlin, 1970.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY
OF WARSAW, PŁOCK BRANCH,
09-400 PŁOCK, POLAND

Received December 19, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Ireneusz Nabiałek

VECTOR SPACES OF REGULAR FUNCTIONAL CONFIGURATIONS*)

Introduction

This paper takes as one's target an extension to use the methods of algebra in investigation of regular functional configurations.

The notion of regular configuration is principal in the theory of conflicts initiated by Z. Pawlak [1]. This notion is extended as regular functional configuration in [2].

A particular kind of transitivity of connections between the objects is basic in the definition of regular configuration. For apex configurations this transitivity is defined through two-elements group $\{1,-1\}$ with ordinary multiplication. A problem posed in this paper consists in investigation of existence a group defining a transitivity of connections between the objects for any regular functional configuration. A positive proof gives possibilities to define the notions of finite field of the degrees of engagement and vector space of regular functional configurations. The notion of generator of a matrix of regular functional configuration (cf. [3] and [4]) is extended.

1. Classes of determined regular functional configurations

Let X be any finite set composed of n elements, μ a measure of Stieltjes on the set of all real numbers R such that $\mu(R) = 1$ and let $N_n = \{1,2,\ldots,n\}$.

*) Research reported here has been partially suported by the Polish Government Grant CPBP 08-15.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Let $C = (X, \mu, \Phi)$ be a determined regular functional configuration in the sense of [2] and [5]. We have then

$$f_{ij}(p) = f_{ji}(p), \tag{1}$$

$$f_{ij}(p) \cdot f_{jk}(p) = f_{ik}(p) \tag{2}$$

for every $i,j,k \in N_n$, $p \in R$ and $(x_i, x_j) \in X^2$, where $\Phi(x_i, x_j) = f_{ij}$, $f_{ij} : R \to \{1; -1\}$ and f_{ij} is a measurable function with respect to μ.

Let F_μ be the set of all measurable functions $f : R \to \{1, -1\}$. For any $f \in F_\mu$ we denote by $\tilde{f}$ a class of the measurable functions equal to f on a set of measure 1 and by $\tilde{F}_\mu$ the set of all classes $\tilde{f}$ such that $f \in F_\mu$. Denote by $\tilde{\Phi}$ a mapping of X^2 into $\tilde{F}_\mu$ satisfied the analogous conditions to (1) and (2). By $\tilde{f}_o$ and $\tilde{f}_1$ we denote the classes of the constant functions $f_o : R \to \{1\}$ and $f_1 : R \to \{-1\}$.

It's evident that if $'$ is ordinary multiplication of functions, then $(\tilde{F}_\mu, \cdot)$ is an abelian group, $\tilde{f}_o$ is its identity element and the order of any element $\tilde{f}$ of $\tilde{F}_\mu$ is 2.

We define the notion of the degrees of engagement α_{ij} of a pair $(x_i, x_j) \in X^2$ in the class $(X, \mu, \tilde{\Phi})$ analogously to [2]:

$$\alpha_{ij} = \int_{-\infty}^{+\infty} \tilde{f}_{ij} \, d\mu,$$

where the integral is calculated for any function of the class $\tilde{f}_{ij}$.

2. Finite fields of degrees of engagement

Consider now any finite subset $\tilde{H}$ of $\tilde{F}_\mu$. By $(\tilde{H})$ denote a subgroup of $(\tilde{F}_\mu, \cdot)$ generated by $\tilde{H}$.

The group $(\tilde{H})$ is finite because the order of any its element is 2. The order of $(\tilde{H})$ is 2^r, where $r \leqslant \operatorname{card} \tilde{H}$. It's known (cf. [6]), that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

then $(\widetilde{H})$ is an additive group of a finite field $((\widetilde{H}),\cdot,*)$ of characteristic 2 and this field is only one with respect to isomorphismes.

For any $\tilde{g}_1,\ldots,\tilde{g}_n$ of $\widetilde{F}_\mu$ it's possible to define a class of determined regular functional configuration $(X,\mu,\widetilde{\Phi})$ as follows

$$\widetilde{\Phi}(x_i,x_j) = \tilde{g}_i\cdot\tilde{g}_j$$

for every $i,j\in N_n$.

Denote by $C_n(\widetilde{H})$ the set

$$\left\{(X,\mu,\widetilde{\Phi}) : \widetilde{\Phi}(x_i,x_j) = \tilde{g}_i\cdot\tilde{g}_j,\ \ \tilde{g}_i,\tilde{g}_j \in (\widetilde{H}) \quad \text{for every} \quad i,j \in N_n\right\}.$$

Definition 1. A subgroup $(\widetilde{H})$ of the group $\widetilde{F}_\mu$ is called a standard subgroup of $\widetilde{F}_\mu$ iff

(3) $$\tilde{f}_1 \in (\widetilde{H}),$$

(4) if $\tilde{f}\neq\tilde{g}$, then $\int_{-\infty}^{+\infty}\tilde{f}\,d\mu \neq \int_{-\infty}^{+\infty}\tilde{g}\,d\mu$ for every $\tilde{f},\tilde{g}\in(\widetilde{H})$.

Lemma 1. For every standard subgroup $(\widetilde{H})$ of $\widetilde{F}_\mu$ there exists a finite field isomorphic to $((\widetilde{H}),\bullet,*)$ such that every its element is a degree of engagement in a class of the set $C_n(\widetilde{H})$.

Proof. Let $\varphi(\tilde{f}) = \int_{-\infty}^{+\infty}\tilde{f}\,d\mu$ for every $\tilde{f}\in(\widetilde{H})$. From the condition (4) φ is an injective mapping of $(\widetilde{H})$ into the interval $\langle -1;1\rangle$.

Let K_r be the image $\varphi[(\widetilde{H})]$ and let card $K_r = 2^r$. From the definition of the set $C_n(\widetilde{H})$ every number α of K_r is a degree of engagement in a class $(X,\mu,\widetilde{\Phi})$ of $C_n(\widetilde{H})$.

We define the composition laws $\circ$ and $*$ in the set K_r as follows

$$\alpha\circ\beta = \varphi[\varphi^{-1}(\alpha)\cdot\varphi^{-1}(\beta)],$$

$$\alpha * \beta = \varphi[\varphi^{-1}(\alpha)*\varphi^{-1}(\beta)],$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $\cdot$ is ordinary multiplication of functions and $*$ is the multiplication in the field $((\tilde{H}),\cdot,*)$. The mapping φ is an isomorphism of $((\tilde{H}),\cdot,*)$ onto $(K_r,\circ,*)$, hence the lemma is proved.

Definition 2. The field K_r defined in the proof of Lemma 1 is called a finite field of degrees of engagement.

We denote $K_r \sim (\tilde{H})$ iff there exists an isomorphism φ of the field K_r onto the field $(\tilde{H})$.

Lemma 2. For every $r \in N$ any finite field of degrees of engagement K_r has the following properties

$$K_1 = \{1,-1\}, \tag{5}$$

$$K_1 \subset K_r, \tag{6}$$

$$\text{if } \alpha \in K_r, \quad \text{then} \quad -\alpha \in K_r, \tag{7}$$

$$0 \notin K_r. \tag{8}$$

Proof. The conditions (5) and (6) are satisfied because 1 is the zero and -1 is the identity element of the field K_r.

If $\alpha \in K_r$, then there exists $\tilde{f} \in (\tilde{H})$ such that $\alpha = \int_{-\infty}^{+\infty} \tilde{f}\, d\mu$ and then $\tilde{f}_1 \cdot \tilde{f} \in (\tilde{H})$ from the condition (3). We have $\int_{-\infty}^{+\infty} \tilde{f}_1 \cdot \tilde{f}\, d\mu = 0$ and $f \neq f_1 \cdot f$, thus the subgroup (H) is not standard.

It's possible to show, that for $r > 1$ a field K_r can be defined on the different sets of degrees of engagement. For $r = 2$, K_2 can be any set $\{1,-1,\alpha,-\alpha,\}$ where $0 < \alpha < 1$.

3. Generators of matrices of regular functional configurations

Let $(K_r,\circ,*)$ be any fixed finite field of degrees of engagement. Let $G \in K_r^n$ and $\alpha \in K_r$. Denote $G = [g_i]_{n\times 1}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We define the following composition laws:

$$G \circ G = [g_{ij}]_{n\times n} = [g_i \circ g_j]_{n\times n} \quad \text{for every } G \text{ of } K_r^n,$$

$$\alpha \circ G = [\alpha \circ g_i]_{n\times 1} \quad \text{for every } \alpha \in K_r \text{ and } G \in K_r^n.$$

A matrix $M = [\alpha_{ij}]_{n\times n}$ is a matrix of class of regular functional configuration $(X,\mu,\tilde{\Phi})$ iff α_{ij} are the degrees of engagement in this class (cf. [4]).

Lemma 3. For every matrix $G \in K_r^n$ the matrix $G \circ G$ is a matrix of class of a regular functional configuration.

Proof. Let $g_1,\dots,g_n \in K_r$ and let $K_r \sim (\tilde{H})$. Then there exist the classes $\tilde{f}_1,\dots,\tilde{f}_n \in (\tilde{H})$ such that $\int_{-\infty}^{+\infty} \tilde{f}_i \, d\mu = g_i$ for every $i \in N_n$. Hence $g_i \circ g_j$ is a degree of engagement of the elements x_i and x_j in the class $(X,\mu,\tilde{\Phi})$ such that $\tilde{\Phi}(x_i,x_j) = \tilde{f}_i \cdot \tilde{f}_j$ for every $i,j \in N_n$. Therefore, the matrix $G \circ G$ is a matrix of this class.

Definition 3. A matrix $G \in K_r^n$ is called a generator over the field K_r of a matrix M of class of regular functional configurations iff $M = G \circ G$.

Denote by $M_n(K_r)$ the set $\left\{ M : M = G \circ G \text{ and } G \in K_r^n \right\}$.

Lemma 4. Let $M \in M_n(K_r)$ and $M = G' \circ G'$. The set

$$\left\{ G \in K_r^n : G = \alpha \circ G' \quad \text{and} \quad \alpha \in K_r \right\}$$

is the set of all generators of the matrix M.

Proof. If $\alpha \in K_r$ and $G \in K_r^n$, then $(\alpha \circ G) \circ (\alpha \circ G) = G \circ G$, hence if $M = G' \circ G'$, then $\alpha \circ G'$ is a generator of the matrix M over the field K_r for every $\alpha \in K_r$.

If $G \circ G = G' \circ G'$, $G = [g_i]_{n\times 1}$ and $G' = [g'_i]_{n\times 1}$, then $g_1 \circ g_i = g'_1 \circ g'_i$ for every $i \in N_n$, thus $g_i = (g_1 \circ g'_1) \circ g'_i$ and we have $G = \alpha \circ G'$, where $\alpha = g_1 \circ g'_1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 5. For every matrix M of $M_n(K_r)$ there exists one generator $G_o = [g_i]_{n\times 1}$ and only one such that $g_1 = 1$.

Proof. Let $M \in M_n(K_r)$ and $M = G' \circ G'$, where $G' = [g'_i]_{n\times 1}$. From Lemma 4, $g'_1 \circ G'$ is only one generator $G_o = [g_i]_{n\times 1}$ of the matrix M such that $g_1 = 1$.

Definition 4. The matrix $\bar{G} = [\bar{g}_i]_{(n-1)\times 1}$ is called a reduced generator of a matrix M of $M_n(K_r)$, iff $M = G \circ G$, $G = [g_i]_{n\times 1}$, $g_1 = 1$ and $\bar{g}_i = \bar{g}_{i-1}$ for every $i = 2,3,\ldots,n$.

Let $A = [a_{ij}]_{k\times m}$, $B = [b_{ij}]_{k\times m}$ and $a_{ij}, b_{ij}, \lambda \in K_r$.

We define the following composition laws

$$A \oplus B = [a_{ij} \circ b_{ij}]_{k\times m}, \tag{9}$$

$$\lambda * A = [\lambda * a_{ij}]_{k\times m}. \tag{10}$$

Theorem. The set $M_n(K_r)$ with the composition laws $\oplus$ and $*$ defined by (9) and (10) is a vector space isomorphic to the vector space K_r^{n-1} over the field K_r.

Proof. Let $\psi : M_n(K_r) \longrightarrow K_r^{n-1}$ be a mapping such that for every M of $M_n(K_r)$, $\psi(M)$ is a reduced generator of M. From Lemmas 3 and 5 the mapping ψ is bijective.

For every $M, M' \in M_n(K_r)$ we have $\psi(M \oplus M') = \psi([g_i \circ g_j \circ g'_i \circ g'_j]_{n\times n})$, where $g_1 = g'_1 = 1$ and then $\psi(M \oplus M') = \bar{G} \oplus \bar{G}'$, where $\bar{G}$ and $\bar{G}'$ are the reduced generators of M and M'.

If $\lambda \in K_r$, then $\psi(\lambda * M) = \psi([\lambda * (g_i \circ g_j)]_{n\times n})$, where $g_1 = 1$. We have $\lambda * (g_i \circ g_j) = (\lambda * g_i) \circ (\lambda * g_j)$, therefore $\psi(\lambda * M) = \lambda * \psi(M)$.

It's proved that ψ is an isomorphism of $(M_n(K_r), \oplus, *)$ onto the vector space K_r^{n-1} over the field K_r.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Vector spaces of determined regular functional configurations

Consider now any standard subgroup $(\widetilde{H})$ of the group $\widetilde{F}_\mu$ and a finite field of degrees of engagement $K_r \sim (\widetilde{H})$.

Let $\varphi: C_n(\widetilde{H}) \longrightarrow K_r^{n-1}$ be a mapping such that for every C of $C_n(\widetilde{H})$, $\varphi(C)$ is a reduced generator of the matrix of C. The mapping φ is bijective. We define an addition of configurations of $C_n(\widetilde{H})$ and a multiplication of configuration by an element of K_r as follows

$$C_1 + C_2 = \varphi^{-1}(\bar{G}_1 \oplus \bar{G}_2), \tag{11}$$

$$\lambda \cdot C = \varphi^{-1}(\lambda * \bar{G}), \tag{12}$$

where $\bar{G}_1$, $\bar{G}_2$ and $\bar{G}$ are the reduced generators of matrices of configurations C_1, C_2 and C. From (11) and (12), the bijective mapping φ is an isomorphism of $C_n(\widetilde{H})$ onto the vector space K_r^{n-1} over the field K_r. Let $C_n(\widetilde{H})$ and $C_m(\widetilde{H})$ be two vector spaces of determined regular functional configurations and let φ and ψ be the isomorphismes of $C_n(\widetilde{H})$ and $C_m(\widetilde{H})$ onto K_r^{n-1} and K_r^{m-1}. For every linear mapping $h: K_r^{n-1} \longrightarrow K_r^{m-1}$, the mapping $\psi^{-1} \circ h \circ \varphi$ is a linear mapping of the vector space $C_n(\widetilde{H})$ onto vector space $C_m(\widetilde{H})$.

REFERENCES

[1] Z. Pawlak: On conflicts, Intern. J. Machine Studies, 21 (1984), 127-134.

[2] I. Nabiałek: Functional configuration and degrees of engagement, Bull. Polon. Acad. Sci. Ser. Math., 35 (1987), 273-278.

[3] W. Żakowski: Matrices of configurations and their applications in theory of conflicts, Bull. Polon. Acad. Sci. Ser. Math., 35 (1987), 1-6.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[4] I. Nabiałek, W. Żakowski: Matrices of regular functional configurations and their some algebraic properties, Bull. Polon. Acad. Sci. Ser. Math. 36 (1988) 419-423.

[5] I. Nabiałek: Formations of degrees of engagement in regular functional configurations, Bull. Polon. Acad. Sci. Ser. Math., 35 (1987), 685-691.

[6] A. Białynicki-Birula: Zarys algebry. Warszawa, PWN 1987.

INSTITUTE OF MATHEMATICS, WARSAW TECHNICAL UNIVERSITY, 00-661 WARSZAWA, POLAND

Received December 20, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol XXIII No 2 1990

Anatolij Dvurečenskij

FRAME FUNCTIONS AND COMPLETENESS

Using the methods of quantum logics, we present a very simple criterion showing that an inner product space S is complete iff it possesses at least one nonzero frame function. As corollary we obtain the criterion that S is complete iff, for some unit vector from the completion of S, the Bessel equality holds for any maximal orthonormal system in S. It generalizes the criterion of S.P. Gudder and S. Holland [10] who required that any maximal orthonormal system to be a basis.

1. Introduction

Let S be an inner product space over the field C of real or complex numbers, and with an inner product $(.,.)$. There are many characterizations of the completeness of an inner product space via algebraic-topological properties ([8-10]), algebraic conditions ([1-3]) or measure-theoretic properties ([5,6,11]).

We introduce three families of closed subspaces of S which will be interesting for us:

(1) F(S) is the set of all $\perp$-closed subspaces of S, i.e., of all subspaces M of S for which we have $M = M^{\perp\perp}$, where $A^{\perp} = \{x \in S: (x,y) = 0 \text{ for any } y \in A\}$, which, in its turn, is an orthocomplemented, complete lattice.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) $V(S)$ is the set of all subspaces M of S such that $M = \{u_i\}^{\perp\perp}$ and $M^{\perp} = \{v_j\}^{\perp\perp}$ for every maximal orthonormal system (MONS, for short) $\{u_i\}$ and $\{v_j\}$ in M and $M^{\perp}$, respectively, which is an orthocomplemented poset.

(3) $E(S)$ is the set of all splitting subspaces of S, i.e., of all subspaces M of S for which the condition $M + M^{\perp} = S$ holds, which is an orthocomplemented orthomodular orthoposet. S is complete iff $E(S)$ is a quantum logic [3].

It is evident that

$$E(S) \subseteq V(S) \subseteq F(S),$$

and S is complete iff $E(S) = F(S)$. Moreover, it is an open problem whether for an incomplete S $E(S) = V(S)$.

A (completely additive) state on $E(S)$ is a mapping $m: E(S) \to [0,1]$ such that (i) $m(S) = 1$; (ii) $m(\bigvee_{t\in T} M_t) = \sum_{t\in T} m(M_t)$ whenever $\{M_t: t \in T\}$ is a system of mutually orthogonal splitting subspaces for which the join $\bigvee_{t\in T} M_t$ exists in $E(S)$. Analogously we define a state on $V(S)$ or $F(S)$, respectively.

The following is true: S is complete iff $E(S)$ ($V(S)$ [4]) ($F(S)$ [5]) possesses at least one (completely additive) state [6].

In the present note we give a numerical characterization of the completeness using the notion of a frame function.

2. The main result

Denote by $\varphi(S)$ the unit sphere in S, that is, $\varphi(S) = \{x \in S: \|x\| = 1\}$. A frame function is a mapping $f: \varphi(S) \to [0, \infty)$ such that there is a finite constant W called the weight of f such that $\sum_i f(x_i) = W$ for any MONS $\{x_i\}$ in S. The notion of frame function for Hilbert spaces has been introduced by Gleason [7].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem. An inner product space is complete iff it possesses at least one nonzero frame function.

Proof. The necessary condition is evident.

For the sufficiency, we may assume without loss of generality that $W = 1$. Let M be a splitting subspaces of S. Let $\{x_i\}$ and $\{y_i\}$ be two MONS's in M and let $\{z_j\}$ be an MONS in $M^{\perp}$. The simple verification shows that $\{x_i\} \cup \{z_j\}$ and $\{y_i\} \cup \{z_j\}$ are two MONS's in S. Then

$$\sum_i f(x_i) + \sum_j f(z_j) = 1 = \sum_i f(y_i) + \sum_j f(z_j) \tag{1}$$

which gives $\sum_i f(x_i) = \sum_i f(y_i)$.

Define a mapping $m: E(S) \longrightarrow [0,1]$ as follows

$$m(M) = \begin{cases} \sum_i f(x_i) & \text{if } M \neq \{0\} \\ 0 & \text{if } M = \{0\}, \end{cases} \tag{2}$$

where $\{x_i\}$ is an MONS in M. We claim to show that m is a (totally additive state on E(S). Indeed, the property (1) entails

$$m(M) + m(M^{\perp}) = 1$$

for any $M \in E(S)$. Let $M = \bigvee_{t \in T} M_t \in E(S)$, where $\{M_t : t \in T\}$ is a system of mutually orthogonal subspaces from E(S), and let $\{x_t^i\}_i$ be an MONS in M_t for any $t \in T$. Then $\bigcup_{t \in T} \{x_t^i\}$ is an MONS in M. Hence

$$m(M) = \sum_{t,i} f(x_t^i) = \sum_t \sum_i f(x_t^i) = \sum_t m(M_t).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In view of the criterion from [6], the existence of a (totally additive) state on $E(S)$ implies the completeness of S. Q.E.D.

Note 1. In the proof of Theorem, we used the simple observation: if $\{x_i\}$ is an MONS in a splitting subspace M, then $\{x_i\}^{\perp\perp} = M$. For any $\perp$-closed subspace this is true iff S is complete. Therefore (1) and (2) may define a (completely additive) state on $V(S)$ but not on $F(S)$.

Corollary. An inner product space S is complete iff there is a unit vector $y \in \bar{S}$ ($\bar{S}$ denotes the completion of S) such that

$$y = \sum_i (y,x_i)\, x_i \tag{3}$$

for any MONS $\{x_i\}$ in S.

Proof. Let us put $f(x) = |(y,x)|^2$, $x \in \varphi(S)$. The straightforward verification shows that f is a frame function with a unit weight. Q.E.D.

Note 2. If, in Corollary, we find a unit vector $y \in S$ such that (3) holds, then we obtain the generalization of the criterion of Gudder and Holland [8] that any MONS in S is a basis. On the other hand, in this case, we have easily that for every $x \in \varphi(S)$

$$x = \sum_i (x,x_i)\, x_i \tag{4}$$

for any MONS $\{x_i\}$ in S. Indeed, if we define a unitary linear operator $U: S \to S$ such that $Ux = y$ and $Uz = z$ for all $z \perp x,y$, then (3) implies (4) for each $x \in S$. Therefore, for this particular case, Corollary may be proved also using the criterion of Gudder and Holland.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] I. Amemiya, H. Araki: A remark on Piron's paper, Publ. Res. Inst. Math. Sci., Ser. A12 (1966-67) 423-427.

[2] G. Cattaneo, G. Marino: Completeness of inner product spaces with respect to splitting subspaces, Letters Math. Phys., 11 (1986) 15-20.

[3] A. Dvurečenskij: Completeness of inner product space and quantum logic of splitting subspaces, Letters Math. Phys., 15 (1988) 231-235.

[4] A. Dvurečenskij: States on families of subspaces of pre-Hilbert space. Letters Math. Phys., 15 (1988).

[5] A. Dvurečenskij, L. Mišík, Jr.: Gleason's theorem and completeness of inner product spaces, Inter. J. Theor. Phys. 27 (1988) 419-428.

[6] A. Dvurečenskij, S. Pulmannová: State on splitting subspaces and completeness for inner product spaces, Inter. J. Theor. Phys., 27 (1988) 1059-1067.

[7] A.M. Gleason: Measures on the closed subspaces of a Hilbert space, J. Math. Mech., 6 (1957) 885-894.

[8] S.P. Gudder: Inner product spaces, Amer. Math. Monthly, 81 (1974) 29-36.

[9] S.P. Gudder: Correction to "Inner product spaces", Amer. Math. Monthly, 82 (1975) 251-252.

[10] S.P. Gudder, S. Holland, Jr.: Second correction to "Inner product spaces", Amer. Math. Monthly, 82 (1975) 818.

[11] J. Hamhalter, P. Pták: A completeness criterion for inner producta spaces, Bull. Lond. Math. Soc., 19 (1987) 259-263.

MATHEMATICAL INSTITUTE, SLOVAK ACADEMY OF SCIENCES,
CS-814 73 BRATISLAVA, CZECHOSLOVAKIA.

Received January 5, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 2 1990

Joanna Ger

A CLOSED EPIGRAPH THEOREM

1. Various generalizations of the notion of convexity are known in the literature. One of them is the notion of d-convexity or convexity in Menger's sense (M-convexity) in a metric space (X,d) (cf. H. Busemann [1], В.П. Солтан [2], Menger [8], [9]). We shall confine ourselves to the notion of convexity in the so called G-space. In papers of В.П. Солтан [2]-[5] and J. Ger [6] the notion of convex function defined on a metric space was introduced. Below we recall those properties of G-spaces which will be useful in the sequel referring to [1] for further details.

Let (X,ρ) be a metric space and let $x,y,z \in X$ be three pairwise distinct points. We shall say that y lies between x and z and write $(x\ y\ z)$ if $\rho(x,z) = \rho(x,y) + \rho(y,z)$.

Definition 1 (cf. Busemann [1], Menger [9]). A metric space (X,ρ) is called M-convex (convex in Menger's sense) iff for every two distinct points $x,z \in X$ there exists a point $y \in X \setminus \{x,z\}$ such that $(x\ y\ z)$.

Definition 2 (cf. Busemann [1], Menger [9]). A metric space (X,ρ) is called finitely compact iff every bounded and infinite subset of X has at least one cluster point.

Alternatively, we say that (X,ρ) is finitely compact iff every bounded and closed subset of X is compact.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

H. Busemann [1] introduced and investigated the notion of a G-space defined as follows:

Definition 3 (cf. Busemann [1]). A finitely compact M-convex metric space (X,ρ) is called a G-space provided that:

1° for every point $p \in X$ there exists a positive number r_p such that for any two points x,y from the ball $K(p,r_p)$ centered at p and with radius r_p, there exists a point $z \in X$ such that $(x\ y\ z)$;

2° for any two distinct points $x,y \in X$ and any points $z_1,z_2 \in X$ such that $(x\ y\ z_1)$, $(x\ y\ z_2)$ and $\rho(y,z_1) = \rho(y,z_2)$ one has $z_1 = z_2$.

Condition 1° is called the axiom of local prolongability, (ALP); condition 2° expresses the uniqueness of prolongation.

In this paper the symbol X always denotes a G-space; the symbols $\mathbf{R}, \mathbf{Q}, \mathbf{N}$ will stand for sets of reals, rationals and positive integers, respectively.

Fix any two distinct points $x,y \in X$. Let $I:[0,\rho(x,y)] \longrightarrow X$ be an isometry such that $I(0) = x$, $I(\rho(x,y)) = y$ or $I(\rho(x,y)) = x$, $I(0) = y$. Then the set $T(x,y) := I([0,\rho(x,y)])$ is called a segment joining the points x and y. Any two distinct points in a G-space X may be joined by a segment contained in X (cf. [1], [9]). Such a segment need not be unique. If this segment is unique then there exists exactly one isometry $I:[0,\rho(x,y)] \xrightarrow{\text{onto}} T(x,y)$ and such that $I(0) = x$ and $I(\rho(x,y)) = y$ (see [6] Remark 3).

Definition 4 (cf. Busemann [1]). A set $D \subset X$ is called convex iff for every two distinct points $x,y \in \mathrm{cl}D$ the segment $T(x,y)$ is unique and $T(x,y) \subset D$ if $x,y \in D$.

Let us note that if D is convex then the sets clD and intD are convex, too. (cf. [1]).

Definition 5 (cf. J. Ger [6]). Let $x,y \in \mathrm{cl}D$, $x \neq y$. Assume $I:[0,\rho(x,y)] \longrightarrow T(x,y)$ to be an isometry such that $I(0) = x$ and $I(\rho(x,y)) = y$. For every $\lambda \in [0,1]$ we define

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\lambda x \oplus (1-\lambda)\, y := I((1-\lambda)\rho(x,y)).$$

Remark (cf. [6] Lemma 1). If $x,y \in \mathrm{cl}D$, $x \neq y$, $\lambda \in [0,1]$ and $z = \lambda x \oplus (1-\lambda)\, y$ then $\rho(z,x) = (1-\lambda)\rho(x,y)$ and $\rho(z,y) = \lambda\rho(x,y)$.

In the whole paper the symbol D denotes a non-empty, open and convex subset of X.

2. The following theorem holds true in any linear topological Baire space E: (cf. R. Ger [7]); if f is a J-convex function defined on an open and convex set $D_o \subset E$ and if the set

$$\mathrm{epi}\, f := \left\{ (x,y) \in D_o \times \mathbf{R} : f(x) \leq y \right\}$$

is closed in $D_o \times \mathbf{R}$ then f is continuous. The goal of the present paper is to show that this result carries over the case of G-spaces. We start with the following

Definition 6 (cf. [6]). A function $f: D \longrightarrow \mathbf{R}$ is called M-convex iff

(I)
$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)\, f(y)$$

for $x,y \in D$ and every $\lambda \in [0,1]$. A function $f: D \longrightarrow \mathbf{R}$ is JM-convex (Jensen M-convex) if (I) holds for all $x,y \in D$ and $\lambda = \frac{1}{2}$.

Now, we may prove the following

Lemma 1. If $f: D \longrightarrow \mathbf{R}$ is JM-convex and if its epigraph

$$\mathrm{epi}\, f := \left\{ (x,y) \in D \times \mathbf{R} : f(x) \leq y \right\}$$

is closed in $D \times \mathbf{R}$, then f is M-convex.

Proof. From Theorem in [6] we get the inequality

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)\, f(y)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

valid for every $\lambda \in [0,1] \cap \mathbb{Q}$ and all $x,y \in D$. It means that $(\lambda x \oplus (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in \operatorname{epi} f$. Let us take an arbitrary $\lambda_0 \in (0,1)$ and let $(\lambda_n)_{n\in N}$ be a rational sequence such that $\lambda_0 = \lim_{n\to\infty} \lambda_n$. Suppose that two distinct points x,y are fixed. From the fact that epi f is closed we obtain

$$\lim_{n\to\infty} (\lambda_n x \oplus (1-\lambda_n)y, \lambda_n f(x) + (1-\lambda_n) f(y)) \in \operatorname{epi} f;$$

or, equivalently, if I is an isometry from $[0,\rho(x,y)]$ onto $T(x,y)$ we get (see Definition 5):

$$\lim_{n\to\infty} (I(1-\lambda_n)\rho(x,y)), \lambda_n f(x) + (1-\lambda_n) f(y)) \in \operatorname{epi} f.$$

Therefore

$$(I((1-\lambda_0)\rho(x,y)), \lambda_0 f(x) + (1-\lambda_0) f(y)) \in \operatorname{epi} f,$$

or, in other words,

$$(\lambda_0 x \oplus (1-\lambda_0) y, \lambda_0 f(x) + (1-\lambda_0) f(y)) \in \operatorname{epi} f.$$

This means that

$$f(\lambda_0 x \oplus (1-\lambda_0)y) \leq \lambda_0 f(x) + (1-\lambda_0) f(y)$$

and ends the proof.

Lemma 2. Let $\lambda_0 \in (0,1)$ and $x_0 \in D$ be arbitrarily fixed points and let $r_0 = r(x_0)$ be the number occurring in Definition 3. If the function $\varphi: D \to X$ is given by the formula

(1) $$\varphi(x) := \lambda_0 x \oplus (1-\lambda_0)x_0, \quad x \in D,$$

then, for every $r \leq r_0$, we have

$$\varphi(K(x_0,r)) = K(x_0,\lambda_0 r).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. The mapping φ given by the formula (1) is a homeomorphism of D onto $\varphi(D)$ (see [6] Lemma 6). We shall show that for every $r \leq r_o$ the following inclusion

$$K(x_o, \lambda_o r) \subset \varphi(K(x_o, r))$$

holds. The opposite inclusion is fulfilled in view of Remark and formula (1). Let $z \in K(x_o, \lambda_o r) \setminus \{x_o\}$ be an arbitrarily fixed point and let $T(z, x_o)$ be the segment joining z and x_o. From Definition 3 we infer that there exists exactly one point y such that $(x_o\ z\ y)$, $z \in T(x_o, y)$ and $\rho(x_o, y) = \frac{1}{\lambda_o}\rho(z, x_o)$ since $\rho(x_o, y) = \frac{1}{\lambda_o}\rho(z, x_o) \leq \frac{1}{\lambda_o}\lambda_o r = r$. It means that $y \in K(x_o, r)$ and $\rho(z, x_o) = \lambda_o \rho(x_o, y)$. Therefore (see Remark) $z = \lambda_o y \oplus (1-\lambda_o)x_o$ and $z = \varphi(y) \in \varphi(K(x_o, r))$.

The following main result yields an analogue of the closed epigraph theorem proved by R. Ger in [7]. The point is that in our case no algebraic structure in the space considered is assumed. On the other hand one cannot treat our Theorem as a direct generalization of R. Ger's result from [7] because he had not assumed the metrizability of the underlying linear space and his functions were vector-valued. Both results however yield "convex analogues" of the classical Banach closed graph theorem.

Theorem. Let $f: D \to \mathbb{R}$ be a JM-convex function. If the epigraph of f is closed in $D \times \mathbb{R}$ then f is continuous in D.

Proof. Fix an $x_o \in D$ and put

$$A := \{x \in D: f(x) - f(x_o) \leq 1\}.$$

For an arbitrary $x \in D$ there exists an $n \in \mathbb{N}$ such that $f(x) - f(x_o) \leq 2^n$. From the fact that f is M-convex (see Lemma 1) we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$f\left(\frac{1}{2^n}x \oplus \left(1 - \frac{1}{2^n}\right)x_o\right) - f(x_o) \leq \frac{1}{2^n}f(x) + \left(1 - \frac{1}{2^n}\right)f(x_o) - f(x_o) \leq 1. \quad *)$$

Therefore, we have

(2) for every $x \in D$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^n}x \oplus \left(1 - \frac{1}{2^n}\right)x_o \in A$.

Let $\varphi_n : \mathrm{cl}D \longrightarrow X$ be a mapping given by the formula

$$\varphi_n(x) := \frac{1}{2^n}x \oplus \left(1 - \frac{1}{2^n}\right)x_o, \quad x \in \mathrm{cl}D. \tag{3}$$

By virtue of (2) we obtain the inclusion

$$D \subset \bigcup_{n\in\mathbb{N}} \varphi_n^{-1}(A \cap D_n) \quad \text{where} \quad D_n := \varphi_n(\mathrm{cl}D), \quad n \in \mathbb{N}.$$

Note that the function φ_n given by (3) has the form (1); consequently Lemma 2 may be applied. Since D is open and nonempty subset of a complete metric space, by the classical theorem of Baire, D is of the second Baire category whence

$$\mathrm{int}\ \mathrm{cl}\ \varphi_n^{-1}(A \cap D_n) \neq \emptyset \tag{4}$$

for some $n \in \mathbb{N}$. We are goint to show that

$$\mathrm{int}\ \mathrm{cl}(A \cap D_n) \neq \emptyset.$$

To this aim we shall first prove the following inclusion

$$\mathrm{cl}\varphi_n^{-1}(A \cap D_n) \subset \varphi_n^{-1}(\mathrm{cl}(A \cap D_n)). \tag{5}$$

---

*) This inequality may also be derived directly from the JM-convexity of f without using Lemma 1 (see Theorem 1 from [6]) because the coefficients occurring here are rational.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Indeed, take an $x \in cl\varphi_n^{-1}(A \cap D_n)$. Then there exists a sequence $(a_k)_{k\in\mathbb{N}}$, $a_k \in A \cap D_n$, $k \in \mathbb{N}$, such that $x = \lim_{k\to\infty} \varphi_n^{-1}(a_k)$. Let $b_k := \varphi_n^{-1}(a_k)$. Then $x = \lim_{k\to\infty} b_k$, and $\varphi_n(b_k) = a_k$. From (3) we have $a_k = \frac{1}{2^n} b_k \oplus \left(1 - \frac{1}{2^n}\right) x_o$. We have also $a_k \in T(b_k, x_o)$ and $\rho(a_k, x_o) = \frac{1}{2^n}\rho(b_k, x_o)$. Since $\lim_{k\to\infty} b_k = x$ we have $\lim_{k\to\infty} a_k = a$ and $a = \frac{1}{2^n} x \oplus \left(1 - \frac{1}{2^n}\right) x_o$ (see [6] Corollary 1) and from (3) we get $a = \varphi_n(x)$ or, equivalently, $x = \varphi_n^{-1}\left(\lim_{k\to\infty} a_k\right) = \varphi_n^{-1}(a) \in \varphi_n^{-1}(cl(A \cap D_n))$.

From Lemma 2 we obtain that φ_n is an open mapping, and so

$$\text{int } \varphi_n^{-1}(cl(A \cap D_n)) \subset \varphi_n^{-1}(\text{int } cl(A \cap D_n)).$$

This, (4) and inclusion (5) imply that

$$\emptyset \neq \text{int } cl(\varphi_n^{-1}(A \cap D_n)) \subset \text{int } \varphi_n^{-1}(cl(A \cap D_n)) \subset \varphi_n^{-1}(\text{int } cl(A \cap D_n)).$$

Therefore

$$U := \text{int } cl(A \cap D_n) \neq \emptyset.$$

Now, we shall prove that the set $A \cap D_n$ is closed in D. To show this let us fix a $z \in D \setminus (A \cap D_n)$; then $(z, f(x_o)+1) \in (D \times \mathbb{R}) \setminus \text{epi } f$. From the fact that the set $(D \times \mathbb{R}) \setminus \text{epi } f$ is open we get the existence of a neighbourhood U_z of the point z and a number $\delta > 0$ such that $(U_z \times (f(x_o)+1-\delta,\ f(x_o)+1+\delta)) \subset (D \times \mathbb{R}) \setminus \text{epi } f$. So, for every $x \in U_z$, we have $(x, f(x_o)+1) \in (D \times \mathbb{R}) \setminus \text{epi } f$ whence $f(x) > f(x_o)+1$. Consequently, $x \in D \setminus A$ which means that $(A \cap D_n)$ is closed in D. Moreover

$$\emptyset \neq U \cap A \cap D_n = U \cap D \cap cl(A \cap D_n) = U \cap D$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and $U \cap D$ is an open, nonempty subset of A. We have shown that int $A \neq \emptyset$. The function f is M-convex and upper bounded on A. From Corollary 2 in [6] we obtain that f is continuous in A.

Corollary. If $f: D \to R$ is JM-convex and lower semicontinuous in D then f is M-convex and continuous in D.

The proof follows from Lemma 1, Theorem and the fact that if f is lower semicontinuous function then epi f is closed (see R. Sikorski [10], exercise 5, p.131).

REFERENCES

[1] H. Busemann: The geometry of geodesic. Academic Press, New York, 1955.

[2] В.П. Солтан: Введение в аксиоматическую теорию выпуклости. Кишинев, 1984.

[3] В.П. Солтан: Аксиоматический подход к теории выпуклых функций, Dokl. Akad. Nauk SSSR 254 (1980), 813-816.

[4] В.П. Солтан, П.С. Солтан: d-выпуклые функции, Dokl. Akad. Nauk SSSR 240 (1979), 555-558.

[5] В.П. Солтан, В.Д. Ченой: Некоторые классы d-выпуклых функций в графе, Dokl. Akad. Nauk SSSR 273 (1983), 1314-1317.

[6] J. Ger: Convex functions in metric spaces, Radovi Mat. 2 (1986), 217-236.

[7] R. Ger: Convex transformations with Banach lattice range, Stochastica 11 (1987), 13-23.

[8] K. Menger: Ergebnisse eines math. Kolloq., Wien, v.1 (931).

[9] K. Menger: Untersuchungen uber allgemeine Metrik I, II, III, Math. Ann. 100 (1928), 75-163.

[10] R. Sikorski: Funkcje rzeczywiste, Tom I. Monografie Mat. 35, PWN (Polish Scientific Publishers), Warszawa 1958.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY IN KATOWICE, 40-007 KATOWICE, POLAND

Received January 6, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol XXIII No 2 1990

Danuta Przeworska-Rolewicz

REMARKS ON MULTIPLIERS FOR RIGHT INVERSES IN D-ALGEBRAS

Detailed studies of multipliers with integral operators in convolutional algebras were given recently by Dimovski [2] and Bozhinov [1]. The purpose of the present note is to indicate that some of these results can be obtained without any topological assumptions in a rather general form.

Denote by $L(X)$ the set of all linear operators with domains and ranges in a linear space X over a field $\mathcal{F}$ of scalars and by $L_o(X)$ the set of operators $A \in L(X)$ with $\operatorname{dom} A = X$.

Through the paper we shall admit the following assumptions, denoted by (A) for the sake of brevity:

(A) Let X be a D-algebra, i.e. a commutative algebra (over $\mathcal{F}$) with a right invertible operator $D \in L(X)$ such that $\operatorname{dom} D$ is subalgebra of X:

$$x, y \in \operatorname{dom} D \text{ implies } xy \in \operatorname{dom} D.$$

We assume that $\ker D \neq \{0\}$. Let F be an initial operator for D corresponding to a right inverse R of D which has property

(1) $$\exists_{0 \neq r \in \ker D} \ \forall_{x \in X} \ Rx = rx \quad (\operatorname{dom} R = X).$$

(for all notations connected with right invertible operators and their properties, cf. the author [3]).

Formula (1) implies that

(2) $$R(xy) = xRy = yRx.$$

Indeed, $R(xy) = r(xy) = xry = xRy$. Similarly, $R(xy) = R(yx) = yRx$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1. Suppose that (A) holds and that $M \in L_o(X)$ satisfies the condition

$$M(rx) = mx \quad \text{for all} \quad x \in X, \quad \text{where} \quad m = Mr. \tag{3}$$

(i) If $MR = RM$ then M is of the form

$$Mx = D(mx) \quad \text{for all} \quad x \in X; \tag{4}$$

(ii) If M is of the form (4) then $MR = RM$ if and only if

$$F(mx) = 0 \quad \text{for all} \quad x \in X. \tag{5}$$

Proof. (i) Suppose that $MR = RM$. Condition (3) implies that for all $x \in X$ we have

$$0 = (MR - RM)x = MRx - RMx = M(rx) - RMx = mx - RMx,$$

i.e. $mx = RMx$. Hence $mx \in \operatorname{dom} D$ for all $x \in X$ and $Mx = DRMx = D(mx)$. (ii) Suppose that M is of the form (4). Then for all $x \in X$ we have

$$(MR - RM)x = M(rx) - RD(mx) = M(rx) - (I-F)(mx) =$$

$$= mx - mx + F(mx) = F(mx).$$

Thus $MR = RM$ if and only if $F(mx) = 0$ for all $x \in X$.

Corollary 1. Suppose that (A) holds and that $M \in L_o(X)$ is a multiplier, i.e.

$$M(xy) = xMy = yMx \quad \text{for all} \quad x, y \in X. \tag{6}$$

Then M satisfies (i) and (ii) of Theorem 1. Indeed, by (6), we have $M(rx) = xMr = xm = mx$ for all $x \in X$, i.e. (3) is satisfied.

Theorem 2. Suppose that (A) holds, $M \in L_o(X)$ satisfies (3) and $MR = RM$. Then $\ker M = \{0\}$ if and only if $m = Mr$ is not a zero divisor.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. By our assumptions, if $Mx = 0$ then $D(mx) = 0$. This implies that $mx = z \in \ker D$. By (5), we get $z = Fz = F(mx) = 0$. Hence $mx = 0$. If m is not a zero divisor, we find $x = 0$ and $\ker M = \{0\}$. If m is a zero divisor then there exists an $x \neq 0$ such that $RMx = MRx = M(rx) = mx = 0$. This implies that $Mx = DRMx = 0$, i.e. $0 \neq x \in \ker M$. Hence $\ker M \neq \{0\}$.

Corollary 2. Suppose that all assumptions of Theorem 2 are satisfied. Then the equation

$$Mx = y, \tag{7}$$

where $y \in X$ is arbitrary, has at most a unique solution.

Theorem 1 can be generalized for higher powers of R in the following manner.

Theorem 3. Suppose that (A) holds, $M \in L_o(X)$ satisfies (3). Let $n \in \mathbb{N}$ be arbitrarily fixed. Then

(i) If $MR^n = R^nM$ then $Mx = D(mx)$ for all $x \in X$, where $m = Mr$ (i.e. M is of the form (4));

(ii) If M is of the form (4) then $MR^n = R^nM$ if and only if $F(mx) = 0$ for all $x \in X$ (i.e. (5) is satisfied).

Proof. (i) By an easy induction we prove that

$$R^nx = r^nx \quad \text{for all} \quad x \in X, \ n \in \mathbb{N}. \tag{8}$$

This implies

$$0 = (MR^n - R^nM)x = MR^nx - R^nMx = M(r^nx) - R^nMx =$$
$$= r^{n-1}xMr - R^nMx = r^{n-1}mx - R^nMx = R^{n-1}(mx) - R^nMx.$$

As before, we conclude that $mx \in \operatorname{dom} D$ for x X and

$$Mx = D^nR^nMx = D^nR^{n-1}(mx) = D(mx) \quad \text{for} \quad x \in X.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(ii) Since M is of the form (4), we find for $x \in X$

$$(MR^n - R^nM)x = M(r^n x) - R^n D(mx) = r^{n-1} xMr - r^{n-1} RD(mx) =$$
$$= r^{n-1} mx - r^{n-1}(I-F)(mx) = r^{n-1} mx - r^{n-1} mx + F(mx) = F(mx).$$

Therefore $MR^n = R^nM$ if and only if $F(mx) = 0$ for all $x \in X$.

Theorem 4. Suppose that condition (A) holds and there exists a $d \in \mathbb{R}$ such that D satisfies a Duhamel-like condition

$$D(xy) = xDy + dxFy \quad \text{for all} \quad x \in X, \ y \in \operatorname{dom} D. \tag{9}$$

Then

$$F(xy) = F(xFy) \quad \text{for all} \quad x \in X, \quad y \in \operatorname{dom} D. \tag{10}$$

Proof. Let $x \in X$, $y \in \operatorname{dom} D$ be arbitrary. By definition of F, (2) and (9), we find

$$F(xy) = (I-RD)xy = xy - R[xDy + dxFy] = xy - R(xDy) -$$
$$- dR(xFy) = xy - xRdy - drxFy = xFy - drxFy = (1-dr)xFy.$$

i.e.

$$F(xy) = (1-dr)xFy \quad \text{for all} \quad x \in X, \quad y \in \operatorname{dom} D. \tag{11}$$

If we put in (11) Fy instead of y, we obtain

$$F(xFy) = (1-dr)xF^2y = (1-dr)xFy = F(xy).$$

Formula (11) immediately implies

Corollary 3. Suppose that all assumptions of Theorem 4 are satisfied. Then

$$F(zx) = (1-dr)zFx \quad \text{for all} \quad z \in \ker D, \quad x \in \operatorname{dom} D. \tag{12}$$

Corollary 4. Suppose that all assumptions of Theorem 4 are satisfied and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(1-dr)x = 0 \quad \text{for all} \quad x \in X. \tag{13}$$

Then

$$F(zx) = 0 \quad \text{for all} \quad z \in \ker D, \quad x \in X. \tag{14}$$

Example 1. Suppose that $X = C(R_+)$, $D = \frac{d}{dt}$, $R = \int_0^t$, $(Fx)(t) = x(0)$. Define the multiplication in X as the convolution

$$(x * y)(t) = \int_0^t x(t-s)y(s)ds \quad \text{for} \quad x,y \in X. \tag{15}$$

Clearly, X is a D-algebra, dom $D = C^1(R_+)$ is a subalgebra of X, $\ker D \neq \{0\}$. Moreover

$$(Rx)(t) = \int_0^t x(s)ds = x * r, \quad \text{where} \quad r(t) \equiv 1.$$

It is also well-known that condition (9) is satisfied with $d = 1$. Thus in this case we have $1 - dr = 0$, i.e. condition (13) is satisfied.

Note that, by the Titchmarsh theorem, X is an algebra without zero divisors.

Corollary 5. Suppose that all assumptions of Theorem 4 are satisfied. Then

$$F(xy) = F[(Fx)(Fy)] \quad \text{for} \quad x,y \in \text{dom } D. \tag{16}$$

Proof. If we put in (10) Fx instead of x, we find $F(yFx) = F[(Fx)(Fy)]$. But, again by (10), we have $F[(Fx)(Fy)] = F(yFx) = F(xy)$.

Theorem 5. Suppose that condition (A) holds, D satisfies (9), $M \in L_0(X)$ and $MR = RM$. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $m \in \operatorname{dom} D$ and $Dm = dm$. In particular, if $d = 0$ then $m \in \ker D$;

(ii) If $d \neq 0$ and equation (7) has a unique solution for every $y \in X$ then $Fm \neq 0$.

Proof. (i) By definition, $m = Mr = D(mr) = mDr + dmFr = dmr = drm = dRm$ (since $Dr = 0$ and $Fr = r$). Hence $m \in \operatorname{dom} D$ and $Dm = dm$. If $d = 0$ then $Dm = 0$.

(ii) By Corollary 2, m is not a zero divisor. If we put in condition (5) Dx instead of x, we obtain $F(mDx) = 0$ for all $x \in \operatorname{dom} D$. Let $y \in X$ be arbitrary and let $x \in \operatorname{dom} D$ be the unique solution of (7). Theorem 4 and Corollary 5 together imply that

$$Fy = FMx = FD(mx) = F(mDx + dmFx) = F(mDx) + dF(mFx) =$$

$$= dF(mFx) = dF[(Fm)(Fx)].$$

If $Fm = 0$ then $Fy = 0$. This is a contradiction with our assumption that $y \in X$ is arbitrary. Hence $Fm \neq 0$.

Corollary 6. Suppose that all assumptions of Theorem 5 are satisfied. If $d \neq 0$ then equation (7) can be written in an equivalent form

$$\left(1 - \frac{d}{2}r\right)mx = \frac{1}{2d}y, \quad \text{where} \quad y \in X \quad \text{is arbitrary.} \tag{17}$$

Proof. By Theorem 5, $Dm = dm$, $Fm \neq 0$. By definition, $Fm = (1-RD)m = m - rDm = m - drm = (1 - dr)m$. Hence for $x \in X$ we have $Mx = D(mx) = xDm + dxFm = x(dm) + dx(1 - dr)m = d(2 - dr)mx = 2d\left(1 - \frac{d}{2}r\right)m$, which implies (17).

Corollary 7. Suppose that assumptions of Theorem 5 are satisfied and $d \neq 0$. If the operator $I - \frac{d}{2}R$ is invertible then equation (17) can be written in an equivalent form

$$mx = \frac{1}{2d}\left(I - \frac{d}{2}R\right)^{-1} y \quad \text{for all} \quad y \in X. \tag{17'}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. By definition, we have $(1 - \frac{d}{2}r)u = (I - \frac{d}{2}R)u$ for all $u \in X$. This and the invertibility of the operator $I - \frac{d}{2}R$ together imply (17').

Proposition 1. Suppose that an operator $D \in L(X)$ satisfies condition (9), $A, B \in L_o(X)$ are multipliers and $Az = z$ for all $z \in \ker D$. Then the operator $D_1 = DA + B$ also satisfies condition (9), i.e.

$$D_1(xy) = yD_1x + dyF_1x \quad \text{for all} \quad x \in \operatorname{dom} D, \quad y \in X,$$

where $F_1 = FA$ is a projection onto $\ker D$ (in particular, $A = I$).

Proof. Since $F_1z = FAz = Fz = z$ for all $z \in \ker D$, we conclude that F_1 is a projection onto $\ker D$. By our assumptions, for all $x \in \operatorname{dom} D$, $y \in X$ we have

$$D_1(xy) = DA(xy) + B(xy) = D(yAx) + yBx = yDAx + d(FAx)y +$$

$$+ yBx = y(DA + b)x + d(FAx)y = yD_1x + dyF_1x.$$

Condition (9) can be written in a symmetric form

$$D(xy) = \frac{1}{2}(xDy + yDx) + \frac{d}{2}(xFy + yFx) \quad \text{for all } x, y \in \operatorname{dom} D. \tag{18}$$

Indeed, the commutativity of X and condition (9) together imply that for $x, y \in \operatorname{dom} D$ we have

$$D(xy) = \frac{1}{2}[D(xy) + D(yx)] = \frac{1}{2}[yDx + dyFx + xDy + dxFy] =$$

$$= \frac{1}{2}(xDy + yDx) + \frac{d}{2}(xFy + yFx).$$

Formula (18) shows that X is a D-algebra with $c_D = \frac{1}{2}$ and with the non-Leibniz component $f_D(x,y) = \frac{d}{2}(xFy + yFx)$. (cf. [3], Chapter 6).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Note. If dim ker $D = 1$ then ker $D = \lin\{r\}$ and $Fx = \varphi(x)r$ for $x \in X$, where φ is a linear functional defined on X and such that $\varphi(r) = 1$.

Theorem 6. Suppose that all assumptions of Theorem 5 are satisfied and $m = Mr$ is not a zero divisor. Consider an initial value problem

$$Dx - ax = y, \quad Fx = x_0, \tag{19}$$

where $a, y \in X$, $x_0 \in \ker D$ are given. Then $x = rv + x_0$ is a solution of (19) if and only if v is a solution of the equation

$$(1 - ra)v = y + ax_0. \tag{20}$$

Both equations have simultaneously a unique solution (or not).

Proof. Consider the auxiliary problem

$$Du = y, \quad Fu = x_0, \tag{21}$$

where $y \in X$, $x_0 \in \ker D$ are given. This problem has a unique solution $u = ry + x_0$. Indeed, write $w = u - x_0$. Then $Fw = Fu - Fx_0 = x_0 - x_0 = 0$, $Dw = D(u - x_0) = Du - Dx_0 = Du = y$. By (9), we get $Mw = D(mw) = mDw + dmFw = mDw = my$. By (5), we have $F(mw) = 0$. Thus, we obtain $mw = mw - F(mw) = (I - F)(mw) = RD(mw) = RMw = rMw = rmy$. This implies that $m(w - ry) = 0$. Since m is not a zero divisor, we conclude that $w - ry = 0$, i.e. $w = ry$ and $u = w + x_0 = ry + x_0$.

If v satisfies the equation (20) and $x = rv + x_0$ then $Dx = D(rv + x_0) = DRv + Dx_0 = v$, $Fx = F(rv + x_0) = FRv + Fx_0 = x_0$ and $Dx - ax = v - a(rv + x_0) = (1 - ar)v - ax_0 = y + ax_0 - ax_0 = y$. Hence, x is a solution of the problem (19). On the other hand, in order to solve (19), write $u = (1-ra)x$. Then we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$Du = Dx - D(rax) = Dx - DR(ax) = Dx - ax = y,$$

$$Fu = F(x - rax) = Fx - FR(ax) = x_o.$$

Hence, u is a solution of the problem (21), i.e. $u = ry + x_o$. By (9), we find

$$Mu = D(mu) = mDu + dmFu = my + dmx_o = m(y + dx_o).$$

But, by our definition, if we write $v = Dx$, we get

$$m(y+dx_o) = Mu = M(x-rax) = Mx - M(rax) = D(mx) - D(mrax) =$$

$$= mDx + dmFx - mraDx - dmraFx = (m-mra)(Dx+dFx) =$$

$$= (m-mra)(v+dx_o) = m(1-ra)(v+dx_o),$$

which implies

$$0 = m[(1-ra)(v+dx_o) - (y+dx_o)] = m[(1-ra)v - y - d(rax_o)].$$

Since m is not a zero divisor, we conclude that

$$(1-ra)v = y+drax_o = y + ax_o\,Dr + dax_o\,Fr = y + D(rax_o) =$$

$$= y + DR(ax_o) = y + ax_o$$

since $Dr = 0$ and $Fr = r$, by our assumption that r ker D. This means that v satisfies (20). Since $v = Dx$, we have $x = Rv + Fx = rv + x_o$. This and (20) together imply that x is a solution of (19).

Theorem 7. Suppose that all assumptions of Theorem 4 are satisfied and $d = 0$. Then

(22) $$r^n \in \ker D^n \quad \text{for every} \quad n \in \mathbb{N}.$$

Proof. By our assumption that $d = 0$, we get from (9) that $D(zx) = zDx$ for all $z \in \ker D$, $x \in X$. This implies

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(23) $$D^n(zx) = zD^n x \quad \text{for all} \quad z \in \ker D, \quad x \in X, \quad n \in \mathbb{N}.$$

By our assumption, $r \in \ker D$, i.e. $Dr = 0$. Suppose that $D^n r^n = 0$ for an arbitrarily fixed $n \geq 2$. Then, by (23), we obtain that

$$D^{n+1} r^{n+1} = D^{n+1}(r \cdot r^n) = rD^{n+1} r^n = rD(D^n r^n) = 0$$

which proves (22).

Theorem 7 permits us to apply all previous considerations to higher powers of R^n in another way. Namely, we have

Corollary 8. Suppose that D, R, F satisfy condition (A) and (9) with $d = 0$, i.e. we have

$$D(xy) = xDy \quad \text{for} \quad x \quad X, \; y \quad \operatorname{dom} D.$$

Let $n \in \mathbb{N}$ be arbitrarily fixed. Then the operators $D_o = D^n$, $R_o = R^n$ and

$$F_o = \sum_{k=0}^{n-1} R^k F D^k$$

satisfy also condition (A) with $r_o = r^n$.

Proof. Theorem 7 implies that $r_o = r^n \in \ker D_o = \ker D^n$. By the Taylor formula, an initial operator F_o for $D_o = D^n$ corresponding to $R_o = R^n$ is

$$F_o = I - R_o D_o = I - R^n D^n = \sum_{k=0}^{n-1} R^k F D^k.$$

By (8), we have $R_o x = R^n x = r^n x = r_o x$ for all $x \in X$.

To conclude our considerations, we should point out that the assumption (1) is, indeed, not very restrictive. This is shown by

Example 2. Suppose that X, D, R, F are defined as in Example 1. We have there: $r(t) \equiv 1$ and $d = 1$. Also, since $\dim \ker D = 1$, we may write $\ker D = \operatorname{lin}\{r\}$. Suppose that $R_1 \neq R$ is a right inverse of D. Then an initial operator F_1 corresponding to R_1 is $F_1 x = \varphi(x) r$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $x \in X$, where φ is a linear functional defined on X such that $\varphi(r)=1$. This implies that $R_1 x = Rx - \varphi(Rx)r = Rx - \varphi(rx)r$ for $x \in X$.

Consider now the so-called Berg-Dimovski convolution determined by the functional φ:

$$(24) \qquad (x \underset{\varphi}{*} y)(t) = \varphi_\tau \left\{ \int_\tau^t x(t+\tau+\sigma)y(\sigma)d\sigma \right\} \quad \text{for} \quad x,y \in X$$

(cf. [2], [1]), where the subscript "τ" means that the functional φ_τ acts on the variable τ. Again X is a D-algebra with respect to the multiplication defined by (24). And we also have

$$(r \underset{\varphi}{*} x)(t) = \varphi_\tau \left\{ \int_\tau^t x(\sigma)d\sigma \right\} = \varphi_\tau \left\{ \int_0^t x(\sigma)d\sigma - \int_0^\tau x(\sigma)d\sigma \right\} =$$

$$= \int_0^t x(\sigma)d\sigma - \varphi_\tau \left\{ \int_0^\tau x(\sigma)d\sigma \right\} = [Rx - \varphi(rx)r](t) = (R_1 x)(t).$$

This means that again condition (1) is satisfied.

REFERENCES

[1] N. Bozhinov: Convolutional representations of commutators and multipliers. Publish. House of the Bulg. Acad. Sci., Sofia, 1988.

[2] I.H. Dimovski: Convolutional calculus. Publish. House of the Bulg. Acad. Sci., Sofia, 1982; 2 nd ed. Kluwer Acad. Publishers, Dordrecht, 1990.

[3] D. Przeworska-Rolewicz: Algebraic analysis. PWN - Polish Scientific Publishers and D. Reidel, Warszawa-Dordrecht, 1988.

INSTITUTE OF MATHEMATICS POLISH ACADEMY OF SCIENCES,
WARSZAWA

Received April 12, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Józef Sidorowicz

О УСЛОВНОМ ПРОЦЕССЕ С ДИФРАКЦИЕЙ НА ГЛАДКОЙ ОБЛАСТИ В R^2

В [1], [2] была получена уточненная логарифмическая асимптотика плотности $p_D^{\pm}(t,x,y)$ винеровского процесса w_s, $0 \leqslant s \leqslant t$, с дифракцией на гладкой и выпуклой области $D \subset R^2$ либо $D \subset R^{\nu}$. Здесь мы изучим двумерный условный процесс $\hat{w}(s)$, $0 \leqslant s \leqslant t$, при $\hat{w}(0) = \left(0, b_0 t^{\frac{2}{3}}\right) = x$, $\hat{w}(t) = \left(\varrho, b_1 t^{\frac{2}{3}}\right) = y$ и $t \to 0$ предполагая, что точки x,y находятся в зоне "грубокой тени" по отношению к D, т.е. отрезок $[x,y]$ пересекает область D. Вначале исследуем общий случай, когда ∂D - произвольный гладкий контур. В частном случае, когда ∂D - постоянной кривизны получаем наиболее интересное следствие - теоремы об отсутствии гауссовости у условного процесса $\hat{w}(s)$, $0 \leqslant s \leqslant t$, по направлению нормали к ∂D.

Перейдем к постановке задачи. Пусть $\vec{\gamma}(u)$ - уравнение контура ∂D, где $u = \delta\varrho$ - натуральный параметр, $\delta = \frac{s}{t}$, $\varrho = |\widetilde{AB}|$ (см. рис. 1). Очевидно, что любая точка $z \in R^2 \setminus D$ определяется

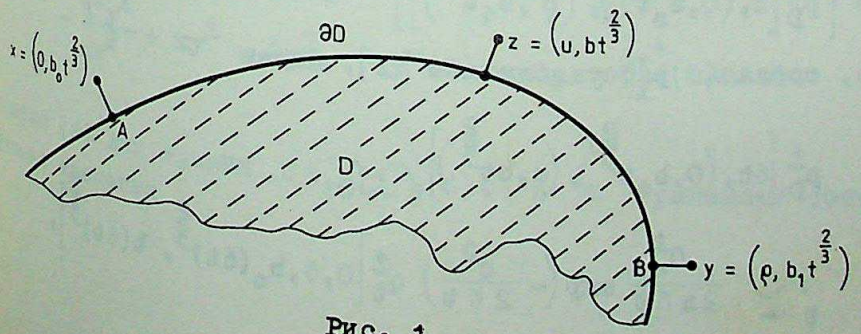


Рис. 1

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

уравнением $\vec{z} = \vec{\gamma}(u) - v\,\vec{n}(u)$, где $\vec{n}(u)$ - единичный вектор нормальный к ∂D в точке u; v - евклидовое расстояние по нормали к ∂D.

Введём систему координат (a,b) (a - вдоль оси u, b - вдоль оси v) с началом в точе $\vec{\gamma}(u)$ и изучим условный процесс

$$w^*(s) = \left[w_1^*(s), w_2^*(s)\right] = \left[\left(\bar{w}_1(s) - \frac{s}{t}\rho\right)t^{-\frac{1}{2}}, \bar{w}_2(s)t^{-\frac{2}{3}}\right],$$

где $\bar{w}_1(s)$ - проекция процесса $\hat{w}(s)$ на ось u; $\bar{w}_2(s)$ - на ось v.

Предположим, что точки x,z и точки z,y находятся в зоне "глубокой тени" по отношению к D. По определению условной вероятности P^* (см. [3]) процесса w_s^*, принимая

$$(*) \qquad c := б\rho + a\sqrt{t},$$

получим

$$P^*\left\{|w_1^*(s)| < a + da,\ |w_2^*(s)| < b + db\right\} = P_z\left\{w_t \in y + dy\right\} \times$$

$$\times P_x\left\{\hat{w}(бt) \in \vec{\gamma}(б\rho) + \left[(a + da)\,t^{\frac{1}{2}},\ (b + db)\,t^{\frac{2}{3}}\right]\right\} \times$$

$$\times \left\{P_x\left\{\hat{w}(t) \in y + dy\right\}\right\}^{-1} = p_D^{\pm}\left[бt, \left(0, b_0 t^{\frac{2}{3}}\right), \left(c, bt^{\frac{2}{3}}\right)\right] \times$$

$$\times p_D^{\pm}\left[(1 - б)t, \left(0, bt^{\frac{2}{3}}\right), \left(\rho - c,\ b_1 t^{\frac{2}{3}}\right)\right] \times$$

$$\times \left\{p_D^{\pm}\left[t, \left(0, b_0 t^{\frac{2}{3}}\right), \left(\rho, b_1 t^{\frac{2}{3}}\right)\right]\right\}^{-1} t^{\frac{7}{6}} da\,db.$$

Далее, согласно рассуждениям в [1], имеем

$$(1a) \qquad p_D^{\pm}\left[бt, \left(0, b_0 t^{\frac{2}{3}}\right), \left(c, bt^{\frac{2}{3}}\right)\right]_{t \to 0}$$

$$\sim \frac{c_1^{\pm}}{2\pi б t} \exp\left(-\frac{c^2}{2бt}\right) q_t^{\pm}\left[0, б, b_0 (бt)^{\frac{2}{3}},\ b(бt)^{\frac{2}{3}}\right],$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(1b)\quad p_D^{\pm}\left[(1-\delta)t,\ \left(0,bt^{\frac{2}{3}}\right),\ \left(\rho-c,b_1t^{\frac{2}{3}}\right)\right]_{t\to 0}\sim \frac{c_2^{\pm}}{2\pi(1-\delta)t}\times$$

$$\times\exp\left[-\frac{(\rho-a)^2}{2(1-\delta)t}\right]q_t^{\pm}\left[\delta,1,b\left((1-\delta)t\right)^{\frac{2}{3}},\ b_1\left((1-\delta)t\right)^{\frac{2}{3}}\right],$$

$$(1c)\quad p_D^{\pm}\left[t,\left(0,b_0t^{\frac{2}{3}}\right),\left(\rho,b_1t^{\frac{2}{3}}\right)\right]_{t\to 0}\sim \frac{c_3^{\pm}}{2\pi t}\exp\left(-\frac{\rho^2}{2t}\right)\times$$

$$\times q_t^{\pm}\left(0,1,b_0t^{\frac{2}{3}},b_1t^{\frac{2}{3}}\right).$$

По лемме в [1] при $\xi_1 = xt^{-\frac{1}{2}}$, $\xi_2 = yt^{-\frac{1}{2}}$ имеем

$$(2)\quad q_t^{\pm}(\delta_1,\delta_2,\xi_1,\xi_2)\underset{t\to 0}{\sim}\left[2\rho^2K(\delta_2\rho)\right]^{\frac{1}{3}}t^{-\frac{1}{6}}\times$$

$$\times\hat{y}_1^{\pm}\left[\left(2\rho^2K(\delta_1\rho)\right)^{\frac{1}{3}}t^{-\frac{2}{3}}x\right]\hat{y}_1^{\pm}\left[\left(2\rho^2K(\delta_2\rho)\right)^{\frac{1}{3}}t^{-\frac{2}{3}}y\right]\times$$

$$\times\left[\frac{K(\delta_2\rho)}{K(\delta_1\rho)}\right]^{\frac{1}{3}}c_0^{\pm}\exp\left[-\lambda_1^{\pm}\left(\frac{\rho}{2t}\right)^{\frac{1}{3}}\int_{\delta_1\rho}^{\delta_2\rho}K^{\frac{2}{3}}(u)du\right],$$

где $\hat{y}_1^{\pm}$ и $\lambda_1^{\pm}$ - нормированная младшая собственная функция и соответствующее собственное значение задачи

$$-\frac{d^2y^{\pm}}{dx^2}+xy^{\pm}=\lambda y^{\pm},\quad y^{-}(0)=0,\quad \frac{dy^{+}}{dx}(0)=0.$$

Формула (2) для времён δt, $(1-\delta)t$ и t принимает соответственно значения

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3a)\qquad q^{\pm}_{\sigma t}\Big[0,\sigma,b_0(\sigma t)^{\frac{2}{3}},b(\sigma t)^{\frac{2}{3}}\Big]\underset{t\to 0}{\sim}\Big[2\sigma^2\rho^2K(\sigma\rho)\Big]^{\frac{1}{3}}\times$$

$$\times(\sigma t)^{-\frac{1}{6}}\,\hat{y}^{\pm}_1\Big[b_0\big(2\sigma^2\rho^2K(0)\big)^{\frac{1}{3}}\Big]\,\hat{y}^{\pm}_1\Big[b\big(2\sigma^2\rho^2K(\sigma\rho)\big)^{\frac{1}{3}}\Big]\times$$

$$\times\left[\frac{K(\sigma\rho)}{K(0)}\right]^{\frac{1}{3}}C^{\pm}_0\exp\left[-\lambda^{\pm}_1\Big(\frac{c}{2\sigma t}\Big)^{\frac{1}{3}}\int_0^c K^{\frac{2}{3}}(u)\,du\right],$$

$$(3b)\qquad q^{\pm}_{(1-\sigma)t}\Big[\sigma,1,b\big((1-\sigma)t\big)^{\frac{2}{3}},b_1\big((1-\sigma)t\big)^{\frac{2}{3}}\Big]\underset{t\to 0}{\sim}\times$$

$$\times\underset{t\to 0}{\sim}\Big[2(1-\sigma)^2\rho^2K(\rho)\Big]^{\frac{1}{3}}\Big[(1-\sigma)t\Big]^{-\frac{1}{6}}\times$$

$$\times\,\hat{y}^{\pm}_1\Big[b\big(2\,(1-\sigma)^2\rho^2K(\sigma\rho)\big)^{\frac{1}{3}}\Big]\,\hat{y}^{\pm}_1\Big[b_1\big(2(1-\sigma)^2\rho^2K(\rho)\big)^{\frac{1}{3}}\Big]\times$$

$$\times\left[\frac{K(\rho)}{K(\sigma\rho)}\right]^{\frac{1}{3}}C^{\pm}_0\exp\left[-\lambda^{\pm}_1\Big(\frac{\rho-c}{2(1-\sigma)t}\Big)^{\frac{1}{3}}\int_c^{\rho}K^{\frac{2}{3}}(u)\,du\right],$$

$$(3c)\qquad q^{\pm}_t\big(0,1,b_0t^{\frac{2}{3}},b_1t^{\frac{2}{3}}\big)\underset{t\to 0}{\sim}\Big[2\rho^2K(\rho)\Big]^{\frac{1}{3}}t^{-\frac{1}{6}}\times$$

$$\times\,\hat{y}^{\pm}_1\Big[b_0\big(2\rho^2K(0)\big)^{\frac{1}{3}}\Big]\,\hat{y}^{\pm}_1\Big[b_1\big(2\rho^2K(\rho)\big)^{\frac{1}{3}}\Big]\times$$

$$\times\left[\frac{K(\rho)}{K(0)}\right]^{\frac{1}{3}}C^{\pm}_0\exp\left[-\lambda^{\pm}_1\Big(\frac{\rho}{2t}\Big)^{\frac{1}{3}}\int_0^{\rho}K^{\frac{2}{3}}(u)\,du\right].$$

Учитывая результаты (1) и (3), плотность $\hat{p}^{\pm}_t$ условной вероятности P^* процесса $w(s)$ имеет вид

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\hat{p}_t^{\pm}\{\cdot\} \underset{t\to 0}{\sim} \frac{\left[2\rho^2 K(\delta\rho)\right]^{\frac{1}{3}} C_D^{\pm}}{2\pi\sqrt{\delta(1-\delta)}} \exp\left[-\frac{a^2}{2\delta(1-\delta)}\right] \times$$

$$\times \hat{y}_1^{\pm}\left[b_0\left(2\delta^2\rho^2 K(0)\right)^{\frac{1}{3}}\right] y_1^{\pm}\left[b\left(2\delta^2\rho^2 K(\delta\rho)\right)^{\frac{1}{3}}\right] \times$$

$$\times \hat{y}_1^{\pm}\left[b\left(2(1-\delta)^2\rho^2 K(\delta\rho)\right)^{\frac{1}{3}}\right] \hat{y}_1^{\pm}\left[b_1\left(2(1-\delta)^2\rho^2 K(\delta\rho)\right)^{\frac{1}{3}}\right] \times$$

$$\times \left\{\hat{y}_1^{\pm}\left[b_0\left(2\rho^2 K(0)\right)^{\frac{1}{3}}\right] \hat{y}_1^{\pm}\left[b_1\left(2\rho^2 K(\rho)\right)^{\frac{1}{3}}\right]\right\}^{-1}.$$

Заметим, что выражение в экспоненте (см. (3а - с))

$$g(a) := -\lambda_1^{\pm}\left[\left(\frac{c}{2\delta t}\right)^{\frac{1}{3}} \int_0^c + \left(\frac{\rho - c}{2(1-\delta)\rho}\right)^{\frac{1}{3}} \int_c^{\rho} - \right.$$

$$\left. - \left(\frac{\rho}{2t}\right)^{\frac{1}{3}} \int_0^{\rho}\right] K^{\frac{1}{3}}(u)\,du = O(1),$$

так как (см. (*))

$$g(a) = -\lambda_1^{\pm}(2t)^{-\frac{1}{3}}\left\{\left[\left(\frac{c}{\delta}\right)^{\frac{1}{3}} - \rho^{\frac{1}{3}}\right] \int_0^c + \right.$$

$$\left. + \left[\left(\frac{\rho - c}{1-\delta}\right)^{\frac{1}{3}} - \rho^{\frac{1}{3}}\right] \int_c^{\rho}\right\} K^{\frac{2}{3}}(u)\,du \underset{t\to 0}{\sim}$$

$$\underset{t\to 0}{\sim} -\lambda_1^{\pm}\left(\frac{\rho}{2t}\right)^{\frac{1}{3}}\left[\left(1 + \frac{a\sqrt{t}}{3\delta\rho} - 1\right) \int_0^c + \right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\left(1-\frac{a\sqrt{t}}{3(1-б)\rho}-1\right)\int_c^{\rho}\Big]K^{\frac{2}{3}}(u)\,du=$$

$$=-\lambda_1^{\pm}\frac{a}{3}\left(\frac{\sqrt{t}}{2\rho^2}\right)^{\frac{1}{3}}\left[б^{-1}\int_0^{c}-(1-б)^{-1}\int_c^{\rho}\right]K^{\frac{2}{3}}(u)\,du=$$

$$=-\lambda_1^{\pm}\frac{a}{3(1-б)}\left(\frac{\sqrt{t}}{2\rho^2}\right)^{\frac{1}{3}}\left(б^{-1}\int_0^{c}-\int_0^{\rho}\right)K^{\frac{2}{3}}(u)\,du\underset{t\to 0}{\sim}$$

$$\underset{t\to 0}{\sim}-\lambda_1^{\pm}\frac{a}{3(1-б)}\left(\frac{\sqrt{t}}{2\rho^2}\right)^{\frac{1}{3}}\int_0^{\rho}\Big[K^{\frac{2}{3}}(бz)-K^{\frac{2}{3}}(z)\Big]dz.$$

Далее заметим (см. [1]), что значение q определяет плотность процесса вдоль координаты b. Но q имеет только уточненную логарифмическую асимптотику. Чтобы показать отсутствие гауссовости по направлению этой координаты, достаточно рассмотреть частный случай, когда $K = \text{const}$. В этом случае получим точную асимптотику $\hat{p}_t^{\pm}\{\cdot\}$. Согласно рассуждениям в [1] по формуле Каца-Фейнмана $q^{\pm}(б_1, б_2, \xi_1, \xi_2)$ является, для $r = \xi_1$, решением задачи

$$\begin{cases}\dfrac{\partial q^{\pm}}{\partial б}=\dfrac{1}{2}\dfrac{\partial^2 q^{\pm}}{\partial r^2}-\dfrac{\rho^2 K}{\sqrt{t}}\,rq^{\pm}, & б\in[б_1, б_2],\\[2ex] q^{\pm}(б_1, б_2, r, \xi_2)=\delta(r-\xi_2),\quad q^{-}\Big|_{r=0}=0,\quad \dfrac{\partial q^{+}}{\partial r}\Big|_{r=0}=0.\end{cases}$$

Повторяя вычисление в [1], имеем

$$q^{\pm}(б_1, б_2, \xi_1, \xi_2)\underset{t\to 0}{\sim}\hat{y}_1^{\pm}\left[\left(\frac{2\rho^2 K}{\sqrt{t}}\right)^{\frac{1}{3}}\xi_1\right]y_1^{\pm}\left[\left(\frac{2\rho^2 K}{\sqrt{t}}\right)^{\frac{1}{3}}\xi_2\right]\times$$

$$\times\left(\frac{2\rho^2 K}{\sqrt{t}}\right)^{\frac{1}{3}}\exp\left\{-\lambda_1^{\pm}\left(\frac{\rho^4}{2t}\right)^{\frac{1}{3}}(б_2-б_1)\right\},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

По аналогии (как выше) в общем случае получаем, что условный процесс при $K = \text{const}$ имеет вид

$$p_t^{\pm}\{\cdot\} \underset{t\to 0}{\sim} \frac{(2K)^{\frac{1}{3}}\rho^{\frac{2}{3}}C_D^{\pm}}{2\pi\sqrt{б(1-б)}} \exp\left[-\frac{a^2}{2б(1-б)}\right] \times$$

$$\times \hat{y}_1^{\pm}\left[b_0(2б^2\rho^2K)^{\frac{1}{3}}\right] \hat{y}_1^{\pm}\left[b(2б^2\rho^2K)^{\frac{1}{3}}\right] \times$$

$$\times \hat{y}_1^{\pm}\left[b\left(2(1-б)^2\rho^2K\right)^{\frac{1}{3}}\right] \hat{y}_1^{\pm}\left[b_1\left(2(1-б)^2\rho^2K\right)^{\frac{1}{3}}\right] \times$$

$$\times \left\{\hat{y}_1^{\pm}\left[b_0(2\rho^2K)^{\frac{1}{3}}\right] y_1^{\pm}\left[b_1(2\rho^2K)^{\frac{1}{3}}\right]\right\}^{-1}.$$

Т е о р е м а. Условный процесс $\hat{w}(s)$, $0 \leqslant s \leqslant t$, при условии $x = \hat{w}(0) = \left(0, b_0 t^{\frac{2}{3}}\right)$, $y = \hat{w}(t) = \left(\rho, b_1 t^{\frac{2}{3}}\right)$, $x,y \in R^2\setminus D$, причем точки x,y находятся в зоне "глубокой тени" по отношению к D, движется с отклонением от кратчайшего пути на вектор, которого одна компонента (вдоль кратчайшей геодезической) является неоднородным гауссовским процессом с дисперсией $б(1-б)$, $б \in [0,1]$, $б = \frac{s}{t}$, вторая (по направлению нормали к ∂D) - некоторый явно описываемый негауссовский процесс.

ЛИТЕРАТУРА

[1] J. Sidorowicz: "Тепловая дифракция" на гладкой выпуклой области, Demonstratio Math. 18 (1985) 527-536.

[2] J. Sidorowicz: Тепловая дифракция на гладкой и выпуклой области в R^3, Demonstratio Math. 22 (1989) 79-84.

[3] С.А. М о л ч а н о в: Диффузионные процессы и риманова геометрия, Uspehi Mat. Nauk 30 (1975) 3-59.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY,
75-620 KOSZALIN, POLAND

Received November 25, 1987.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Andrzej Walendziak

THE KUROŠ - ORE PROPERTY, REPLACEABLE IRREDUNDANT DECOMPOSITIONS

1. Introduction

In the present paper we give a generalization of the following classical result due to R.P. Dilworth ([1], Theorem 7.7): If L is a semimodular algebraic lattice satisfying the descending chain condition, then L has the Kuroš-Ore property if and only if L is locally modular.

Moreover, here we prove that if L is an upper continuous semimodular lattice satisfying the descending chain condition, then L has the Kuroš-Ore property if and only if L has replaceable irredundant decompositions.

2. Basic notions

Let L be a complete lattice. Lattice join, meet, inclusion and proper inclusion are denoted respectively by the symbols $\vee$, $\wedge$, $\leq$ and $<$. Let 0 be the least element of L, and 1 be the greatest element of L.

An element $m \in L$ is called irreducible iff, for all $x,y \in L$ $m = x \wedge y$ implies $m = x$ or $m = y$. M(L) is defined to be the set of all irreducible elements of L.

If a is an element of the lattice L, then a representation $a = \bigwedge T$ with $T \subseteq M(L)$ is called a (meet) decomposition of a. A decomposition $a = \bigwedge T$ is irredundant if $\bigwedge(T-\{t\}) \neq a$, for all $t \in T$.

A complete lattice L is said to have the Kuroš-Ore property iff, each element of L has a finite irredundant

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

decomposition and whenever an $a \in L$ has two irredundant decompositions $a = w_1 \wedge \ldots \wedge w_n = \Lambda T$, then $|T| = n$.

We say that a complete lattice L has replaceable irredundant decompositions if each element of L has at least one irredundant decomposition, and whenever $a = \Lambda T = \Lambda R$ are two irredundant decompositions of an element $a \in L$, for each $t_o \in T$ there exists an $r_o \in R$ such that $a = \Lambda(T - \{t_o\}) \wedge r_o$.

A complete lattice L is called upper continuous iff, for every $a \in L$ and for every chain $C \subseteq L$, $a \wedge VC = (a \wedge c : c \in C)$.

For two elements $a,b \in L(a<b)$ we define

$$b/a: = \{x \in L : a \leq x \leq b\}.$$

If $b/a = \{a,b\}$, then we say that b covers a, in notation, $a \prec b$. A lattice L is said to be strongly atomic if, for every pair of elements $a, b \in L$ with $a < b$, there exists an element $p \in b/a$ such that $a \prec p$. In a complete strongly atomic lattice L, for each $a \in L$, let P_a denote the set of all elements covering a, and we set $u_a: = VP_a$.

If for every $a \in L$, the sublattice u_a/a is modular, then we say that L is locally modular.

3. Some lemmas

We start this section by the following

Lemma 1. If a,b are elements of an upper continuous strongly atomic lattice L and $a \not\geq b$, then there exists an element $m \in M(L)$ such that $m \geq a$ and $m \not\geq b$.

Proof. Since L is strongly atomic and $a < a \vee b$, there exists an element $p \in L$ such that $a \prec p \leq a \vee b$.
Let $T:= \{x \in L : x \geq a, x \not\geq p\}$. T is nonempty, since $a \in T$. Let C be a chain in T. Then upper continuity yields

$$p \wedge VC = V(p \wedge c : c \in C) = a.$$

Thus $V C \in T$, and T contains a maximal element m by Zorn's lemma. Clearly, $m \in M(L)$, $m \geq a$ and $m \not\geq b$.

In the proofs of Theorems 3.7 and 7.3 from [1] it was not used that L is an algebraic lattice, only that L is upper continuous. Therefore, Lemmas 2 and 3 below can be proved

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

analogously, and their proofs will be omitted.

Lemma 2. If an upper continuous strongly atomic lattice L has the property that, for all $a,b \in L$, $a \wedge b \prec a,b$ imply $a,b \prec a \vee b$, then L is semimodular.

In view of this lemma, every locally modular, upper continuous, strongly atomic lattice is semimodular.

Lemma 3. If a,b,p_1,p_2 are elements of a locally modular, upper continuous, strongly atomic lattice L, and if $p_1,p_2, \in P_a$, $b \wedge (p_1 \vee p_2) = a$ and $p_1 \vee b = p_2 \vee b$, then $p_1 = p_2$.

Let L be an upper continuous strongly atomic lattice L. By Theorem 1 in [3], every element of L is a meet of irreducible elements. Now, from Theorem 11 in [5] we get at once

Lemma 4. Let L be an upper continuous strongly atomic lattice. Then L has replaceable irredundant decompositions if L satisfies the following condition:

(*) For every $x \in L$ and every $m \in M(L)$, $x \wedge m$ is irreducible in the lattice $x/x \wedge m$.

The next lemma is generalization of [2] (Corollary, p.19).

Lemma 5. A locally modular, upper continuous, strongly atomic lattice L has replaceable irredundant decompositions.

Proof. Now we shall prove that L satisfies (*). Suppose on the contrary that there exists an element $x \in L$ and an element $m \in M(L)$ such that $a = x \wedge m$ is not irreducible in x/a. Therefore, x/a contains two distinct atoms p_1 and p_2. Obviously we have

$$a \leq m \wedge (p_1 \vee p_2) \leq m \wedge x = a, \text{ i.e.,} \tag{1}$$

$$m \wedge (p_1 \vee p_2) = a.$$

By Lemma 2, L is semimodular. Therefore, $m \prec m \vee p_1$ and $m \prec m \vee p_2$, and as $m \in M(L)$ and hence covered by a unique element,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) $$m \vee p_1 = m \vee p_2.$$

From (1) and (2) we conclude by Lemma 3 that $p_1 = p_2$. This contradiction shows that L satisfies condition (•). Then, from Lemma 4 it follows that L has replaceable irredundant decompositions.

For our investigations we need the following concept. A subset A of a complete lattice L is said to be independent if, for all $a \in A$, $a \wedge V(A-\{a\}) = 0$.

Lemma 6. If L is a semimodular, upper continuous lattice satisfying the descending chain condition, and if L has the Kuroš-Ore property, then L satisfies the following condition, defined in [1] (p.53):

(**) For all $x, y \in L$, if the sublattice $x \vee y/x$ has exactly one atom, then the sublattice $y/x \wedge y$ has exactly one atom.

Proof. The lattice L is strongly atomic, because L satisfies the descending chain condition. Let $x,y \in L$ be such that $x \vee y/x$ has exactly one atom and $y/x \wedge y$ contains two distinct atoms p_1 and p_2. We set $a := x \wedge y$. Since $p_1 \neq p_2$, the set $\{p_1,p_2\}$ is independent subset of $1/a$. By Zorn's lemma, upper continuity implies the existence of maximal independent subset of P_a containing $\{p_1,p_2\}$. Since L satisfies the descending chain condition, any maximal independent set of P_a is necessarily finite. Therefore, the set $\{p_1,p_2\}$ is contained in some finite maximal independent set $\{p_1,p_2,p_3,\ldots,p_n\}$ of elements covering a. By 6.5 [1] we have

$$u_a = p_1 \vee \ldots \vee p_n.$$

Let i be an element of the set $I = \{1,2,\ldots,n\}$. We set

$$\bar{p}_i := V(p_j : j \in I - \{i\})$$

By semimodularity of L we obtain that $\bar{p}_i \prec u_a$. From Lemma 1 there is an irreducible element $w_i \geq \bar{p}_i$ such that $w_i \not\geq u_a$, i.e.., $w_i \wedge u_a = \bar{p}_i$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Suppose that $a < w_1 \wedge \ldots \wedge w_n$. Since L is strongly atomic there is an element $q \in L$ such that $a \prec q \leq w_1 \wedge \ldots \wedge w_n$. Consequently, $q \leq u_a = p_1 \vee \ldots \vee p_n$, and for every $i \in I$

$$q \leq u_a \wedge w_i = \bar{p}_i. \tag{3}$$

Obviously, $q \neq p_i$ for each $i \in I$. Let Q be a minimal subset of $\{p_1, \ldots, p_n\}$ such that $q \leq VQ$, and let p_k be an element of Q. By semimodularity of L and the minimality of the set Q,

$$V(Q-\{p_k\}) \prec VQ \text{ and } q \neq V(Q-\{p_k\}).$$

Therefore, $q \vee V(Q-\{p_k\}) = VQ$. From this and (3) we have $p_k \leq VQ = q \vee V(Q - \{p_k\}) \leq V(Q - \{p_k\}) \vee \bar{p}_k = \bar{p}_k$. This contradiction shows that

$$a = w_1 \wedge \ldots \wedge w_n.$$

Observe that this decomposition of a is irredundant. Indeed, for each $i \in I$,

$$\Lambda(w_j : j \neq i) \geq \Lambda(\bar{p}_j : j \neq i) \geq p_i > a.$$

Let p be a single atom in the sublattice $x \vee y/x$. Then, $x \vee p_1 = x \vee p_2 = p$ and hence $p = x \vee p_1 \vee p_2$. By Lemma 1 there exists an irreducible element $w \geq x$ such that $w \wedge p = x$. Consequently,

$$w \wedge (p_1 \vee p_2) \leq w \wedge (x \vee p_1 \vee p_2) = w \wedge p = x,$$

and $w \wedge (p_1 \vee p_2) \leq p_1 \vee p_2 \leq y$. Therefore, $w \wedge (p_1 \vee p_2) \leq x \wedge y = a$. Then,

$$w \wedge (p_1 \vee p_2) = a. \tag{4}$$

Suppose that $a < w \wedge w_3 \wedge \ldots \wedge w_n$. Consequently, there exists an element $r \in L$ with $a \prec r \leq w \wedge w_3 \wedge \ldots \wedge w_n$. Hence $r \in P_a$, and therefore $r \leq u_a = p_1 \vee \ldots \vee p_n$. Let R be a minimal subset of $\{p_1, \ldots, p_n\}$ such that $r \leq VR$. Note that $r \neq p_1 \vee p_2$, otherwise $a < r \leq w \wedge (p_1 \vee p_2)$, contrary to (4). Then, there is an integer $j \in \{3,4,\ldots,n\}$ such that $p_j \in R$. By semimodularity of L and the minimality of the set R we obtain $r \vee V(R - \{p_j\}) = VR$. Hence

$$p_j \leq VR = r \vee V(R - \{p_j\}) \leq \bar{p}_j \vee V(R - \{p_j\} = \bar{p}_j.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This contradiction shows that $a = w \wedge w_3 \wedge \ldots \wedge w_n$. Then we have two decompositions of a:

$$a = w_1 \wedge \ldots \wedge w_n = w \wedge w_3 \wedge \ldots \wedge w_n,$$

where the left - hand decomposition is irredundant. Since $n \neq n-1$, L does not have the Kuroš-Ore property. Therefore the proof is complete.

4. The main result

In this section we will prove the following

Theorem. If L is a semimodular upper continuous lattice satisfying the descending chain condition, then the following statements are equivalent:

(i) L is locally modular.

(ii) L has replaceable irredundant decompositions.

(iii) L has the Kuroš-Ore property.

(iv) L satisfies condition (**).

Proof. (i) implies (ii). Since L satisfies the descending chain condition, L is strongly atomic. By Lemma 5, (ii) follows from (i).

(ii) implies (iii). For any element $a \in L$, all the irredundant decompositions of a are finite, because L satisfies the descending chain condition. By (ii), for each element $a \in L$, the number of irreducible elements in any irredundant decomposition of a is unique. Therefore, L has the Kuroš-Ore property.

(iii) implies (iv). Obvious by Lemma 6.

(iv) implies (i). Let a be an element of L. By upper continuity of L, there is a maximal independent subset P of P_a. From 6.5 [1] it follows that $u_a = VP$. Since L satisfies the descending chain condition, the set P is finite. Then, the lattice u_a/a is of finite length.

By the proof of Theorem 7.6 [1] (p.55), every element of u_a/a is a finite join of elements covering a.

Now we prove that u_a/a is a point lattice, i.e., for every pair of elements $x > y$ in u_a/a, x is the join of elements

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

covering y.

Let $x,y \in u_a/a$ with $y < x$. Let write x in the form $x = p_1 \vee \ldots \vee p_n$, where $p_i \in P_a$ for $i = 1,2,\ldots,n$, $p_1,\ldots,p_k \leq y$ and $p_{k+1},\ldots,p_n \not\leq y$. Hence

$$x = (y \vee p_{k+1}) \vee \ldots \vee (y \vee p_n),$$

and by semimodularity, $y \prec y \vee p_i$ for $i = k+1,\ldots,n$. Then u_a/a is a point lattice.

Finally, we prove that the lattice u_a/a is lower semimodular, i.e., its dual is semimodular. Let $x,y \in u_a/a$ and $x \prec x \vee y$. Then the sublattice $x \vee y/x$ has exactly one atom. By condition (**), the sublattice $y/x \wedge y$ has exactly one atom. Since $u_a^{\vee}/a$ is a point lattice, y is a join of elements covering $x \wedge y$. Therefore, $x \wedge y \prec y$, and hence u_a/a is a lower semimodular lattice.

Thus u_a/a is of finite length and it both semimodular and lower semimodular. Now, by Corollary 3 ([4], p.174) we conclude that u_a/a is modular, and therefore (i) is satisfied. This completes the proof.

Remark. Since every algebraic lattice is upper continuous, from this theorem it follows Theorem 7.7 of [1].

Examples. Let L be the lattice diagrammed in Figure 1. L is a finite semimodular lattice which does not have replaceable irredundant decompositions. This example shows that in Lemma 5, "local modularity" cannot be replaced by "semimodularity".

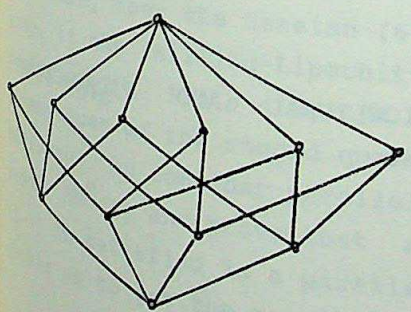


Figure 1

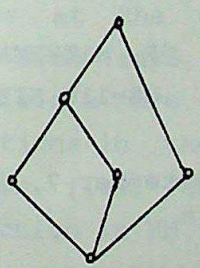


Figure 2

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Figure 2 represents a nonsemimodular lattice which has the Kuroš-Ore property but which is not locally modular. This example shows that the assumption of semimodularity cannot be dropped in the previous theorem.

Now we observe that in the theorem, the expression "lattice satisfying the descending chain condition" cannot be replaced by "strongly atomic lattice". Consider, for example, the lattice P(X) of all subsets of an infinite set X. P(X) does not have the Kuroš-Ore property, but it satisfies conditions (i), (ii) and (iv) of the theorem.

REFERENCES

[1] **P. Crawley, R. P. Dilworth:** Algebraic theory of lattices, Prentice Hall, Englewood Cliffs (N.J), 1973.

[2] **R. P. Dilworth, P. Crawley:** Decomposition theory for lattices without chain conditions, Trans. Amer. Math. Soc. 96 (1960) 1-22.

[3] **M. Erne:** On the existence of decompositions in lattices, Algebra Universalis 16(1983) 338-343.

[4] **G. Grätzer:** General lattice theory, Birkhäuser, Basel, 1978.

[5] **G. Richter:** The Kuroš-Ore theorem,finite and infinite decompositions, Studia Sci. Math. Hungar. 17(1982) 243-250.

DEPARTMENT OF MATHEMATICS, AGRICULTURAL AND PEDAGOGICAL UNIVERSITY, 08-110 SIEDLCE, POLAND

Received September 7, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Marek Rochowski

PARALLEL CROSS SECTIONS AND CANONICAL FRAMES IN NORMAL BUNDLES OF IMMERSED MANIFOLDS IN EUCLIDEAN SPACE

Introduction

The notations in this note coincide with that in [1]. Let

(1.1) $x : M^n \longrightarrow E^{n+N}$

denote an immersion of an n-dimensional manifold M^n in the Euclidean (n+N)-dimensional space E^{n+N}. In [1] for every $p \in M^n$ we have defined a canonical orthonormal frame

(1.2) $x(p)\tilde{e}_{n+1}\ldots\tilde{e}_{n+N}$

in the normal space $N_pM^n(x) \subset E^{n+N}$ to the surface $x(M^n)$ at $x(p)$. The uniqueness of the construction of $\tilde{e}_{n+\rho}$, $1 \le \rho < N$, in (1,2) depend on whether the Hessian defined by (6,9) in [1] is different from zero. In this note we prove that if

(1,3) $x(M^n) \subset S^{n+N-1} \subset E^{n+N}$

and (1,2) is defined by means of the Killing-Lipschitz curvature, then the Hessian (6,2) in [1] is different from zero, if the Killing-Lipschitz curvature at the canonical vector $\tilde{e}_{n+\rho}$, $1 \le \rho < N$, is different from zero and the eigenvalues of the second quadratic form $x.d^2x$ are mutually different. We consider parallel cross sections in the normal bundle $NM^n(x)$ and prove that an immersion (1.1) such that (1,3) is satisfied is a parallel cross section in $NM^n(x)$. This enable us to get the result.

2. Preliminaries

By TM^n we denote the tangent bundle of M^n. By

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.1) $x(p)e_1 \cdots e_n e_{n+1} \cdots e_{n+N}$

we denote an orthonormal frame in E^{n+N} such that e_i, $1 \le i \le n$, is a basis of the tangent space $T_p M^n$ and e_r, $n+1 \le r \le n+N$, is a basis of the normal space $N_p M^n(x)$, $p \in M^n$. The indices are submitted to the convention

$$1 \le i,j,k,l \le n, \quad n+1 \le r,s,t,u \le n+N, \quad 1 \le \rho,\delta,\tau \le N.$$

We have

(2.2) $dx = \omega_i e_i$, $de_i = \omega_{ik} e_k + \omega_{ir} e_r$, $de_r = \omega_{ri} e_i + \omega_{rs} e_s$,

where repeated indices denote summation. The equations of structure have the form

$$\text{(2.3)} \quad \begin{cases} d\omega_i = \omega_k \wedge \omega_{ki}, \\ d\omega_{ik} = \omega_{ij} \wedge \omega_{jk} + \omega_{ir} \wedge \omega_{rk}, \\ d\omega_{ir} = \omega_{ij} \wedge \omega_{jr} + \omega_{is} \wedge \omega_{sr}, \\ d\omega_{rs} = \omega_{rj} \wedge \omega_{js} + \omega_{rt} \wedge \omega_{ts}. \end{cases}$$

The matrix (A_{rij}) of the second quadratic form of $x(M^n)$ evaluated in the direction of the vector e_r is defined by

(2.4) $\omega_{ir} = A_{rij}\omega_j$, $A_{rij} = A_{rji}$, $1 \le i,j \le n$.

Manifolds and functions are supposed to be differentiable, i.e. of class C^∞.

3. Parallel cross sections in a normal bundle $NM^n(x)$

The absolute differential in $NM^n(x)$, where x is the immersion (1.1), is defined by

(3.1) $De_r = \omega_{rs} e_s$.

Let $U \subset M^n$ denote an open set diffeomorphic with the unit disk $K^n \subset E^n$. By

(3.2) $\bar{e}_r : U \longrightarrow NM^n(x)$, $n+1 \le r \le n+N$,

we denote local cross sections. We have

(3.3) $D\bar{e}_r = \bar{\omega}_{rs}\bar{e}_s$, $\bar{\omega}_{rs} = \bar{A}_{sri}\omega_i$, $\bar{A}_{sri} = -\bar{A}_{rsi}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The forms $\bar{\omega}_{rs}$ are defined in $U \subset M^n$ and therefore can be represented by means of ω_i. Hence, $\bar{A}_{sri}$ in (3.3) are functions which depend on $p \in U$ and (linearly) on secondary parameters a_{ij}, $1 \le i,j \le n$, where (a_{ij}) denotes an orthogonal matrix which fix $e_i = a_{ij}\bar{e}_j$ by means of local cross sections

(3.4) $\quad \bar{e}_i : U \longrightarrow TM^n, \qquad 1 \le i \le n.$

The functions $\bar{A}_{sri}$ are the connection components of the connection ∇ in $NM^n(x)$ which corresponds to the absolute differential D. Indeed, we have

$$\nabla_{e_i}\bar{e}_r = \Gamma_{irs}\bar{e}_s\ , \quad D\bar{e}_r = \nabla_{e_i}\bar{e}_r\omega_i = \Gamma_{irs}\bar{e}_s\omega_i.$$

Hence, from (3.3) we get $\bar{A}_{sri} = \Gamma_{irs}$. Let

(3.5) $\quad e = \bar{a}_r\bar{e}_r\ ,$

where $\bar{a}_r$ are functions defined in U and $\bar{e}_r$ are local cross sections (3.2). The local cross section (3.5) is called parallel, if

(3.6) $\quad De = 0.$

From (3.3), (3.5) and (3.6) we get

$$De = d\bar{a}_r\bar{e}_r + \bar{a}_r D\bar{e}_r = (d\bar{a}_s + \bar{a}_r\bar{\omega}_{rs})\bar{e}_s = 0.$$

Hence (3.5) is parallel cross section, if and only if

(3.7) $\quad d\bar{a}_r + \bar{a}_s\bar{\omega}_{sr} = 0.$

The system of partial differential equations equivalent with (3.7) together with the equations $\omega_r = 0$, $n+1 \le r \le n+N$, define all immersions which admit a parallel cross section (3.5). Exterior differentiation of (3.7) lead to the integrability conditions. We have

(3.8) $\quad d(d\bar{a}_r + \bar{a}_s\bar{\omega}_{sr}) = \bar{a}_t(d\bar{\omega}_{tr} - \bar{w}_{ts} \wedge \bar{\omega}_{sr}) = 0.$

If we choose the cross sections (3.2) such that $e = \bar{a}\bar{e}_t$, where $\bar{a} \neq 0$ and $t, n+1 \le t \le n+N$, denotes a fixed integer, then from (3.8) we get

$$\bar{a}(d\bar{\omega}_{tr} - \bar{\omega}_{ts} \wedge \bar{\omega}_{sr}) = 0$$

and since such a choice is possible for every t, $n+1 \le t \le n+N$, we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.9) $d\bar{\omega}_{tr} = \bar{\omega}_{ts} \wedge \bar{\omega}_{sr}$.

From the last equations of structure (2.3) and (3.9) we get

(3.10) $\bar{\omega}_{ri} \wedge \bar{\omega}_{is} = 0$.

From (2.4) and (3.10) we get

(3.11) $\bar{A}_{rik}\bar{A}_{sij} = \bar{A}_{rij}\bar{A}_{sik}$.

We have

Proprosition 1. If for a fixed r, $n+1 \leq r \leq n+N$, the cross section (3.2) is parallel and the eigenvalues of the matrix $(\bar{A}_{rij})$ are different, then there exists such a frame

(3.12) $x(p)\bar{e}_1 \ldots \bar{e}_n$

in T_pM^n that

(3.13) $\bar{A}_{sij} = 0$ for $i \neq j$ and every s, $n+1 \leq s \leq n+N$.

Proof. We define the vectors $\bar{e}_i$, $1 \leq i \leq n$, of (3.12) such that they determine the princip axes of $(\bar{A}_{rij})$. An orthogonal change of coordinates in T_pM^n to the frame (3.12) transforms $(\bar{A}_{rij})$ to a diagonal form. The transformed matrixes are denote by the same symbol $(\bar{A}_{sij})$, $n+1 \leq s \leq n+N$. We have by assumption

(3.14) $\bar{A}_{rii} \neq \bar{A}_{rjj}$, $\bar{A}_{rij} = 0$ for $i \neq j$.

Since $\bar{e}_r$ is a parallel cross section, (3.11) is satisfied. From (3.11) and (3.14) we get (3.13) as asserted.

Proprosition 2. For every immersion (1.1) such that (1.3) is satisfied we have

(3.15) $Dx = 0$.

Proof. Let (2.1) denote a frame such that e_r, $n+2 \leq r \leq n+N$, are tangent to S^{n+N-1}. Then

(3.16) $x = e_{n+1}$

and from (2.2) we have

(3.17) $dx = \omega_i e_i = de_{n+1} = \omega_{n+1,i}e_i + \omega_{n+1,r}e_r$.

Hence we get

(3.18) $\omega_i = \omega_{n+1,i}$, $\omega_r = \omega_{n+1,r} = 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From (3.1) (3.16) and (3.18) we get (3.15).

4. Parallel cross sections and canonical frames

We define canonical cross sections in $NM^n(x)$ as explained in [1], where x is the immersion (1.1), by means of the Killing-Lipschitz curvature (see (4.4) in [1])

(4.1) $$L_n(p,e_r) = (-1)^n \det(A_{rij}),$$

where A_{rij} are defined by (2.4). Let

(4.2) $$x(p)\tilde{e}_{n+1}\cdots\tilde{e}_{n+\rho}\bar{e}_{n+\rho+1}\cdots\bar{e}_{n+N}$$

denotes an orthonormal frame in $N_pM^n(x)$, where $p \in U$ and $U \subset M^n$ is an open set diffeomorphic with the unit disk $K^n \subset E^n$, such that $L_n(p,\tilde{e}_{n+\sigma})$, $1 \le \sigma \le \rho$, is the maximal value of the restriction $L_n/_{S_p^{N-\sigma}}$, where $S_p^{N-\sigma} \subset N_pM^n(x)$ is the $(N-\sigma)$-dimensional unit sphere contained in the linear subspace of $N_pM^n(x)$ orthogonal to the vectors $\tilde{e}_{n+1},\dots,\tilde{e}_{n+\sigma-1}$. We suppose that the Hessian

(4.3) $$\det(h^\sigma_{nru})_{n+\sigma+1 \le r,u \le n+N}, \quad 1 \le \sigma < \rho,$$

defined by (6.9) in [1], where we set $\rho = \sigma$, $k = n$, is different from zero. Then from Corollary 1 in [1] it follows that

(4.4) $$\tilde{e}_{n+\sigma} : U \longrightarrow NM^n(x), \quad 1 \le \sigma < \rho,$$

are differentiable cross sections.

For $k = n$ we get from (6.9) in [1]

(4.5) $$h^\sigma_{nru} = (-1)^{n+1}\,\delta_{ru}\,n\det(\tilde{A}_{n+\rho,pq})_{1\le p,q\le n} +$$

$$+ (-1)^n \sum_{l<m} \det(\tilde{A}_{n+\rho,lq},\dots,\bar{A}_{rlq},\dots,\bar{A}_{umq},\dots,\tilde{A}_{n+\rho,nq}) +$$

$$+ \det(\tilde{A}_{n+\rho,lq},\dots,\bar{A}_{ulq},\dots,\bar{A}_{rmq},\dots,\tilde{A}_{n+\rho,nq}),$$

where $1 \le l < m \le n$, $n+\rho+1 \le r$, $u \le n+N$, and q, $1 \le q \le n$, denotes the number of the column in the determinants on the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

right of (4.5). By

$$\bar{e}_{n+\tau} : U \longrightarrow NM^n(x), \qquad \rho+1 \le \tau \le N, \tag{4.6}$$

we denote in (4.2) arbitrary differentiable cross sections.

We have

Proprosition 3. If the eigenvalues of the matrix $(\tilde{A}_{n+\rho,ij})$ of the second quadratic form evaluated in the direction of the canonical vector $\tilde{e}_{n+\rho}$ are all positive or all negative for every $p \in U$, then

$$\det(h^{\rho}_{nru})_{n+\rho+1 \le r,u \le n+N} \tag{4.7}$$

is different from zero for every $p \in U$ and there exists a cross section

$$\tilde{e}_{n+\rho}: U \longrightarrow NM^n(x). \tag{4.8}$$

Proof. We transform the matrix

$$(h^{\rho}_{uru}), \quad n+\rho+1 \le r,u \le n+N, \tag{4.9}$$

to the diagonal form as follows. By means of an orthogonal transformation of the tangent space T_pM^n we transform $(\tilde{A}_{n+\rho,ij})$ to the diagonal matrix, which for simplicity is denoted by the same symbol. Hence we set

$$\tilde{A}_{n+\rho,ij} = 0 \quad \text{for } i \ne j. \tag{4.10}$$

From (4.5) and (4.10) we get

$$\begin{aligned} h^{\rho}_{nru} = (-1)^{n+1}\delta_{ru}\,n\det(\tilde{A}_{n+\rho,pq})_{1\le p,q\le n} + \\ + (-1)^n \sum_{l<m} \tilde{A}_{n+\rho,11}\cdot\ldots\cdot\tilde{A}_{n+\rho,l-1,l-1}\tilde{A}_{n+\rho,l+1,l+1}\cdots \\ \cdots\tilde{A}_{n+\rho,m-1,m-1}\tilde{A}_{n+\rho,m+1,m+1}\cdots\tilde{A}_{n+\rho,nn}\cdot(\bar{A}_{rll}\bar{A}_{umm} + \\ + \bar{A}_{ull}\bar{A}_{rmm} - 2\bar{A}_{ulm}\bar{A}_{rml}), \text{ where } 1 \le l < m \le n. \end{aligned} \tag{4.11}$$

For a fixed $p \in U$ we choose the vectors $\bar{e}_r$, $n+\rho+1 \le r \le n+N$, of (4.2) as canonical vectors defined by means of the function (see (4.3) in [1])

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$L_2(p,e_r) = \sum_{i<j} (A_{rii}A_{rjj} - A^2_{rij}).$$

Such vectors exist, although they are in general not uniquely defined. For simplicity we denote them by the same symbols $\bar{e}_r$. From (4.19) in [1] we get for $r \neq u$, $n+\rho+1 \leq r,u \leq n+N$,

(4.12) $$\bar{A}_{rll}\bar{A}_{umm} + \bar{A}_{rmm}\bar{A}_{ull} - 2\bar{A}_{ulm}\bar{A}_{rml} = 0.$$

From (4.11) and (4.12) we get

(4.13) $$h^{\rho}_{nrr} = (-1)^{n+1}n\det(\tilde{A}_{n+\rho,pq})_{1\leq p,q\leq n} +$$

$$+ (-1)^n 2\sum_{l<m} \tilde{A}_{n+\rho,11}\cdots\tilde{A}_{n+\rho,l-1,l-1}\tilde{A}_{n+\rho,l+1,l+1}\cdots$$

$$\cdots\tilde{A}_{n+\rho,m-1,m-1}\tilde{A}_{n+\rho,m+1,m+1}\cdots\tilde{A}_{n+\rho,nn}\cdot(\bar{A}_{rll}\bar{A}_{rmm} - \bar{A}^2_{rlm}),$$

(4.14) $$h^{\rho}_{nru} = 0 \quad \text{for} \quad r \neq u, \quad n+\rho+1 \leq r,u \leq n+N.$$

The formula (4.13) can be written in the form

(4.15) $$h^{\rho}_{nrr} = (-1)^{n+1}2\tilde{A}_{n+\rho,11}\cdots\tilde{A}_{n+\rho,nn}\left(\frac{n}{2} - \sum_{l<m}(a_l a_m - b_{lm})\right),$$

where

(4.16) $$a_l = \frac{\bar{A}_{rll}}{\tilde{A}_{n+\rho,ll}}, \quad b_{lm} = \frac{\bar{A}^2_{rlm}}{\tilde{A}_{n+\rho,ll}\tilde{A}_{n+\rho,mm}}.$$

Since $\tilde{e}_{n+\rho}$ is defined by means of L_n and $(\tilde{A}_{n+\rho,ij})$ is by (4.10) a diagonal matrix, we get from (4.27) in [1]

(4.17) $$\sum_{i=1}^{n} \tilde{A}_{n+\rho,11}\cdot\ldots\cdot\bar{A}_{rii}\cdot\ldots\cdot\tilde{A}_{n+\rho,nn} = 0, \quad n+\rho+1 \leq r \leq n+N.$$

From (4.16) and (4.17) we get

(4.18) $$\sum_{i=1}^{n} a_i = 0.$$

From (4.18) it follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4.19)\qquad \sum_{1<m} a_1 a_m = -\sum_{l=2}^{n} a_l^2 - \sum_{2\le l<m} a_l a_m = -\frac{1}{2}\sum_{l=2}^{n} a_l^2 - \frac{1}{2}\Big(\sum_{l=2}^{n} a_l\Big)^2 \le 0.$$

From (4.15), (4.16) and (4.19) it follows

$$(4.20)\qquad h^{\rho}_{nrr} \neq 0 \quad \text{for every } r,\ n+\rho+1 \le r \le n+N.$$

From (4.14) and (4.20) follows that (4.7) is different from zero for every $p \in U$ and this together with the fact that (4.3) are different from zero and therefore that (4.4) are uniquely defined for every $p \in U$ follows that $\tilde{e}_{n+\rho}$ is uniquely defined for every $p \in U$ and consequently from Corollary 1 in [1] it follows that (4.8) exists.

Finally we have

Proprosition 4. Let (1.1) denote an immersion which admits a parallel cross section (3.5), where e is a unit vector. We suppose that the eigenvalues of the second quadratic form $e(p).d^2x(p)$ are mutually different for every $p \in U \subset M^n$. If

$$(4.21)\qquad L_n(p,\tilde{e}_{n+\rho}) \neq 0 \text{ for every } p \in U,$$

then (4.7) is different from zero for every $p \in U$.

Proof. Let

$$(4.22)\qquad x(p)e_{n+1}\cdots e_{n+N}\ ,\quad p \in U,$$

denotes an orthonormal frame in $N_pM^n(x)$ such that

$$(4.23)\qquad e_r\colon U \longrightarrow NM^n(x),\quad n+1 \le r \le n+N,$$

are differentiable cross sections and

$$(4.24)\qquad e = e_{n+1}\ .$$

From Proposition 1 it follows that there exists such a frame (3.12) in the tangent space T_pM^n that

$$(4.25)\qquad A_{rij} = 0 \quad \text{for } i \neq j \text{ and every } r,\ n+1 \le r \le n+N,$$

where A_{rij} are coefficients of the second quadratic form $e_r.d^2x$. From (3.34) in [1] we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(4.26) $\bar{A}_{rij} = a_{rs}A_{sij}$, $n+\rho+1 \le r \le n+N$,

where $\bar{A}_{rij}$ are coefficients of the second quadratic form $\bar{e}_r.d^2x$ and $\bar{e}_r$ are vectors of (4.2). From (4.25) and (4.26) it follows

(4.27) $\bar{A}_{rij} = 0$ for $i \ne j$ and every r, $n+\rho+1 \le r \le n+N$.

From (4.16) and (4.27) it follows

(4.28) $b_{lm} = 0$ for $1 \le l < m \le n$.

From (4.1) and (4.21) it follows that all eigenvalues fo the matrix $(A_{n+\rho,ij})$ are different from zero. From (4.14), (4.15), (4.16), (4.19) and (4.28) the assertion of Proposition 4 follows.

As an immediate consequence of Propositions 2 and 4 we get the following

Corollary. If the immersion (1.1) satisfies (1.3), the second quadratic form $x.d^2x$ has mutually different eigenvalues and (4.21) is satisfied, then (4.7) is different from zero for every $p \in U$.

REFERENCES

[1] M. Rochowski: A scheme of generating canonical frames in normal bundles of immersed manifolds in Euclidean spaces, Demonstratio Math. 23 (1990) 271-300.

Author's address: Fleiner Strasse 76c, 7000 Stuttgart 40, BRD/RFN

Received September 9, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1. I

Appl
Chebysch
in the i
many aut
[S1], [G
found an
Recently
analysis
(like e.
of opti
orthogon
[T1], [W

In [
optimal
(without
the linea

In t
particula
for the a
problem
-invariar
equation
lemmas c
approxima
index obt

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Jerzy Pełczewski

APPLICATION OF LEGENDRE POLYNOMIALS METHOD FOR SOME OPTIMAL CONTROL PROBLEMS

1. Introduction

Application of orthogonal polynomials (Legendre, Laguerre, Chebyschew, Jacobi) and of the general orthogonal polynomials in the investigation of dynamic systems has been presented by many authors. According to the properties of these polynomials [S1], [G1] the operational matrices of integration have been found and applied for the solution of differential equations. Recently these results have been used in dynamic systems analysis (like e.g. [G2], [W1]) and in variational problems (like e.g. [C1], [C3], [H1], [H2], [H3]). Also in the solution of optimal control problems the methods based on the orthogonal polynomials have been applied - like e.g. [S1], [T1], [W1], [Y1].

In [S1] Shih and Kung present a general approach to the optimal control problem with quadratic performance index (withouth terminal cost function) for a system described by the linear integrodifferential equations.

In this paper the author confined himself to the particular case of the shifted Legendre polynomials approach for the approximate solution of the optimal quadratic control problem with terminal cost function for a linear time-invariant system without inequality constraints. The state equation was reduced to an algebraic matrix equation and the lemmas concerning the existence and uniqueness of its approximate solution were presented. With the performance index obtained in the form of an algebraic function the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

considered optimal control problem has the solution corresponding to the infimum of the above function in presence of equality constraints.

This result can be used for an effective numerical procedure and has the advantage (in comparison with the method based on Riccati equation) that the matrix $\underline{P}$ in the quadratic form with the control vector in the performance index can be positive semidefinite, and not necessarily positive definite.

2. Problem statement

Let us consider the linear time - invariant dynamic system described by the state equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{1}$$

where $\underline{x}(\cdot)$, $\underline{u}(\cdot)$ are the state and control vectors of dimensions $n \times 1$, $r \times 1$ respectively and A, B are constant $n \times n$ and $n \times r$ matrices. The problem is to find the optimal control $\underline{u}^*(t)$, which transfers the above system from the given initial state at $t = 0$ to the close neighbourhood of the desired final state $\underline{x}_d$ at $t = t_f$ - while minimizing the functional (the optimality criterion) I_c of the form

$$I_c = S(\underline{x}(t_f), \underline{x}_d) + \int_0^{t_f} \varphi(\underline{x}(t), \underline{u}(t))dt \tag{2}$$

where S, φ are scalar functions

$$S : R^n \longrightarrow R, \quad \varphi : R^{n+r} \longrightarrow R.$$

The following assumptions are accepted:

1° The vectors $\underline{u}(t)$, $\underline{x}(t)$ are unconstrained:

$$\underline{u}(t) \in R^r \quad \forall t \in [0, t_f], \tag{3}$$

$$\underline{x}(t) \in R^n \quad \forall t \in [0, t_f]. \tag{4}$$

2° The matrix A in (1) is nonsingular, i.e. $\det A \neq 0$.

The approximate solution of this optimal control problem can be found by application of the shifted Legendre polynomials methods. The vectors $\underline{u}(t)$, $\underline{x}(t)$ are written in the form

$$\underline{u}(t) = F\underline{s}_m(t) \tag{5}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(6) $\underline{x}(t) = D\underline{s}_m(t)$

where F and D are following matrices of dimensions $r \times m$ and $n \times m$ respectively

(7) $$F = \begin{bmatrix} f_{10} & f_{11} & \cdots & f_{1\,m-1} \\ f_{20} & f_{21} & \cdots & f_{2\,m-1} \\ \vdots & \vdots & & \vdots \\ f_{r0} & f_{r1} & \cdots & f_{r\,m-1} \end{bmatrix},$$

(8) $$D = \begin{bmatrix} d_{10} & d_{11} & \cdots & d_{1\,m-1} \\ d_{20} & d_{21} & \cdots & d_{2\,m-1} \\ \vdots & \vdots & & \vdots \\ d_{n0} & d_{n1} & \cdots & d_{n\,m-1} \end{bmatrix},$$

and $\underline{s}_m(t)$ denotes the vector of dimension $m \times 1$

(9) $$\underline{s}_m(t) = [s_0(t)s_1(t)\ldots s_{m-1}(t)]^T$$

whose components $s_i(t)$, $i=0,1,\ldots,m-1$ are shifted Legendre polynomials on the interval $[0,t_f]$ of the form

(10) $$s_i(t) = \sum_{j=0}^{i} \frac{(-1)^j(2i-j)!}{j![(i-j)!]^2} \left(\frac{t}{t_f}\right)^{i-j}$$

by substituting (5) and (6) into (1) and after integration of (1) we obtain

(11) $$D\underline{s}_m(t) - \underline{x}(0) = A \int_0^t D\underline{s}_m(\tau)d\tau + B \int_0^t F\underline{s}_m(\tau)d\tau.$$

According to the properties of the Legendre polynomials we can write (11) in the form

(12) $$D\underline{s}_m(t) = \underline{x}(0) + ADH_m\underline{s}_m(t) + BFH_m\underline{s}_m(t)$$

where H_m is constant $m \times m$ operational matrix for the integration of the orthogonal vector $\underline{s}_m(t)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(13)\quad H_m = \frac{t_f}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & & \vdots \\ . & . & . & . & . & . & & . & \dots & 0 & -\frac{1}{2m-3} & 0 & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & 0 & . & \dots & . & . & \dots & 0 & 0 & -\frac{1}{2m-1} & 0 \end{bmatrix}$$

with $s_o(t) = 1$ we can write

$$(14)\quad x(0) = V(0)\underline{s_m}(t)$$

where $V(0)$ is the following $n \times m$ matrix

$$(15)\quad V(0) = \begin{bmatrix} x_1(0) & 0 & \dots & 0 \\ x_2(0) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ x_n(0) & 0 & \dots & 0 \end{bmatrix}$$

and substituting (14) into (12) we obtain

$$(16)\quad D = V(0) + ADH_m + BFH_m$$

being the approximate expression of the state equation (1).

In the case where the control vector $\underline{u}(t)$ is given we can find the matrix F satisfying (5) and solve the equation (16) with the unknown matrix D. Next, according to (6) we obtain the state vector $\underline{x}(t)$ corresponding to this control. The existence and uniquenes of the equation's (16) solution are given by the following two lemmas.

Lemma 1. The solution D of the matrix equation (16) exists if and only if there is a nonsingular matrix T satisfying the formula

$$(17)\quad \begin{bmatrix} A^{-1} & 0 \\ 0 & H_m \end{bmatrix} = T \begin{bmatrix} A^{-1} & C \\ 0 & H_m \end{bmatrix} T^{-1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where

(18) $$C = A^{-1}V(0) + A^{-1}BFH_m.$$

Proof. After the multiplication of the equation (16) by A^{-1} we obtain (with C from (18))

(19) $$A^{-1}D - DH_m = C.$$

It is known [L1] that solution of the matrix equation

(20) $$LY + YM = C$$

(with unknown Y) exists if and only if the matrices

$$\begin{bmatrix} L & 0 \\ 0 & -M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} L & C \\ 0 & -M \end{bmatrix}$$

are similar i.e. [L1]

$$\begin{bmatrix} L & 0 \\ 0 & -M \end{bmatrix} = T \begin{bmatrix} L & C \\ 0 & -M \end{bmatrix} T^{-1}$$

where T is the nonsingular matrix. Hence with $L = A^{-1}$, $M = -H_m$ we conclude that the solution D of the equation (19) and consequently of the equation (16) exists if and only if the similarity condition (17) is satisfied.

Lemma 2. The equation (16) has unique solution if and only if the matrices A^{-1} and $-H_m$ have no eigenvalues in common.

Proof. The equation of the form (20) has [L1] the unique solution if and only if the condition

(21) $$\sigma(L) \cap \sigma(-M) = 0$$

is satisfied.

Here $\sigma(L)$ and $\sigma(-M)$ are the spectras (sets of all distinct eigenvalues) of matrices L and -M, 0 is a null vector. Hence with $L = A^{-1}$, $M = -H_m$ we conclude that the solution D of the equation (19) and also (16) is unique if and only if the intersection of $\sigma(A^{-1})$ and $\sigma(H_m)$ is empty, then A^{-1} and $-H_m$ have no eigenvalues in common.

That completes the proof.

In the considered optimal control problem the equality

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

constraint is given by the equation (1) which is approximatively represented by (16).

The criterion (2) can be expressed as function of matrices F, D and of the vector $\underline{s}_m(t)$. In order to obtain this form we substitute into (2) $\underline{u}(t)$, $\underline{x}(t)$ from (5), (6). We notice that the final state can be represented in the form

$$\underline{x}(t_f) = D\underline{s}_m(t_f) \tag{22}$$

or

$$\underline{x}(t_f) = D\underline{v} \tag{23}$$

where with $s_i(t_f) = 1$ the m dimensional vector $\underline{v}$ is

$$\underline{v} = [1\ 1\ \ldots\ 1]^T. \tag{24}$$

Finally, after all these substitutions we have

$$I_c = S(D\underline{v}, \underline{x}_d) + \int_0^{t_f} \varphi(D\underline{s}_m(t), F\underline{s}_m(t))dt \tag{25}$$

and our optimal problem becomes:

Find the minimum of I_c from (25) with respect to F, D while satisfying the equality constraint (16).

If the form of the function $\varphi(D\underline{s}_m(t), F\underline{s}_m(t))$ enable us to integrate it effectively then according to the properties of Legendre polynomials we obtain from (25)

$$I_c = \psi(F, D, \underline{x}(0), \underline{x}_d, t_f). \tag{26}$$

Hence in this case the variational problem will be reduced to the minimization (with respect to F, D) of the function (26) with the equality constraint (16).

After finding the matrices F*, D* corresponding to the global minimum of I_c we obtain the approximate formula of the optimal control and state vectors $\underline{u}^*(t)$, $\underline{x}^*(t)$ from (5) and (6)

$$\underline{u}^*(t) = F^*\underline{s}_m(t) \tag{27}$$

$$\underline{x}^*(t) = D^*\underline{s}_m(t). \tag{28}$$

This solution exists if for $F = F^*$ the condition in Lemma 1 is satisfied.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Application to the optimal control problem with quadratic criterion

The results obtained above can be applied for the optimal control of a linear system (1) with the quadratic criterion

$$I_{qc} = \frac{1}{2}[\underline{x}(t_f) - \underline{x}_d]^T G[\underline{x}(t_f) - \underline{x}_d] + \frac{1}{2}\int_0^{t_f} [\underline{x}^T(t)Q\underline{x}(t) + \underline{u}^T(t)P\underline{u}(t)]dt \tag{29}$$

in the case where assumptions 1°, 2° hold. In (29) we denoted by G, P, Q constant matrices at least positive semidefinite of dimensions $n \times n$, $n \times n$, $r \times r$ respectively. We shall prove the following theorem.

Theorem. The approximate formula for the optimality criterion (29) for a system described by (1) is

$$I_{qc} = \frac{1}{2}[D\underline{v} - \underline{x}_d]^T G[D\underline{v} - \underline{x}_d] + \frac{1}{2} t_f \sum_{i=1}^{m-1} \frac{e_{(i+1)(i+1)}}{2i+1} \tag{30}$$

where $e_{(i+1)(i+1)}$, $i=0,1,\ldots,m-1$ are elements on the main diagonal of the following $m \times m$ matrix E.

$$E = D^T Q D + F^T P F. \tag{31}$$

Proof. We substitute $\underline{u}(t)$, $\underline{x}(t)$, $\underline{x}(t_f)$ from (5), (6), (23) respectively into (29) and obtain

$$I_{qc} = \frac{1}{2}[D\underline{v} - \underline{x}_d]^T G[D\underline{v} - \underline{x}_d] + \frac{1}{2}\int_0^{t_f} \underline{s}_m^T(t) E \underline{s}_m(t)dt \tag{32}$$

according to (9) we obtain

$$\underline{s}_n^T(t)E\underline{s}_m(t) = s_0(t)\sum_{i=0}^{m-1} e_{1(i+1)}s_i(t) + s_1(t)\sum_{i=0}^{m-1} e_{2(i+1)}s_i(t) + \ldots + s_{m-1}(t)\sum_{i=0}^{m-1} e_{m(i+1)}s_i(t).$$

Hence according to the property of Legendre polynomials

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(33)\qquad \int_0^{t_f} s_i(t)s_k(t)dt = \begin{cases} 0 & \text{for } k\neq i \\ \dfrac{t_f}{2i+1} & \text{for } k=i \end{cases}$$

the second term on the right side of (32) becomes

$$(34)\qquad \int_0^{t_f} \underline{s_m}^T(t)E\underline{s_m}(t)dt = t_f e_{11} + \frac{t_f}{3} e_{22} + \frac{t_f}{5} e_{33} +$$

$$+ \ldots + \frac{t_f}{2m-1} e_{mm} = t_f \sum_{i=0}^{m-1} \frac{e_{(i+1)(i+1)}}{2i+1} .$$

Substituting (34) into (32) we obtain (30). That completes the proof.

We conclude that the considered optimal control problem with quadratic criterion was reduced to the minimization (with respect to F, D) of the function (30) with the equality constraint (16).

4. Final remarks

The advantage of the presented procedure is related to the possibility of the numerical methods effective application.

The obtained results are approximate because we consider the Legendre polynomials with the finite number of terms: $s_0(t)$ to $s_{m-1}(t)$. But by the appropriate choice of this number we will assure the accuracy of computation results - sufficient for the investigation of real systems.

It is well known that the optimal control problem with quadratic criterion for the linear time - invariant system can be solved by the usual method based on Riccati matrix equation. But it should be underlined that the proposed approach with Legendre polynomials is more general. Namely the inverse of the matrix P from (29) does not appear in presented results. Hence P can be positive semidefinite and not necessarily positive definite, like in the case where we must solve the Riccati equation.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[C1] R.Y. Chang, M.L. Wang: Shifted Legendre direct method for variational problems. J. Optimization Theory Appl. 19 (1983) 299-307.

[C2] R.Y. Chang, S.Y. Yang, M.L. Wang: Solutions of linear dynamic systems by generalized orthogonal polynomials, Internat. J. Systems Sci. 17 (1986) 1727-1740.

[C3] J.H. Chou, I.R. Horng: Shifted Jacobi series analysis of linear two-point boundary value problems, Internat. J. Systems Sci. 19 (1988) 161-169.

[G1] I.S. Gradszteyn, I.M. Ryzhik: Tables of Integrals, Series and Products (in Russian) Moskwa 1971.

[H1] I.R. Horng, I.R. Chou: Shifted Chebyshev direct method for solving variational problems, Internat. J. Systems Sci. 16 (1985) 855-861.

[H2] I.R. Horng, J.H. Chou: Solution of linear two-point boundary-value problems via shifted Chebyshev series, Internat. J. Systems Sci. 18 (1987) 293-300.

[H3] C. Hwang, Y.P. Shih: Laguerre Series Direct Method for Variational Problems, J. Optimization Theory Appl. 39 (1983) 143-149.

[L1] P. Lancaster, M. Tismenetsky: The Theory of Matrices, Academic Press, Orlando Fla, 1985.

[S1] D.H. Shih, F.C. Kung: Optimal Control of Deterministic Systems via Shifted Legendre Polynomials, IEEE Transactions on Automatic Control, vol. AC-31, 5 (1986) 451-454.

[S2] G. Szego: Orthogonal Polynomials. American Math. Society, New York 1975.

[T1] S.C. Tsay, T.T. Lee: Analysis and optimal control of linear time varying systems via general orthogonal polynomials. Internat. J. Systems Sci. 18 (1987)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1579-1594.

[W1] M.L. Wang, R.Y. Chang: Solution of linear dynamic system with initial or boundary value conditions by shifted Legendre approximations. Internat. J. Systems Sci. 14 (1983) 343-353.

[Y1] C.Y. Yang, C.K. Chen: Analysis and optimal control of time varying systems via polynomial series, Internat. J. Systems Sci. 19 (1988) 289-310.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ, 90-238 ŁÓDŹ, POLAND

Received October 25, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII NO 3 1990

Vijai Gupta, P.N. Agrawal, A. Sahai and T.A.K. Sinha

L_p-APPROXIMATION BY COMBINATION OF MODIFIED SZASZ-MIRAKYAN OPERATORS

1. Introduction and preliminary results

For $f \in L_p[0,\infty)$ $(p \geq 1)$, Kasan et al. [3] proposed a sequence of modified Szász-Mirakyan operators defined as

$$S_n(f,t) = \int_0^\infty W(n,t,u) f(u)\,du, \tag{1.1}$$

where

$$W(n,t,u) = n\sum_{\nu=1}^{\infty} p_{n\nu}(t) p_{n\nu}(u) \ ;$$

$$p_{n\nu}(t) = e^{-nt}\frac{(nt)^\nu}{\nu!} \ .$$

It turns out that however smooth the function may be, the order of approximation by these operators is $0(n^{-1})$. In order to improve this order of approximation, we apply the technique of linear combination to (1.1):

$$S_n(f,k,t) = \begin{vmatrix} 1 & d_0^{-1} & \dots & d_0^{-k} \\ 1 & d_1^{-1} & \dots & d_1^{-k} \\ \cdot & \cdot & \dots & \cdot \\ 1 & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} S_{d_0 n}(f,t) & d_0^{-1} & \dots & d_0^{-k} \\ S_{d_1 n}(f,t) & d_1^{-1} & \dots & d_1^{-k} \\ \cdot & \cdot & \dots & \cdot \\ S_{d_k n}(f,t) & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix} \tag{1.2}$$

where d_i, $i = 0,1,\dots,k$ are distinct positive integers. Combination of this type were first considered by Rathore [5] and May [4], and it is easily seen that they can be alternately written in the following form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$S_n(f,k,t) = \sum_{j=0}^{k} C(j,k)(S_{d_j n}(f,t)$$

where $C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}$, $k \neq 0$ and $C(0,0) = 1$.

Throughout this paper, let $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$, $0 < a < b < \infty$; $I_i = [a_i, b_i]$, $i = 1,2,3$ and $[\beta]$ denote the integral part of β.

In the proof of our main results, we have used the Steklov means briefly introduced below:

Let $1 \leq p < \infty$, $f \in L_p[0,\infty)$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as

$$f_{\eta,m}(u) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\left(f(u)+(-1)^{m-1}\Delta_{\sum_{i=1}^{m}}^{m} f(u)\right)du_1 du_2 \ldots du_m,$$

where $u \in I_1$ and $\Delta_h^m f(u)$ is the mth order forward difference of the function f with step length h. It follows [2,6] that

(a) $f_{\eta,m}$ has derivatives up to order m, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m-1)}$ exists a.e. and belongs to $L_p(I_1)$;

(b) $\| f_{\eta,m}^{(r)} \|_{L_p(I_2)} \leq M_r \eta^{-r} \omega_r(f,n,p,I_1)$, $r = 1,2,\ldots,m$;

(c) $\| f - f_{\eta,m} \|_{L_p(I_2)} \leq M_{m+1}\omega_m(f,n,p,I_1)$;

(d) $\| f_{\eta,m} \|_{L_p(I_2)} \leq M_{m+2} \| f \|_{L_p(I_1)}$;

(e) $\| f_{\eta,m}^{(m)} \|_{L_p(I_2)} \leq M_{m+3}\eta^{-m} \| f \|_{L_p(I_1)}$,

where AC([a,b]) stands for the class of absolutely continuous functions on [a,b] and M_i's are certain constants that depend on i but are independent of f and n.

Let BV([a,b]) denote the set of all functions of bounded variation on [a,b]. Then the seminorm $\| f \|_{BV([a,b])}$ is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

defined by the total variation of f on [a,b].

Lemma 1 (see [2]). Let the function $T_{n,m}(t)$ (m being a non-negative integer) be defined as

$$T_{n,m}(t) = n\sum_{\nu=0}^{\infty} p_{n\nu}(t)\int_0^{\infty} p_{n\nu}(u)(u-t)^m du,$$

then we have the following recurrence relation

$$nT_{n,m+1}(t) = tT'_{n,m}(t) + (m+1)T_{n,m}(t) + 2mtT_{n,m-1}(t), \quad m \in N.$$

Consequently

(a) $T_{n,m}(t)$ is a polynomial in t and n^{-1} of degree $[\frac{m}{2}]$ in t and m in n^{-1};

(b) For every $t \in [0,\infty)$, $T_{n,m}(t) = 0(n^{-[(m+1)/2]})$.

Therefore on using Hölder's inequality, we have

(1.3) $S_n(|u-t|^r,t) = 0(n^{-r/2})$

for each $r > 0$ and for every fixed $t \in [0,\infty)$.

Lemma 2. For $p \in N$ and n sufficiently large there holds $S_n((u-t)^p,k,t) = n^{-(k+1)}\{Q(p,k,t) + o(1)\}$, $t \in [0,\infty)$ where $Q(p,k,t)$ is a certain polynomial in t of degree [p/2].

Proof. From Lemma 1, for sufficiently large n we can write

$$S_n((u-t)^p,t) = \frac{p_0(t)}{n^{[(p+1)/2]}} + \frac{p_1(t)}{n^{[(p+1)/2]+1}} + \ldots + \frac{p_{[p/2]}(t)}{n^p},$$

where p_i, s are certain polynomial in t of degree [i/2].

Using the above in (1.2) we obtain the required result.

Lemma 3 (see [1]). Let $1 \le p < \infty$, $f \in L_p[a,b]$, $f^{(k)} \in AC[a,b]$ and $f^{(k+1)} \in L_p[a,b]$. Then

$$\|f^{(j)}\|_{L_p[a,b]} \le K_j(\|f^{(k+1)}\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}),$$

$j = 1,2,\ldots,k$, where K_j, s are certain constans depending only on j, k, p, a and b.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Main results

Theorem 1. Let $f \in L_p[0,\infty)$, $p > 1$. If f has $2k+2$ derivatives on I_1 with $f^{(2k+1)} \in AC(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then for all n sufficiently large

$$\|S_n(f,k,\cdot) - f\|_{L_p(I_2)} \le Mn^{-(k+1)}\{\|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)}\}$$

where M is a constant independent of f and n.

Proof. By the hypothesis, for $t \in I_2$ and $u \in I_1$

$$f(u) = \sum_{j=0}^{2k+1} \frac{(u-t)^j}{j!} f^{(j)}(t) + \frac{1}{(2k+1)!}\int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)dw.$$

Hence we can write

$$(2.1)\quad f(u) = \sum_{j=0}^{2k+1} \frac{(u-t)^j}{j!} f^{(j)}(t) +$$

$$+ \frac{1}{(2k+1)!}\int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)dw + F(u,t)(1-\Phi(u)),$$

where $\Phi(u)$ stands for the characteristic function of I_1 and

$$F(u,t) = f(u) - \sum_{j=0}^{2k+1} \frac{(u-t)^j}{j!} f^{(j)}(t),$$

for all $u \in [0,\infty)$ and $t \in I_2$. Using (2.1) in (1.2), we have

$$S_n(f,k,t) - f(t) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(t)}{j!} S_n((u-t)^j,k,t) +$$

$$+ \frac{1}{(2k+1)!} S_n(\Phi(u)\int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)dw,k,t) +$$

$$+ S_n(F(u,t)(1-\Phi(u)),k,t) = \Sigma_1 + \Sigma_2 + \Sigma_3 .$$

In view of Lemma 2 and [1]

$$\|\Sigma_1\|_{L_p(I_2)} \le C_1 n^{-(k+1)}\left(\sum_{j=1}^{2k+1} \|f^{(j)}\|_{L_p(I_2)}\right) \le$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq C_2 n^{-(k+1)}(\|f^{(j)}\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)}) .$$

To estimate L_2, let h_f be the Hardy-Littlewood majorant [7] of $f^{(2k+2)}$ on I_1. Use of Hölder's inequality and (1.3) lead to:

$$J_1 \equiv |S_n(\Phi(u)\int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)dw, t)| \leq$$

$$\leq S_n(\Phi(u)|\int_t^u |u-w|^{2k+1}|f^{(2k+2)}(w)|dw|, t) \leq$$

$$\leq S_n(\Phi(u)|u-t|^{2k+1}\ |\int_t^u |f^{(2k+2)}(w)|dw|, t) \leq$$

$$\leq S_n(\Phi(u)(u-t)^{2k+2}|h_f(u)|, t) \leq$$

$$\leq \{S_n(|u-t|^{(2k+2)q}\Phi(u), t)\}^{1/q} . \{S_n(|h_f(u)|^p\Phi(u), t)\}^{1/p} \leq$$

$$\leq C_3 n^{-(k+1)}\{S_n(|h_f(u)|^p\Phi(u), t)\}^{1/p} \leq$$

$$\leq C_3 n^{-(k+1)}(\int_{a_1}^{b_1} W(n,t,u)|h_f(u)|^p du)^{1/p} .$$

Fubini's theorem and [8, Chap. 2] imply that

$$|J_1|^p_{L_p(I_2)} \leq C_3 n^{-(k+1)p}\int_{a_2}^{b_2}\int_{a_1}^{b_1} W(n,t,u)|h_f(u)|^p du dt \leq$$

$$\leq C_3 n^{-(k+1)p}\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} W(n,t,u)dt\right)|h_f(u)|^p du \leq$$

$$\leq C_3 n^{-(k+1)p}\int_{a_1}^{b_1}|h_f(u)|^p du \leq$$

$$\leq C_3 n^{-(k+1)p}\|h_f(u)\|^p_{L_p(I_1)} \leq$$

$$\leq C_4 n^{-(k+1)p}\|f^{(2k+2)}\|^p_{L_p(I_1)} .$$

Therefore, $\|J_1\|_{L_p(I_2)} \leq C_4 n^{-(k+1)}\|f^{(2k+2)}\|_{L_p(I_1)}$. Consequently,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|\Sigma_2\|_{L_p(I_2)} \le C_5 n^{-(k+1)} \|f^{(2k+2)}\|_{L_p(I_1)}.$$

For $u \in [0,\infty)\setminus[a_1,b_1]$, $t \in I_2$ there exists a $\delta > 0$ such that $|u-t| \ge \delta$. Thus

$$|S_n(F(u,t)(1-\Phi(u)),t| \le \delta^{-(2k+2)} S_n(|F(u,t)|(u-t)^{2k+2},t) \le$$

$$\le \delta^{-(2k+2)} S_n((|f(u)| + \sum_{j=0}^{2k+1} \frac{|u-t|^j}{j!}|f^{(j)}(t)|)(u-t)^{2k+2},t) =$$

$$\le \delta^{-(2k+2)} [S_n(|f(u)|(u-t)^{2k+2},t) +$$

$$+ \sum_{j=1}^{2k+1} \frac{|f^{(j)}(t)|}{j!} S_n(|u-t|^{2k+2+j},t)] = J_2 + J_3 .$$

Using Hölder's inequality and (1.3) we get

$$|J_2| \le \delta^{-(2k+2)} (S_n(|f(u)|^p,t))^{1/p} (S_n(|u-t|^{(2k+2)q},t))^{1/q} \le$$

$$\le C_6 n^{-(k+1)} (S_n(|f(u)|^p,t))^{1/p} .$$

Again, applying Fubini's theorem, we get

$$\int_{a_2}^{b_2} |J_2|^p dt \le C_6 n^{-p(k+1)} \int_{a_2}^{b_2} \int_0^\infty W(n,t,u)|f(u)|^p du dt \le$$

$$\le C_7 n^{-p(k+1)} \|f\|^p_{L_p[0,\infty)} .$$

Thus

$$\|J_2\|_{L_p(I_2)} \le C_7 n^{-(k+1)} \|f\|_{L_p[0,\infty)} .$$

Moreover, using (1.3) and [1], we get

$$\|J_1\|_{L_p(I_2)} \le C_8 n^{-(k+1)} \sum_{j=0}^{2k+1} \|f^{(j)}\|_{L_p(I_2)} \le$$

$$\le C_8 n^{-(k+1)} (\|f\|_{L_p(I_2)} + \|f^{(2k+2)}\|_{L_p(I_2)}) .$$

Combining the estimates of J_2 and J_3, we are led to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|\Sigma_2\|_{L_p(I_2)} \le C_9 n^{-(k+1)}(\|f\|_{L_p[0,\infty)} + \|f^{(2k+2)}\|_{L_p(I_2)}) .$$

Hence we obtain the result.

Theorem 2. Let $f \in L_1[0,\infty)$. If f has 2k+1 derivatives in I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large we have

$$\|S_n(f,k,\cdot)-f\|_{L_1(I_2)} \le Mn^{-(k+1)}(\|f^{(2k+1)}\|_{BV(I_1)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)}) ,$$

where M is a constant independent of f and n.

Proof. By the given assumptions on f, for almost all $t \in I_2$ and for all $u \in I_1$, we have

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w).$$

We can write

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) +$$

$$+ \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w)\Phi(u) + F(u,t)(1-\Phi(u)) ,$$

where $\Phi(u)$ denotes the characteristic function of I_1 and

$$F(u,t) = f(u) - \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t),$$

for almost all $t \in I_2$ and for all $u \in [0,\infty)$. Hence, we obtain

$$S_n(f,k,t)-f(t) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} S_n((u-t)^i,k,t) +$$

$$+ \frac{1}{(2k+1)!} S_n\left(\int_t^u (u-w)^{2k+1} df^{(2k+1)}(w)\Phi(u),k,t\right) +$$

$$+ S_n(F(u,t)(1-\Phi(u)),k,t) = J_1 + J_2 + J_3 .$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Applying Lemma 2 and [1] we get

$$\|J_1\|_{L_1(I_2)} \le C_1 n^{-(k+1)}(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)}).$$

Further, we have

$$K \equiv \|S_n(\int_t^u (u-w)^{2k+1} df^{(2k+1)}(w)\Phi(u),t)\|_{L_1(I_2)} \le$$

$$\le \int_{a_2}^{b_2}\int_{a_1}^{b_1} W(n,t,u)|u-t|^{2k+1}|\int_t^u |df^{(2k+1)}(w)|\,|dudt\,.$$

For each n there exists a non-negative integer $r = r(n)$ such that $rn^{-1/2} < \max(b_1-a_2, b_2-a_1) \le (r+1)n^{-1/2}$. Then we have

$$K \le \sum_{l=0}^{r}\int_{a_2}^{b_2}\Big\{\int_{t+ln^{-1/2}}^{t+(l+1)n^{-1/2}} \Phi(u)W(n,t,u)|u-t|^{2k+1}\Big(\int_t^{t+(l+1)n^{-1/2}} \Phi(w) \times$$

$$\times |df^{(2k+1)}(w)|\Big)du + \int_{t-(l+1)n^{-1/2}}^{t-ln^{-1/2}} \Phi(u)W(n,t,u)|u-t|^{2k+1} \times$$

$$\times \Big(\int_{t-(l+1)n^{-1/2}}^{t} \Phi(w)|df^{(2k+1)}(w)|\Big)du\Big\}dt\,.$$

Let $\Phi_{t,c,d}(w)$ denote the characteristic function of the interval $[t-cn^{-1/2}, t+dn^{-1/2}]$ where c,d are nonnegative integers. Then we have

$$K \le \sum_{l=1}^{r}\int_{a_2}^{b_2}\Big\{\int_{t+ln^{-1/2}}^{t+(l+1)n^{-1/2}} \Phi(u)W(n,t,u)l^{-4}n^2|u-t|^{2k+5} \times$$

$$\times \Big(\int_t^{t+(l+1)n^{-1/2}} \Phi(w)\Phi_{t,0,l+1}(w)|df^{(2k+1)}(w)|\Big)du +$$

$$+ \int_{t-(l+1)n^{-1/2}}^{t-ln^{-1/2}} \Phi(u)W(n,t,u)l^{-4}n^2|u-t|^{2k+5} \times$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\times\left(\int_{t-(l+1)n^{-1/2}}^{t}\Phi(w)\Phi_{t,l+1,0}(w)\left|df^{(2k+1)}(w)\right|\right)du\Big)dt +$$

$$+\int_{a_2}^{b_2}\int_{-n^{-1/2}}^{a_1+n^{-1/2}}\Phi(u)W(n,t,u)|u-t|^{2k+1}\times$$

$$\times\left(\int_{t-n^{-1/2}}^{t+n^{-1/2}}\Phi(u)\Phi_{t,1,1}(w)\left|df^{(2k+1)}(w)\right|\right)dudt \le$$

$$\le\sum_{l=1}^{r}\left(1^{-4}n^{2}\int_{a_2}^{b_2}\left(\int_{t+ln^{-1/2}}^{t+(l+1)n^{-1/2}}\Phi(u)W(n,t,u)|u-t|^{2k+5}\times\right.\right.$$

$$\times\left(\int_{a_1}^{b_1}\Phi_{t,0,l+1}(w)\left|df^{(2k+1)}(w)\right|\right)du +$$

$$+\int_{t-(l+1)n^{-1/2}}^{t-ln^{-1/2}}\Phi(u)W(n,t,u)|u-t|^{2k+5}\times$$

$$\times\left(\int_{a_1}^{b_1}\Phi_{t,l+1,0}(w)\left|df^{(2k+1)}(w)\right|\right)du\Big)dt +$$

$$+\int_{a_2}^{b_2}\int_{-n^{-1/2}}^{a_1+n^{-1/2}}\Phi(u)W(n,t,u)|u-t|^{2k+1}\times$$

$$\times\left.\left(\int_{a_1}^{b_1}\Phi_{t,1,1}(w)\left|df^{(2k+1)}(w)\right|dudt\right)\right).$$

By using Lemma 1 and Fubini's theorem in the next step we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$K \le C_2 n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{t,0,l+1}(w)\,|df^{(2k+1)}(w)|\,dt +$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{t,l+1,0}(w)\,|df^{(2k+1)}(w)|\,dt\Big) +$$

$$+ \int_{a_2}^{b_2}\int_{a_1}^{b_1} \Phi_{t,1,1}(w)\,|df^{(2k+1)}(w)|\,dt\Big) =$$

$$= C_3 n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \Phi_{t,0,l+1}(w)\,dt\Big)|df^{(2k+1)}(w)| +$$

$$+ \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \Phi_{t,l+1,0}(w)\,dt\Big)|df^{(2k+1)}(w)| +$$

$$+ \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \Phi_{t,1,1}(w)\,dt\Big)|df^{(2k+1)}(w)|\Big) \le$$

$$\le C_4 n^{-(2k+1)/2}\Big(\sum_{l=1}^{r} l^{-4}\Big(\int_{a_1}^{b_1}\int_{w-(l+1)n^{-1/2}}^{w} dt\Big)|df^{(2k+1)}(w)| +$$

$$+ \int_{a_1}^{b_1}\Big(\int_{w}^{w+(l+1)n^{-1/2}} dt\Big)|df^{(2k+1)}(w)| +$$

$$+ \int_{a_1}^{b_1}\Big(\int_{w-n^{-1/2}}^{w+n^{-1/2}} dt\Big)|df^{(2k+1)}(w)|\Big) \le C_5 n^{-(k+1)}\|f^{(2k+1)}\|_{BV(I_1)},$$

Hence,

$$\|J_2\|_{L_1(I_2)} \le C_6 n^{-(k+1)}\|f^{(2k+1)}\|_{BV(I_1)},$$

where the constant on the right side depends on k.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For all $u \in [0,\infty)\setminus[a_1,b_1]$, $t \in I_2$, we can chose a $\delta > 0$ such that $|u-t| \geq \delta$. Then

$$\|S_n(F(u,t)(1-\Phi(u)),t)\|_{L_1(I_2)} \leq$$

$$\leq \int_{a_2}^{b_2}\int_0^{\infty} W(n,t,u)|f(u)|(1-\Phi(u))dudt +$$

$$+ \sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty} W(n,t,u)|f^{(i)}(t)||u-t|^i(1-\Phi(u))dudt = J_4+J_5, .$$

For sufficiently large u, there exixt positive constants M_0 and C_7 such that $\frac{(u-t)^{2k+2}}{u^{2k+2}+1} > C_7$ for all $u \geq M_0$, $t \in I_2$.

Also, by Fubini's theorem,

$$J_4 = \left(\int_0^{M_0}\int_{a_2}^{b_2} + \int_{M_0}^{\infty}\int_{a_2}^{b_2}\right)W(n,t,u)|f(u)|(1-\Phi(u))dtdu = J_6+J_7.$$

Now, using Lemma 1 we have

$$J_6 = \delta^{-(2k+2)}\int_0^{M_0}\int_{a_2}^{b_2} W(n,t,u)|f(u)|(u-t)^{2k+2}dtdu \leq$$

$$\leq C_8 n^{-(k+1)}\left(\int_0^{M_0}|f(u)|du\right)$$

and

$$J_7 = \frac{1}{C_7}\int_{M_0}^{\infty}\int_{a_2}^{b_2} W(n,t,u)\frac{(u-t)^{2k+2}}{u^{2k+2}+1}|f(u)|dtdu \leq$$

$$\leq C_9 n^{-(k+1)}\int_{M_0}^{\infty}|f(u)|du .$$

Combining the estimates of J_6 and J_7, we get

$$J_4 \leq C_{10} n^{-(k+1)}\|f\|_{L_1[0,\infty)} .$$

Further, using (1.3) and [1] we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$J_5 \le \delta^{-(2k+2)} \sum_{i=0}^{2k+1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^{\infty} W(n,t,u) |f^{(i)}(t)| |u-t|^{2k+i+2} du dt \le$$

$$\le C_{11} n^{-(k+1)} \left(\sum_{i=0}^{2k+1} \|f^{(i)}\|_{L_1(I_2)} \right) \le$$

$$\le C_{11} n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right).$$

The above estimates of J_4 and J_5 lead us to

$$\|S_n(F(u,t)(1-\Phi(u)),t)\|_{L_1(I_2)} \le$$

$$\le C_{12} n^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right).$$

Consequently we obtain

$$\|J_3\|_{L_1(I_2)} \le C_{13} n^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \right).$$

Finally combining the estimates of J_1, J_2 and J_3, we obtain the required result.

Theorem 3. Let $f \in L_p[0,\infty)$, $p \ge 1$. Then for all n sufficiently large

$$\|S_n(f,k,\cdot)-f\|_{L_p(I_2)} \le$$

$$\le M_k \left(\omega_{2k+2}(f, n^{-1/2}, p, I_1) + n^{-(k+1)} \|f\|_{L_p[0,\infty)} \right),$$

Where M_k is a constant independent of f and n.

Proof. Let $f_{\eta,2k+2}(u)$ be the Steklov mean of $(2k+2)$th order corresponding to $f(u)$ where $\eta > 0$ is sufficiently small and $f(u)$ is defined as zero outside $[0,\infty)$. Then we have

$$\|S_n(f,k,\cdot)-f\|_{L_p(I_2)} \le \|S_n(f-f_{\eta,2k+2},k,\cdot)\|_{L_p(I_2)} +$$

$$+ \|S_n(f-f_{\eta,2k+2},k,\cdot) - f_{\eta,2k+2}\|_{L_p(I_2)} +$$

$$+ \|f_{\eta,2k+2}-f\|_{L_p(I_2)} = \Sigma_1 + \Sigma_2 + \Sigma_3 .$$

To estimate Σ_1, let $\Phi(u)$ be the characteristic function of I_3. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$S_n((f-f_{\eta,2k+2})(u),t) = S_n(\Phi(u)\ (f-f_{\eta,2k+2})(u),t) +$

$+ S_n((1-\Phi(u))\ (f-f_{\eta,2k+2})(u),t) = \Sigma_4 + \Sigma_5$.

The following is true for $p = 1$, the truth of the same for $p > 1$ follows from Hölder's inequality.

$$\int_{a_2}^{b_2} |\Sigma_4|^p\, dt \le \int_{a_2}^{b_2}\int_{a_3}^{b_3} W(n,t,u)\,|(f-f_{\eta,2k+2})(u)|^p\, dudt .$$

Now on applying Fubini's theorem, we get

$$\int_{a_2}^{b_2} |\Sigma_4|^p\, dt \le \int_{a_2}^{b_2}\int_{a_3}^{b_3} W(n,t,u)\,|(f-f_{\eta,2k+2})(u)|^p\, dtdu \le$$

$$\le \|f-f_{\eta,2k+2}\|^p_{L_p(I_3)} .$$

Hence

$$\|\Sigma_4\|_{L_p(I_2)} \le \|f-f_{\eta,2k+2}\|_{L_p(I_3)} .$$

Using Hölder's inequality, (1.3) and Fubini's theorem we get the following for $p \ge 1$

$$\|\Sigma_5\|_{L_p(I_2)} \le C_1 n^{-(k+1)} \|f-f_{\eta,2k+2}\|_{L_p[0,\infty)} .$$

Now, use of Jensen's inequality and Fubini's theorem lead us to

$$\|f_{\eta,2k+2}\|_{L_p[0,\infty)} \le C_2 \|f\|_{L_p[0,\infty)} .$$

Hence

$$\|\Sigma_5\|_{L_p(I_2)} \le C_3 n^{-(k+1)} \|f\|_{L_p[0,\infty)} .$$

Again by the property (c) of Steklov means, we get

$$\Sigma_1 \le C_4(\omega_{2k+2}(f,\eta,p,I_1) + n^{-(k+1)}\|f\|_{L_p[0,\infty)}) .$$

It is well known that,

$$|f^{(2k+1)}_{\eta,2k+2}|_{BV(I_3)} = \|f^{(2k+2)}_{\eta,2k+2}\|_{L_1(I_3)} .$$

Therefore, by virtue of Theorem 1 ($p > 1$), Theorem 2 ($p = 1$)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and Lemma 3 we have

$$\Sigma_2 \le C_5 n^{-(k+1)} (\|f^{(2k+2)}_{\eta,2k+2}\|_{L_p(I_3)} + \|f_{\eta,2k+2}\|_{L_p[0,\infty)}) \le$$

$$\le C_6 n^{-(k+1)} (\eta^{-(2k+2)} \omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)}) ,$$

in view of the properties of Steklov means.

To estimate Σ_3, we use the Steklov means property (c), and obtain that

$$\Sigma_3 \le C_6 \omega_{2k+2}(f,\eta,p,I_1) .$$

Then the required result follows. This ends the proof of Theorem 3.

Acknowledgement. The authors are extremely grateful to the referee for the critical review of the paper as also for pointing out some errors and omissions.

REFERENCES

[1] **S. Goldberg, A. Meir:** Minimum moduli of ordinary differential operators, Proc. London Math. Soc. 3, 23(1971), 1-15.

[2] **E. Hewitt, K. Stromberg:** Real and Abstract Analysis. McGraw-Hill, New York (1956).

[3] **H.S. Kasana, G. Prasad, P.N. Agraval, A. Sahai:** On modified Szász-Operators, Conf. on Math. Anal. and Appl. Kuwait (1985).

[4] **C.P. May:** Saturation and inverse theorems for combinations of a class of exponential type operators, Canad. J. Math., 28(1976),1224-1250.

[5] **R.K.S. Rathore:** Linear combinations of linear positive operators and generating relations in special functions, Ph.D. Thesis, IIT Delhi (India)(1973).

[6] **A. Timan:** Theory of Approximation of Functions of Real Variable, Macmillan, N.Y. (1963).

[7] **B. Wood:** L_p-Approximation by linear combination of integral Bernstein type operators, Anal. Numer. Theor. Approx. 1,13(1984),65-72.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[8] A. Zygmund: Trigonometrical Series, Dover Publications Inc. N.Y. (1955).

V. Gupta, P.N. Agraval, A. Sahai: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROORKEE, ROORKEE-247667 /U.P/ INDIA;
T. A. K. Sinha: DEPARTMENT OF MATHEMATICS, S.M.D. COLLEGE, POONPOON, PATNA-800001 /BIHAR/, INDIA.

Received November 20, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

B. G. Pachpatte

ON AN INTEGRAL INEQUALITY IN TWO VARIABLES

1. Introduction

In a celebrated paper of 1960, Z. Opial [2] proved an integral inequality which has received wide attention and has been the subject of various generalizations [1, pp. 154-162]. In 1967 E. K. Godunova and V. I. Levin [1, Theorem 13, p. 159] obtained a very interesting generalization of Opial's inequality in the following form

$$(1)\qquad \int_a^b \Phi'(|u(x)|)|u'(x)|dx \le \Phi\left(\int_a^b |u'(x)|dx\right),$$

where $u(x)$ is absolutely continuous function defined on $[a,b]$ with $u(a) = 0$ and $\Phi(r)$ is a convex increasing function for $r \ge 0$ and $\Phi(0) = 0$. In 1982 G. S. Yang [6] proved the following two independent variable analogue of the Opial inequality [2].

Let $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$ be continuous functions on $[a,b]\times[c,d]$ and satisfy $f(a,t) = f(b,t) = f_1(s,c) = f_1(s,d) = 0$ for $a \le s \le b$, $c \le t \le d$. Then

$$(2)\qquad \int_a^b\int_c^d |f(s,t)||f_{12}(s,t)|dtds \le M\int_a^b\int_c^d |f_{12}(s,t)|^2 dtds,$$

where $M = \frac{(a-b)(d-c)}{8}$ and $f_1(s,t) = \frac{\partial}{\partial s}f(s,t)$, $f_{12}(s,t) = \frac{\partial^2}{\partial s\partial t}f(s,t)$.

Inequalities of the form (2) have an interest in their own right and a number of new integral inequalities of the type (2) can be found in a series of recent papers [3]-[5]. The main purpose of the present note is to establish a new

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

integral inequality in two independent variables which claim its origin to the inequality (1) and in the special cases yields the inequality (2) and its generalization given by the present author in [3].

2. Statement of results

Throughout we use the notation $g'(x) = \frac{d}{dx}g(x)$, $D_1h(s,t) = \frac{\partial}{\partial s}h(s,t)$, $D_2h(s,t) = \frac{\partial}{\partial t}h(s,t)$, $D_2D_1h(s,t) = \frac{\partial^2}{\partial t\partial s}h(s,t)$, $\Delta = [a,b]\times[c,d]$, $\Delta_1 = [a,\alpha]\times[c,\beta]$, $\Delta_2 = [a,\alpha]\times[\beta,d]$, $\Delta_3 = [\alpha,b]\times[c,\beta]$, $\Delta_4 = [\alpha,b]\times[\beta,d]$ for $a \le \alpha \le b$, $c \le \beta \le d$ and

$$L[f,D_2D_1f,H,H'] = [H(|f(s,t)|) + H'(|f(s,t)|)|f(s,t)|]|D_2D_1f(s,t)|,$$

$$I_1D_2D_1f = \int_a^\alpha\int_c^\beta |D_2D_1f(s,t)|dtds,\quad I_2D_2D_1f = \int_a^\alpha\int_\beta^d |D_2D_1f(s,t)|dtds,$$

$$I_3D_2D_1f = \int_\alpha^b\int_c^\beta |D_2D_1f(s,t)|dtds,\quad I_4D_2D_1f = \int_\alpha^b\int_\beta^d |D_2D_1f(s,t)|dtds,$$

where g, h, f, H are suitable functions defined on the respective domains of their definitions.

Our main result is embodied in the following

Theorem 1. Let $f(s,t)$, $D_1f(s,t)$, $D_2D_1f(s,t)$ be real-valued continuous functions on Δ and $f(s,t) = f(b,t) = D_1f(s,c) = D_1f(s,d) = 0$ for $a \le s \le b$, $c \le t \le d$. Let $H(r)$ be a nonnegative, continuous, nondecreasing function for $r \ge 0$ such that $H'(r)$ exists, nonnegative, continuous and nondecreasing for $r \ge 0$. Then

$$(3)\qquad \int_a^b\int_c^d L[f,D_2D_1f,H,H']dtds \le \sum_{j=1}^{4} H(I_jD_2D_1f)(I_jD_2D_1f),$$

for $a \le \alpha \le b$, $c \le \beta \le d$.

As a special case of Theorem 1, we have

Corollary. Assume that in the hypothesis of Theorem 1 we have $H(r) = r^p$, where $p \ge 1$ is a constant. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4)\quad \int_a^b\int_c^d |f(s,t)|^p |D_2D_1f(s,t)|dtds \le M\int_a^b\int_c^d |D_2D_1f(s,t)|^{p+1}dtds,$$

where $M = \frac{1}{p+1}\left\{\frac{(b-a)(d-c)}{4}\right\}^p$.

We note that the inequality (4) is established by the present author in [3, p. 265] and in the special case when $p = 1$ it reduces to the inequality (2) given by G.S. Yang [6].

3. Proofs of Theorem 1 and Corollary 2

In order to establish the inequality (3) we obtain the estimates in the following four cases.

Case I. Let $(s,t) \in \Delta_1$ and define

$$(5)\quad z(s,t) = \int_a^s\int_c^t |D_2D_1f(u,v)|dvdu .$$

From (5) we have

$$(6)\quad D_1z(s,t) = \int_c^t |D_2D_1f(s,v)|dv , \quad D_2z(s,t) = \int_a^s |D_2D_1f(u,t)|du.$$

From (6) we observe that for each fixed s, $z(s,t)$ is nondecreasing for t on $[c,\beta]$. Since $f(a,t) = 0$ and $D_1f(s,c) = 0$ for $(s,t) \in \Delta_1$, we have

$$(7)\quad |f(s,t)| \le \int_a^s |D_1f(u,t)|du ,$$

$$(8)\quad D_1f(s,t) \le \int_c^t |D_2D_1f(s,v)|dv = D_1z(s,t) .$$

From (7) and (8) we observe that

$$(9)\quad |f(s,t)| \le \int_a^s |D_1f(u,t)|du = z(s,t) .$$

From (9), (6), (5) and the hypotheses of Theorem 1 we observe that

$$(10)\quad \int_a^\alpha\int_c^\beta L[f,D_2D_1f,H,H']dtds \le$$

$$\le \int_a^\alpha\int_c^\beta [H(z(s,t)) + H'(z(s,t))z(s,t)]|D_2D_1f(s,t)|dtds \le$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq \int_a^{\alpha}\left\{[H(z(s,\beta)) + H'(z(s,\beta))z(s,\beta)] \int_c^{\beta} |D_2D_1f(s,t)|dt\right\}ds =$$

$$= \int_a^{\alpha}[H(z(s,\beta)) + H'(z(s,\beta))z(s,\beta)]D_1z(s,\beta)ds =$$

$$= \int_a^{\alpha}\frac{d}{ds}[H(z(s,\beta))z(s,\beta)]ds = H(z(\alpha,\beta))z(\alpha,\beta) =$$

$$= H(I_1D_2D_1f)(I_1D_2D_1f).$$

Case II. Let $(s,t) \in \Delta_2$ and define

$$z(s,t) = \int_a^s\int_t^d |D_2D_1f(u,v)|dvdu . \tag{11}$$

Case III. Let $(s,t) \in \Delta_3$ and define

$$z(s,t) = \int_s^b\int_c^t |D_2D_1f(u,v)|dvdu . \tag{12}$$

Case IV. Let $(s,t) \in \Delta_4$ and define

$$z(s,t) = \int_a^b\int_t^d |D_2D_1f(u,v)|dvdu . \tag{13}$$

Now by following similar arguments to those in the proof of Case I, but with suitable modifications (see [5], p. 53-54), we obtain the following estimates in Cases II, III and IV, respectively

$$\int_a^{\alpha}\int_{\beta}^d L[f,D_2D_1f,H,H']dtds \leq H(I_2D_2D_1f)(I_2D_2D_1f) , \tag{14}$$

$$\int_{\alpha}^b\int_c^{\beta} L[f,D_2D_1f,H,H']dtds \leq H(I_3D_2D_1f)(I_3D_2D_1f) , \tag{15}$$

and

$$\int_{\alpha}^b\int_{\beta}^d L[f,D_2D_1f,H,H']dtds \leq H(I_4D_2D_1f)(I_4D_2D_1f) . \tag{16}$$

Adding (10), (14), (15) and (16) we get the desired inequality in (3). This completes the proof of Theorem 1.

By taking $H(r) = r^p$, $p \geq 1$ is a constant and hence $H'(r) = pr^{p-1}$ in (3), we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(17)\quad (p+1)\int_a^b\int_c^d |f(s,t)|^p |D_2D_1f(s,t)|\,dt\,ds \le \sum_{j=1}^{4} (I_jD_2D_1f)^{p+1}\ .$$

Now by using the Hölder's inequality with indices $p+1$, $\frac{p+1}{p}$ to each integral on the right side of (17) and then taking $\alpha = \frac{a+b}{2}$ and $\beta = \frac{c+d}{2}$, we get the required inequality in (4). This completes the proof of Corollary 2.

REFERENCES

[1] D.S. Mitrinović: Analytic inequalities. Springer-Verlag, Berlin 1970.

[2] Z. Opial: Sur une inégalité, Ann. Polon Math. 8 (1960), 29-32.

[3] B.G. Pachpatte: On Opial type inequalities in two independent variables, Proc. Royal Soc. Edinburgh 100A (1985), 263-270.

[4] B.G. Pachpatte: On two inequalities similiar to Opial's inequality in two independent variables, Periodica Math. Hungarica 18 (1987), 137-141.

[5] G. Pachpatte: On two independent variable Opial type integral inequalities, J.Math. Anal. Appl. 125 (1987), 47-57.

[6] G.S. Yang: Inequality of Opial type in two variables, Tamakang J. Math. 13 (1982), 255-259.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MARATHWADA UNIVERSITY, AURANGABAD 431 004, (MAHARASHTRA), INDIA

Received December 1st, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Wanda Sznitko

ON SINGULAR INTEGRAL EQUATION IN EUCLIDEAN SPACE E_3

1. Introduction

Let S be a closed Lapunov surface bounding a domain D^+ in the space E_3. The complement of the set $D^+ \cup S$ to the space E_3 will be denote by D^-.

We introduce the following notations (see [1], p. 168):

$$\mathcal{D}(X,Y,Z) = \begin{bmatrix} 0 & X & Y & Z \\ X & 0 & -Z & Y \\ Y & Z & 0 & -X \\ Z & -Y & X & 0 \end{bmatrix}, \quad \mathcal{D}^*(X,Y,Z) = \begin{bmatrix} 0 & X & Y & Z \\ X & 0 & Z & -Y \\ Y & -Z & 0 & X \\ Z & Y & -X & 0 \end{bmatrix}$$

and

$$M(A,Q) = -\mathcal{D}^*\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) \frac{1}{|A-Q|}\, \mathcal{D}(\alpha,\beta,\gamma), \tag{1}$$

where $|A-Q|$ denotes the Euclidean distance between the points $A(X,Y,Z) \in D^+ \cup D^-$ and $Q(\xi,\eta,\zeta) \in S$; and $N(\alpha,\beta,\gamma)$ is the versor normal to the surface S at the point Q, directed outside the domain D^+.

Assume that the elements $q_i(Q)$, $i = 1,2,3,4$, of the vector-function $q(Q) = [q_1(Q), q_2(Q), q_3(Q), q_4(Q)]$ are defined and continuous on S. The vector-function

$$F(A) = \frac{1}{4\pi} \int_S M(A,Q)\, q(Q)\, dS_Q\,, \tag{2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

called a Cauchy type integral, is defined for every $A \in D^+ \cup D^-$. It is known ([1], p. 170]) that (2) is holomorphic in $D^+ \cup D^-$, i.e. it satisfies the elliptic system of equations

$$\mathcal{D}\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F(A) = 0.$$

Moreover, if $q(Q) \in H$, i.e. $q_i(Q)$, $i = 1,2,3,4$, satisfies Hölder condition on S, then there exists a singular integral in the sense of Cauchy principal value

$$\frac{1}{4\pi} \int_S M(P,Q) q(Q) dS_Q \tag{3}$$

which can be expressed by the formula

$$\frac{1}{4\pi} \int_S M(P,Q) q(Q) dS_Q = \frac{1}{2} q(P) + \frac{1}{4\pi} \int_S M(P,Q) [q(Q) - q(P)] dS_Q, \tag{4}$$

The Hilbert problem in the space E_3 consists in finding a vector-function $\phi(P)$, holomorphic separately in D^+ and D^-, vanishing at infinity and satisfying for each $P \in S$ the condition

$$\phi^+(P) = G\phi^-(P) + g(P), \tag{5}$$

where $g(P)$ is a given vector-function and G is a matrix with constant elements having the form

$$G = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ -\beta_2 & \beta_1 & -\beta_4 & \beta_3 \\ -\beta_3 & \beta_4 & \beta_1 & -\beta_2 \\ -\beta_4 & -\beta_3 & \beta_2 & \beta_1 \end{bmatrix}. \tag{6}$$

It is known (see [1], p.176) that the only solution of this problem is the vector-function

$$\phi(A) = \frac{X(A)}{4\pi} \int_S M(A,Q) (X^+)^{-1} q(Q) dS_Q \tag{7}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where the function

$$X(A) = \begin{cases} X^+ = G, & \text{when } A \in D^+, \\ X^- = E, & \text{when } A \in D^-, \end{cases}$$

satisfies the condition $X^+(P) = GX^-(P)$, $P \in S$, (E denotes here the unit matrix).

The aim of this paper is to give (in a closed form) solution of certain singular integral equation with iterative singular operators. One-dimensional singular integral equation of this type was examined in [2].

2. Solution of singular equation

Consider the equation

$$(8) \quad A_0(F^+A_1F^+A_2F^+A_3q)(P) + B(F^-B_1F^-B_2q)(P) = f(P), \quad p \in S,$$

where

$$(F^{\pm}q)(P) = \frac{q(P)}{2} \pm \frac{1}{4\pi}\int_S M(P,Q)q(Q)dS_Q,$$

$A_0, A_1, A_2, A_3, B, B_1, B_2$ are constant matrices of the form (6) and $f(P) \in H$ is a given vector-function.

We shall find the solution of (8) in the set H and shall show that it can be reduced to successive solution of five Hilbert boundary value problems. Writing (8) in the form

$$(9) \quad (F^+A_1F^+A_3q)(P) + A^{-1}B(F^-B_1F^-B_2q)(P) = A_0^{-1}f(P)$$

and taking into account the fact that $A_0^{-1}B$ has the form (6), we state, that (9) can be considered as a Hilbert boundary condition. Denoting by

$$A^{-1}B = \begin{cases} X^+(P), & P \in D^+, \\ E, & P \in D^-, \end{cases}$$

we see that the solution of (9) has the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(10) $$(F^+A_1F^+A_2F^+A_3q)(P) = X^+(F^+B^{-1}f)(P), \quad P \in S,$$

and

(11) $$(F^-B_1F^-B_2q)(P) = (F^-B^{-1}f)(P), \quad P \in S.$$

Taking into account the equality

$(F^+\varphi)(P) = f(P) \iff \varphi(P) = f(P) + g^-(P)$, where $g^-(P)$ is an arbitrary vector-function holomorphic in D^-, we state that equality (10) holds iff there exists a vector-function $\nu_1(P) \in H$ such that

(12) $$A_1F^+A_2F^+A_3q)(P) = X^+(F^+B^{-1}f)(P) + (F^-\nu_1)(P), \quad P \in S.$$

Let us rewrite (12) in the form of Hilbert boundary condition

(13) $$(X_1^+)^{-1}(F^+A_2F^+A_3q)(P) - (F^-\nu_1)(P) = h_1(P), \quad P \in S,$$

where $X_1^+ = A_1^{-1}$, $h_1(P) = X^+(F^+B^{-1}f)(P)$. Vector-function $q(P)$ satisfies (13) iff

(14) $$(F^+A_2F^+A_3q)(P) = X_1^+(F^+h_1)(P), \quad P \in S,$$

and

(15) $$(F^-\nu_1)(P) = -(F^+h_1)(P), \quad P \in S.$$

Further, we have that $q(P)$ satisfies (14) iff there exists a vector-function $\nu_2(P) \in H$ such that

(16) $$(X_2^+)^{-1}(F^+A_3q)(P) - (F^-\nu_2)(P) = h_2(P), \quad P \in S,$$

where $X_2^+ = A_2^{-1}$, $h_2 = (F^+A_1^{-1}F^+h_1)(P)$. Similarly, the vector-function $q(P)$ satisfies (16) iff

(17) $$(F^+A_3q)(P) = X_2^+(F^+h_2)(P), \quad P \in S,$$

and

(18) $$(F^-\nu_2)(P) = -(F^-h_2)(P), \quad P \in S,$$

hold. Finally, (17) holds iff there exists a vector-function $\nu_3(P) \in H$ such that

(19) $$q(P) = A_3^{-1}A_2^{-1}(F^+h_2)(P) + A_3^{-1}(F^-\nu_3)(P), \quad P \in S.$$

Let us now consider (11). Making use of the equality

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(F^-\varphi)(P) = f(P) \iff \varphi(P) = h^+(P) + f(P),$$

where $h^+(P)$ is an arbitrary vector-function holomorphic in D^+, we state that $q(P)$ satisfies (11) iff there exists a vector-function $\mu_1(P) \in H$ such that

$$B_1^{-1}(F^+\mu_1)(P) - (F^-B_2q)(P) = h_3(P), \quad P \in S, \tag{20}$$

where $h_3(P) = -B_1^{-1}(F^-B^{-1}f)(P)$. The equality (20) holds iff

$$(F^+\mu_1)(P) = B_1(F^+h_3)(P), \quad P \in S, \tag{21}$$

and

$$(F^-B_2q)(P) = -(F^-h_3)(P), \quad P \in S. \tag{22}$$

We see that $q(P)$ is a solution of (22) iff

$$q(P) = -B_2^{-1}(F^{-1}h_3)(P) + B_2^{-1}(F^+\mu_2)(P), \quad P \in S, \tag{23}$$

where $\mu_2(P)$ is a vector-function from H.

Equating right-hand sides of (19) and (23), we have

$$A_3B_2^{-1}(F^+\mu_2)(P) - (F^-\nu_3)(P) = h_4(P), \quad P \in S, \tag{24}$$

where $h_4(P) = -A_3B_2^{-1}(F^-h_3)(P) - A_2^{-1}(F^+h_2)(P)$.

Equality (24) holds iff

$$(F^+\mu_2)(P) = B_2A_3^{-1}(F^+h_4)(P), \quad P \in S, \tag{25}$$

and

$$(F^-\nu_3)(P) = -(F^-h_4)(P), \quad P \in S. \tag{26}$$

Finally, from (19) and (26) we get the solution of (8) in the form

$$q(P) = A_3^{-1}A_2^{-1}(F^+h_2)(P) - A_3^{-1}(F^-h_4)(P).$$

More general equation

$$A(F^+A_1F^+A_2\cdots F^+A_mq)(P) + B(F^-B_1F^-B_2\cdots F^-B_nq)(P) = f(P),$$

where $A, A_1, \ldots, A_m, B, B_1, \ldots, B_n$ are constant matrices of the form (6), the given vector-function $f(P) \in H$ and the unknown vector-function $q(P) \in H$, can be solved in the same way.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] А. Бицадзе; Краевые задачи для еллиптических уравнений второго порядка. Москва, 1966.

[2] И.И. Коняк; О решении в замкнутой форме одномерных сингулярных интегральных уравнений, Diff. Urav. 12 (1981) 2224-2237.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received December 12, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Antoni Sadowski

CONCERNING SOME SEQUENCES OF ITERATES IN R^n- SPACE

Introduction

In the examination of Goursat-like problems, via reducing them to functional equations and then finding the solution in series form, an important role is played by the properties of the sequences of iterates that appear in the formula defining the said solution. Such properties for the problems in $\mathbb{R}^2$-space were examined by A.Bielecki and J.Kisyński ([1], Lemma 3; see also [6], p.103) in case of a 2nd order partial differential equation and then by A.Borzymowski ([2], lemma 4) and M.W.Michalski ([8], lemma 1.4) in case of some equations of higher order. Paper [3] of A.Borzymowski and the present author dealt with the properties of the sequences of iterates in R^3- space (it is worth noting that they differ from those in R^2) which were applied to Goursat-like problems in [4] and [5]. In this paper we extend the results of [3] to the case of R^n- space with arbitrary $n \geq 3$. An application of the present outcomes will be the aim of our next paper.

1. Let $\mathbb{N}$ denote the set of all positive integers and let $n \geq 3$ be a fixed element of $\mathbb{N}$. We consider n functions $f_i : \Omega_i \to \langle 0, A_i \rangle$ $(i = 1,2,\ldots,n)$, where $\Omega_i = \bigtimes_{j=1}^{n-1} \langle 0, A_{I_i(j)} \rangle$,

$$I_i(j) = \begin{cases} j & \text{for } 1 \leq j \leq i-1 \quad (2 \leq i \leq n) \\ j+1 & \text{for } i \leq j \leq n-1 \quad (1 \leq i \leq n-1) \end{cases}$$

with $A_\nu < \infty$ being some positive numbers.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We make the following assumption

(H) The functions f_i $(i = 1,2,\dots,n)$ are continuous and there is a constant $\delta \in (0, \min_{1\le i\le n} A_i)$ such that

$$f_i(x^{(i)}) \le \min_{1\le j\le n-1} x_j^i \tag{1}$$

$(x^{(i)} \in \Omega_i;\ x^{(i)} = (x_1^i,\dots,x_{n-1}^i);\ x_j^i = x_{I_i(j)};\ i = 1,2,\dots,n;$

$j = 1,2,\dots,n-1;\ 0 \le \min_{1\le j\le n-1} x_j^i < \delta;\ i = 1,2,\dots,n).$

Moreover, we assume that the surfaces S_i given by the equations

$$x_i = f_i(x^{(i)})$$

$(x^{(i)} \in \Omega_i;\ i = 1,2,\dots,n)$, respectively, satisfy the condition

$$S_\mu \cap S_\nu = \{ x \in \Omega:\ x_\mu = x_\nu = 0 \}$$

$(\mu, \nu = 1,2,\dots,n;\ \mu \ne \nu)$, where $\Omega = \times_{j=1}^{n} <0, A_j>$.

2. Introduce the functions $a_{r,j}^i : \Omega_i \to \mathbb{R}$ defined by the relations

$$a_{r,j}^i(x^{(i)}) = \begin{cases} x_r^j & r \ne i \\ f_i(x^{(i)}) & r = i \end{cases} \tag{2}$$

$(x^{(i)} \in \Omega_i;\ 1\le i<j\le n;\ r = 1,2,\dots,n-1)$

$$a_{r,j}^i(x^{(i)}) = \begin{cases} x_r^j & r \ne i-1 \\ f_i(x^{(i)}) & r = i \end{cases} \tag{3}$$

$(x^{(i)} \in \Omega_i;\ 1\le j<i\le n;\ r = 1,2,\dots,n-1).$

We consider the sequences $\{z^\nu_{\vec{k}(m)}\}$ and $\{u^\nu_{\vec{k}(m),j}\}$ defined by the formulas

$$z^\nu_{\vec{k}(m)}(x^{(\nu)}) = (z^\nu_{1,\vec{k}(m)}(x^{(\nu)}),\dots,z^\nu_{n-1,\vec{k}(m)}(x^{(\nu)})) \tag{4}$$

where

$$z^\nu_{r,\vec{k}(m)}(x^{(\nu)}) = a^{k_{m-1}}_{r,k_m}(z^\nu_{\vec{k}(m-1)}(x^{(\nu)}))$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $m = 2,3,\ldots$; $r = 1,2,\ldots,n-1$;

(4') $z^{\nu}_{r,\vec{k}(1)}(x^{(\nu)}) = a^{k_0}_{r,k_1}(x^{(\nu)})$ for $r = 1,2,\ldots,n-1$

$(x^{(\nu)} \in \Omega_\nu$; $\vec{k}(m) = (k_1,k_2,\ldots,k_m)$; $1 \le k_l \le n$; $k_l \ne k_{l-1}$; $l = 1,2,\ldots,m$; $k_0 = \nu$; $\nu = 1,2,\ldots,n)$

and

(5) $u^{\nu}_{\vec{k}(m),j}(x^{(\nu)}) = (u^{\nu}_{1,\vec{k}(m),j}(x^{(\nu)}),\ldots,u^{\nu}_{n-1,\vec{k}(m),j}(x^{(\nu)}))$,

where

(5') $u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = a^{k_m}_{r,j}(z^{\nu}_{\vec{k}(m)}(x^{(\nu)}))$ for $m = 1,2,\ldots$

$(\vec{k}(m) = (k_1,\ldots,k_m)$; $1 \le k_l \le n$; $k_l \ne k_{l-1}$; $l = 1,2,\ldots,m$; $k_0 = \nu$; $j \ne k_m$; $j = 1,2,\ldots,n$; $r = 1,2,\ldots,n-1$; $x^{(\nu)} \in \Omega$; $\nu = 1,2,\ldots,n)$.

It can be easily observed that

(6) $z^{\nu}_{\vec{k}(m)}(x^{(\nu)}) = u^{\nu}_{\vec{k}(m-1),k_m}(x^{(\nu)})$

$(x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$; $m = 2,3,\ldots)$.

The aim of this paper is to prove the following theorem

Theorem. The relation

(7) $\lim_{m\to\infty} \max_{\vec{k}(m)} [\min_{1\le r\le n-1} (z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}))] = 0$

$(\nu = 1,2,\ldots,n)$, holds good, the convergence in (7) being uniform.

The proof of the Theorem will be preceded by some lemmas.

Lemma 1. For each constant $\delta_0 \in (0, \min_{1\le i\le n} A_i)$ there is a constant $\lambda_{\delta_0} \in (0,1)$, such that the following inequality holds good

(8) $f_i(x^i_1,\ldots,x^i_{s-1},f_{I_i(s)}(x^{(I_i(s))}),x^i_{s+1},\ldots,x^i_{n-1}) \le \lambda_{\delta_0} x_i$

$((x^{(I_i(s))}) \in \Omega_{I_i(s)}$; $x_i \ge \delta_0$; $i = 1,2,\ldots,n$; $s = 1,2,\ldots,n-1)$.

Proof. First, we will prove a weaker inequality

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(9)\quad f_i(x^i_1,\dots,x^i_{s-1},f_{I_i(s)}(x^{(I_i(s))}),x^i_{s+1},\dots,x^i_{n-1}) < x_i$$

$((x^{(I_i(s))}) \in \Omega_{I_i(s)};\ x_i > 0;\ i = 1,2,\dots,n;\ s = 1,2,\dots,n-1)$.

Let us suppose that inequality (9) is not true. Then, for some values of i and s ($1\le i\le n$, $1\le s\le n-1$) there is a point

$$\bar{x}^{(I_i(s))} \in \overset{n-1}{\underset{l=1}{\times}} (0,A_{I_I(s)}(l)>$$ such that

$$(10)\quad f_i(\bar{x}^i_1,\dots,\bar{x}^i_{s-1},f_{I_i(s)}(\bar{x}^{(I_i(s))}),\bar{x}^i_{s+1},\dots,\bar{x}^i_{n-1}) \ge \bar{x}_i .$$

Consider an auxiliary function $g_{i,s} : <0,A_i> \longrightarrow \mathbb{R}$, given by

$$(11)\quad g_{i,s}(x_i) = f_i(\bar{x}^i_1,\dots,\bar{x}^i_{s-1},f_{I_i(s)}(y^{I_i(s)}(x_i)),\bar{x}^i_{s+1},\dots,\bar{x}^i_{n-1})-x_i$$

where

$$y^{(I_i(s))}(x_i) = (y_1^{I_i(s)}(x_i),\dots,y_{n-1}^{I_i(s)}(x_i)) \in \Omega_{I_i(s)} : y_i(x_i)=x_i;$$

$y_j(x_i) = \bar{x}_j$ $(j = 1,2,\dots,n;\ i\ne j\ne I_i(s))$.

Basing on inequality (10) and on the fact that $f_i : \Omega_i \longrightarrow <0,A_i>$ we may conclude that

$$g_{i,s}(\bar{x}_i) \ge 0;\quad g_{i,s}(A_i) \le 0.$$

Thus, by the continuity of $g_{i,s}$ there is a point $\bar{x}^*_i \in <\bar{x}_i,A_i>$ such that

$$(12)\quad g_{i,s}(\bar{x}^*_i) = 0 .$$

Let us consider a point $\overset{*}{y}(\overset{*}{y}_1,\dots,\overset{*}{y}_n) \in \Omega$ with the coordinates defined by

$$\overset{*}{y}_i = \bar{x}^*_i;\ \overset{*}{y}_{I_i(s)} = f_{I_i(s)}(y^{(I_i(s))}(\bar{x}^*_i));\ \overset{*}{y}_j = \bar{x}_j$$

$(1\le j\le n;\ i\ne j\ne I_i(s))$.

It follows from (11),(12) and the definition of $\overset{*}{y}$ that $\overset{*}{y} \in S_{I_i(s)} \cap S_i$, whence by assumption (H) we have $\bar{x}^*_i = \overset{*}{y}_i = 0$, which contradicts the relations $\bar{x}^*_i \ge \bar{x}_i > 0$.

Thus, inequality (9) is valid.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, for fixed i,s ($1\le i\le n$, $1\le s\le n-1$) and δ_0 ($0<\delta_0<\min_{1\le k\le n} A_k$) let us consider the set $\tilde{B}^{\delta_0}_{i,s}$ defined as follows

$$\tilde{B}^{\delta_0}_{i,s} = \overset{n-1}{\underset{l=1}{\times}} \tilde{B}_{I_{I_i(s)}(l),i,s}, \quad \text{where}$$

$$\tilde{B}_{i,i,s} = \langle \delta_0, A_i \rangle;\ \tilde{B}_{j,i,s} = \langle 0, A_j \rangle$$

($j = I_{I_i(s)}(l) \neq i$; $l = 1,2,\dots,n-1$).

Applying the well known Weierstrass theorem to the function $\tilde{g}_{i,s} : \tilde{B}^{\delta_0}_{i,s} \longrightarrow \mathbb{R}$ given by

$$\tilde{g}_{i,s}(x^{(I_i(s))}) = f_i(x^i_1,..,x^i_{s-1}, f_{I_i(s)}(x^{I_i(s)}(x_i)), x^i_{s+1},..,x^i_{n-1})x_i^{-1}$$

and using inequality (9) we can assert that there is the required constant $\lambda_{\delta_0} \in (0,1)$ such that inequality (8) is valid. Q.E.D.

We are now going to prove some lemmas concerning the sequences $\left\{z^\nu_{\vec{k}(m)}\right\}$ and $\left\{u^\nu_{\vec{k}(m),j}\right\}$.

Lemma 2. Let m and ν be fixed positive integers such that $\nu\le n$, $\vec{k}(m)$ - the multi-index introduced on p.2., and x a point of Ω_ν.

If $k_m < k_{m-1}$, then

$$(13)\quad \begin{cases} z^\nu_{r,\vec{k}(m)}(x^{(\nu)}) = z^\nu_{r,\vec{k}(m-1)}(x^{(\nu)}) & \text{for } 1\le r\le k_m-1 \text{ or } k_{m-1}\le r\le n-1 \\ z^\nu_{r,\vec{k}(m)}(x^{(\nu)}) = z^\nu_{r+1,\vec{k}(m-1)}(x^{(\nu)}) & \text{for } k_m\le r\le k_{m-1}-2 \\ z^\nu_{r,\vec{k}(m)}(x^{(\nu)}) = f_{k_{m-1}}(z^\nu_{\vec{k}(m-1)}(x^{(\nu)})) & \text{for } r = k_{m-1}-1 \end{cases}$$

and if $k_{m-1} < k_m$, then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(14)\quad \begin{cases} z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) & \text{for } 1\le r\le k_{m-1}-1 \text{ or } k_m\le r\le n-1 \\ z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r-1,\vec{k}(m-1)}(x^{(\nu)}) & \text{for } k_{m-1}<r\le k_m-1 \\ z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = f_{k_{m-1}}(z^{\nu}_{\vec{k}(m-1)}(x^{(\nu)})) & \text{for } r = k_{m-1} \end{cases}$$

Proof. We will prove relation (13). Let us introduce the notation

$$(15)\quad y^{(k_{m-1})}_{\vec{k}(m-1)}(x^{(\nu)}) = z^{\nu}_{\vec{k}(m-1)}(x^{(\nu)}) .$$

In virtue of relations (4),(4′),(3) and (15) we have

$$z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = y^{k_m}_{r,\vec{k}(m-1)}(x^{(\nu)}) = y^{k_{m-1}}_{r,\vec{k}(m-1)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)})$$

when $1\le r\le k_m-1$ or $k_{m-1}\le r\le n-1$

$$z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = y^{k_m}_{r,\vec{k}(m-1)}(x^{(\nu)}) = y^{k_{m-1}}_{r+1,\,\vec{k}(m-1)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r+1,\vec{k}(m-1)}(x^{(\nu)}) \text{ when } k_m\le r\le k_{m-1}-2$$

$$z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = f_{k_{m-1}}(y^{(k_{m-1})}_{\vec{k}(m-1)}(x^{(\nu)})) = f_{k_{m-1}}(z^{\nu}_{\vec{k}(m-1)}(x^{(\nu)}))$$

when $r = k_{m-1}-1$.

From the above equalities it follows that relations (13) hold good. The validity of formulas (14) can be proved in a similar way by using formulas (4),(4′),(3) and (15)).

Lemma 3. Let δ_0 and λ_{δ_0} be constants appearing in lemma 1. Then, the elements of the sequences $\{u^{\nu}_{\vec{k}(m),j}\}$ (see (5) and (5′)) satisfy the following inequalities[1]

---

[1] The symbol $\delta_{r,s}$ ($r,s\in \mathbb{N}\cup\{0\}$) is the Kronecker delta.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(16)\quad u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) \le (\lambda_{\delta_0})^{\delta_{r,k_m-1}} u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)})$$

when $1\le k_{m-1}<j<k_m\le n$ or $1\le j<k_{m-1}<k_m\le n$ or $1\le j<k_m<k_{m-1}\le n$;

$$(17)\quad u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) \le (\lambda_{\delta_0})^{\delta_{r,k_m}} u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)})$$

when $1\le k_m<j<k_{m-1}\le n$ or $1\le k_m<k_{m-1}<j\le n$ or $1\le k_{m-1}<k_m<j\le n$;

$((x^{(\nu)}) \in \Omega_\nu;\ m = 2,3,\ldots;\ r = 1,2,\ldots,n-1;$

$\min_{1\le r\le n-1} u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) \ge \delta_0;\nu = 1,2,\ldots,n)$

$$(18)\quad u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) \le (\lambda_{\delta_0})^{\delta_{r,k_m-1}} u^{\nu}_{r,\vec{k}(m-2),j}(x^{(\nu)})$$

when $1\le j=k_{m-1}<k_m\le n$;

$$(19)\quad u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) \le (\lambda_{\delta_0})^{\delta_{r,k_m}} u^{\nu}_{r,\vec{k}(m-2),j}(x^{(\nu)})$$

when $1\le k_m<j=k_{m-1}\le n$;

$((x^{(\nu)}) \in \Omega_\nu;\ m = 3,4,\ldots;\ r = 1,2,\ldots,n-1;$

$\min_{1\le r\le n-1} u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) \ge \delta_0;\nu = 1,2,\ldots,n).$

Proof. We will prove inequality (16) in the case when $1\le k_{m-1}<j<k_m\le n$. Denote

$$(20)\quad w^{(k_m)}_{\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{\vec{k}(m)}(x^{(\nu)}).$$

In virtue of (5), (5′), (15), (20) and Lemmas 1 and 2 we have

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = w^{j}_{r,\vec{k}(m)}(x^{(\nu)}) = w^{k_m}_{r,\vec{k}(m)}(x^{(\nu)}) =$$
$$= z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } 1\le r\le k_{m-1}-1;$$

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = w^{j}_{r,\vec{k}(m)}(x^{(\nu)}) = w^{k_m}_{r,\vec{k}(m)}(x^{(\nu)}) =$$
$$= z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = f_{k_{m-1}}(z^{\nu}_{\vec{k}(m-1)}(x^{(\nu)})) \text{ when } r=k_{m-1};$$

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = w^{j}_{r,\vec{k}(m)}(x^{(\nu)}) = w^{k_m}_{r,\vec{k}(m)}(x^{(\nu)}) =$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r-1,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } k_{m-1}<r\leq j-1;$$

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = w^{j}_{r,\vec{k}(m)}(x^{(\nu)}) = w^{k_m}_{r+1,\vec{k}(m)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r+1,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } j\leq r\leq k_m-2;$$

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = f_{k_m}(w^{(k_m)}_{\vec{k}(m)}(x^{(\nu)})) = f_{k_m}(y^{k_m}_{1,\vec{k}(m-1)}(x^{(\nu)}),\ldots$$

$$\ldots, y^{k_m}_{s-1,\vec{k}(m-1)}(x^{(\nu)}), f_{k_{m-1}}(y^{(k_{m-1})}_{\vec{k}(m-1)}(x^{(\nu)})),$$

$$y^{k_m}_{s+1,\vec{k}(m-1)}(x^{(\nu)}),\ldots,y^{k_m}_{n-1,\vec{k}(m-1)}(x^{(\nu)})) \leq$$

$$\lambda_{\delta_0} y^{k_{m-1}}_{k_m-1,\vec{k}(m-1)}(x^{(\nu)}) = \lambda_{\delta_0} z^{\nu}_{k_m-1,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } r=k_m-1;$$

$$u^{\nu}_{r,\vec{k}(m),j}(x^{(\nu)}) = w^{j}_{r,\vec{k}(m)}(x^{(\nu)}) = w^{k_m}_{r,\vec{k}(m)}(x^{(\nu)}) =$$

$$z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)}) = z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } k_m\leq r\leq n-1.$$

Let us observe that by definitions (5) and (5′) we have (see (15))

$$u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) = y^{j}_{r,\vec{k}(m-1)}(x^{(\nu)}) = y^{k_{m-1}}_{r,\vec{k}(m-1)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } 1\leq r\leq k_{m-1}-1;$$

$$u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) = f_{k_{m-1}}(z^{\nu}_{\vec{k}(m-1)}(x^{(\nu)})) \quad \text{when } r=k_{m-1};$$

$$u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) = y^{j}_{r,\vec{k}(m-1)}(x^{(\nu)}) = y^{k_{m-1}}_{r-1,\vec{k}(m-1)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r-1,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } k_{m-1}<r\leq j-1;$$

$$u^{\nu}_{r,\vec{k}(m-1),j}(x^{(\nu)}) = y^{j}_{r,\vec{k}(m-1)}(x^{(\nu)}) = y^{k_{m-1}}_{r,\vec{k}(m-1)}(x^{(\nu)}) =$$

$$= z^{\nu}_{r,\vec{k}(m-1)}(x^{(\nu)}) \quad \text{when } j\leq r\leq n-1.$$

On joining the above-obtained relations we easily conclude that relation (16) is valid in the case when $1\leq k_{m-1}<j<k_m\leq n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The proof of the remaining inequalities appearing in the thesis of Lemma 3 is similar (cf. Lemma 1.6 in [9]).

3. Proof of the Theorem

For a fixed value of ν ($\nu = 1,2,\ldots,n$) consider a function sequence $F_1^\nu : \Omega_\nu \to \mathbb{R}$ defined in the following way

$$F_1^\nu(x^{(\nu)}) = \max_{\substack{\vec{k}(1),\,1\le j\le n \\ j\ne k_1}} \left[\prod_{r=1}^{n-1} u^\nu_{r,\vec{k}(1),j}(x^{(\nu)}) \right]$$

($x^{(\nu)} \in \Omega_\nu$; $\vec{k}(1) = (k_1,\ldots,k_1)$; $k_s \ne k_{s-1}$; $s = 1,2,\ldots,1$; $k_0 = \nu$; $\nu = 1,2,\ldots,n$; $1 = 1,2,\ldots$).

It easily results from assumption (H) and Lemma 3 that for each constant $0<\delta_1<\min(1,\delta)$ there is a number $1_0 \in \mathbb{N}$ such that

(22) $F_1^\nu(x^{(\nu)}) < \delta_1^{n-1}$ $\quad (x^{(\nu)} \in \Omega_\nu;\ \nu = 1,2,\ldots,n;\ 1 > 1_0)$.

(to this end it is sufficient to use Lemma 3 with $\delta_0 = \delta_1/A^{n-2}$, where $A = \max(1, \max_{1\le i\le n} A_i)$). Hence $1_0 = 21_1$, where 1_1 is a positive integer satisfying the inequality

$$(\lambda_{\delta_0})^{1_1} < \delta_0^{n-1}.$$

Thus

(23) $$\max_{\substack{\vec{k}(1),\,1\le j\le n \\ j\ne k_1}} \left[\min_{1\le r\le n-1} u^\nu_{r,\vec{k}(1),j}(x^{(\nu)}) \right] < \delta_1$$

($x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$; $1 > 1_0$).

It follows from definitions (5) and (5′), assumption (H) and the proof of Lemma 3 that

(24) $F^\nu_{1+1}(x^{(\nu)}) \le F_1^\nu(x^{(\nu)})$

($x^{(\nu)} \in \Omega_\nu$; $\nu = 1,2,\ldots,n$; $1 \ge 1_0$).

Using relations (22), (24) and the well known Dini's theorem (see [7], p.114) we may conclude that the sequence F_1^ν ($\nu = 1,2,\ldots,n$) is uniformly convergent to the function equal identically to zero. Thus (6) implies the thesis of our

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem. Q.E.D.

As an immediate consequence of the above-proved Theorem we get the following

Corollary 1. If inequality (1) is replaced by the condition

$$f_i(x^{(i)}) \leq k^* \min_{1\leq j\leq n-1} x_j^i \tag{25}$$

$$(x^{(i)} \in \Omega_i;\ \min_{1\leq j\leq n-1} x_j^i < \delta \in (0, \min_{1\leq i\leq n} A_i);\ k^* \in (0,1))$$

then there is a positive integer m_0 such that the following inequality

$$\max_{\vec{k}(m)} \left[\min_{1\leq r\leq n-1} (z^{\nu}_{r,\vec{k}(m)}(x^{(\nu)})) \right] \leq \text{const}(k^*)^m \tag{26}$$

$(x^{(\nu)} \in \Omega_\nu; \nu = 1,2,\ldots,n;$ and const is a constant independent of m) holds good for $m > m_0$.

REFERENCES

[1] **A. Bielecki, J. Kisyński:** Sur le problème de E. Goursat relatif à l'équation $\frac{\partial^2 z}{\partial x \partial y} = f(x,y)$, Ann. Univ. Mariae Curie-Skłodowska,A 10 (1956), 99-126.

[2] **A. Borzymowski:** Concerning a Goursat problem for some partial differential equation of order 2p, Demonstratio Math. 18 (1985), 253-277.

[3] **A. Borzymowski, A. Sadowski:** On some sequences of iterates, Discussiones Math. 8 (1986), 85-91.

[4] **A. Borzymowski, A. Sadowski:** A Goursat problem for Mangeron's polyvibrating equation in $\mathbb{R}^3$, Applicable Anal. 23 (1987), 319-332.

[5] **A. Borzymowski, A. Sadowski:** A boundary value problem for the Mangeron polyvibrating equation, Rend. Mat. 6 (1986), 565-585.

[6] **M. Kuczma:** Functional equations in a single variable, PWN Warszawa 1968.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[7] K. Maurin: Analysis, Part I: Elements, PWN Warszawa 1976.

[8] M. W. Michalski: Goursat-like boundary value problems for some partial differential equations of even order (in Polish), Doctoral Dissertation, Warsaw Technical University , Warsaw 1985.

[9] A. Sadowski: Goursat-like boundary value problems for some partial differential equations with the number of independent variables greater than two (in Polish), Doctoral Thesis, Warsaw Technical University, Warsaw 1986.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, PLOCK BRANCH, PLOCK, POLAND.

Received December 19, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Andrzej Rogowski

NECESSARY CONDITIONS FOR OPTIMALITY IN PROBLEMS WITH HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

In present paper we consider a hyperbolic partial differential equation of the form

$$(1.1)\quad \sum_{\alpha=0}^{\bar\alpha} A_{\bar\alpha-\alpha}(x)\,\partial^{|\alpha|}z(x)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2} + B(x)u(x) = 0,$$

$$x = (x_1,x_2) \in P,$$

$$z(x) = (z^1(x),\dots,z^m(x)),\quad u(x) = (u^1(x),\dots,u^l(x)),$$

$$P = [0,1]\times[0,1],\quad \bar\alpha = (\bar\alpha_1,\bar\alpha_2) \text{ - fixed multiindex,}$$

with Darboux-type boundary conditions

$$(1.2)\quad \begin{cases} \partial^{\bar\alpha_1-1+\alpha_2} z(x_1,0)/\partial^{\bar\alpha_1-1}\partial(x_2)^{\alpha_2} = 0, & x_1\in[0,1]\ \alpha_2=0,1,\dots,\bar\alpha_2-1,\\ \partial^{\alpha_1} z(0,x_2)/\partial(x_1)^{\alpha_1} = 0, & x_2\in[0,1],\ \alpha_1=0,1,\dots,\bar\alpha_1-1, \end{cases}$$

and with constraints

$$(1.3)\quad u(\cdot) \in U.$$

We are concerned with the minimum of a functional of the form

$$(1.4)\quad J(z,u) = \int_P f(x,z(x),u(x))\,dx.$$

Here $\alpha = (\alpha_1,\alpha_2)$, $\bar\alpha = (\bar\alpha_1,\bar\alpha_2)$, $0 = (0,0)$, are multiindices, $\alpha\le\bar\alpha$ if $\alpha_1\le\bar\alpha_1$ and $\alpha_2\le\bar\alpha_2$. $A_\alpha(x)$, $0\le\alpha\le\bar\alpha$, are nm-matrices with measurable and essentially bounded components, $A_0 = \mathrm{id}$; f satisfies some smooth-convex

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

assumptions (see §3).

In section 2 we give some definitions and preliminary results. The section 3 contains the main result. We show that the optimization problem (1.1) - (1.4) can be written in terms of Ioffe-Tikhomirov extremum principle [3] and then we obtain the Pontryagin-type necessary conditions.

In section 4 we make a number of remarks on the results obtained, particularly, in relation to the previous papers by Suryanarayana [4], A.I. Egorov [6].

2. Definitions and some lemmas

Let $P = [0,1] \times [0,1]$ and $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) \geq (1,1)$ be a given multiindex. $L^1_m(P)$, $L^1_m(0,1)$ denote usual Banach spaces of Lebesgue-summable functions defined on P, [0,1], respectively, with values in $\mathbb{R}^m$.

Definition 2.1 ([1]). We say that the function $z : P \longrightarrow \mathbb{R}^m$ is absolutely continuous when the function of an interval F (defined $F([x^1_1, x^1_2] \times [x^2_1, x^2_2]) = z(x^1_2, x^2_2) - z(x^1_1, x^2_2) - z(x^1_2, x^2_1) + z(x^1_1, x^2_2)$) associated with z is an absolutely continuous function of an interval (see [7]) and $z(\cdot, 0)$, $z(0, \cdot)$ are absolutely continuous functions of the variable.

Definition 2.2 ([1]). We say that the function $z : P \longrightarrow \mathbb{R}^m$ is absolutely continuous of order $\bar{\alpha}$ ($\bar{\alpha} \geq (1,1)$) when z is absolutely continuous (Def. 2.1) and there exist continuous partial derivatives of z in Int P (in classical sense) up to

$$\partial^{|\bar{\alpha}|-2} z(x) / \partial (x_1)^{\bar{\alpha}_1 - 1} \partial (x_2)^{\bar{\alpha}_2 - 1}$$

which can be continuously extended onto the boundary of P to be absolutely continuous in sense of Definition 2.1.

The class of all absolutely continuous functions $z : P \longrightarrow \mathbb{R}^m$ of order $\bar{\alpha}$ ($\bar{\alpha} \geq (1,1)$) we denote by $AC^{\bar{\alpha}}_m(P)$. In case $\bar{\alpha} = (1,1)$ we also write $AC_m(P)$ instead of $AC^{(1,1)}_m(P)$.

The following representation formula is proved in [1].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proposition 2.1. The function $z : P \longrightarrow \mathbb{R}^m$ belongs to $AC_m(P)$ if and only if there exist summable functions $l^{1,1} \in L_m^1(P)$, $l^{0,1}, l^{1,0} \in L_m^1(0,1)$ and a constant $c^0 \in \mathbb{R}^m$ such that

$$z(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} l^{1,1}(r,s)\,ds\,dr + \int_0^{x_1} l^{1,0}(r)\,dr + \int_0^{x_2} l^{0,1}(s)\,ds + c^0 \tag{2.1}$$

for all $(x_1,x_2) \in P$.

Let $T^{(1,0)}$, $T^{(0,1)}$ be operators defined on $L_m^1(P)$ as follows

$$(T^{(1,0)}l)(x_1,x_2) = \int_0^{x_1} l(r,x_2)\,dr,$$

$$(T^{(0,1)}l)(x_1,x_2) = \int_0^{x_2} l(x_1,s)\,ds, \quad (x_1,x_2) \in P,$$

$$T^{(0,0)}l = l.$$

If $\alpha \geq 0$ is a multiindex then

$$\underbrace{T^{(1,0)} \cdot \ldots \cdot T^{(1,0)}}_{\alpha_1 \text{ times}} \cdot \underbrace{T^{(0,1)} \cdot \ldots \cdot T^{(0,1)}}_{\alpha_2 \text{ times}}.$$

Proposition 2.2. The function $z : P \longrightarrow \mathbb{R}^m$ belongs to $AC_m^{\bar\alpha}(P)$ $(\bar\alpha \geq (1,1))$ if and only if there exist functions $l^{\bar\alpha} \in L_m^1(P)$, $l^{(\bar\alpha_1,\alpha_2)} \in L_m^1(0,1)$, $\alpha_2 = 0,1,\ldots,\bar\alpha_2-1$, $l^{(\alpha_1,\bar\alpha_2)} \in L_m^1(0,1)$, $\alpha_1 = 0,1,\ldots,\bar\alpha_1-1$ and constants $c^\alpha \in \mathbb{R}^m$, $0 \leq \alpha \leq (\bar\alpha_1-1,\bar\alpha_2-1)$ such that

$$z(x_1,x_2) = (T^{\bar\alpha_1\bar\alpha_2}l^{\bar\alpha})(x_1,x_2) + \sum_{\substack{\alpha=(\bar\alpha_1,\alpha_2)\\ \alpha_2=0,\ldots,\bar\alpha_2-1}} T^{\alpha_1\alpha}l^{\alpha}(x_1) + \sum_{\substack{\alpha=(\alpha_1,\bar\alpha_2)\\ \alpha_1=0,\ldots,\bar\alpha_1-1}} T^{\alpha_1\alpha}l^{\alpha}(x_2) + \sum_{\alpha=0}^{(\bar\alpha_1-1,\bar\alpha_2-1)} T^{\alpha}c^{\alpha}, \quad (x_1,x_2) \in P. \tag{2.2}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Induction. The assertion holds for $\bar{\alpha} = (1,1)$ by Proposition 2.1. We assume that the formula (2.2) is valid for any $\alpha \leq \tilde{\alpha}$, $\tilde{\alpha}$ fixed. Let $\bar{\alpha} = (\tilde{\alpha}_1+1, \tilde{\alpha}_2)$ and $z \in A_m^{\bar{\alpha}}(P)$. Then there exists $\partial z(x_1,x_2)/\partial x_1$ on P (as continuous extension from Int P) and belongs to $A_m^{\tilde{\alpha}}(P)$, what immediately follows from Definition 2.2. Moreover,

$$z(x_1,x_2) = \int_0^{x_1} \partial z(r,x_2)/\partial x_1 dr + g(x_2), \quad (x_1,x_2) \in P, \tag{2.3}$$

where

$$g(x_2) = (T^{(0,\tilde{\alpha}_2)} h^{\tilde{\alpha}_2})(x_2) + \sum_{\alpha_2=0}^{\tilde{\alpha}_2-1} T^{(0,\alpha_2)} d^{\alpha_2}, \tag{2.4}$$

$$h^{\tilde{\alpha}_2} \in L_m^1(0,1), \quad d^{\alpha_2} \in \mathbb{R}^m, \quad \alpha_2 = 0,\dots,\tilde{\alpha}_2-1,$$

(see Def. 2.2 and Def. 2.1). Taking into account the representation formula (2.2) for $\partial z/\partial x_1$ with replacing $\bar{\alpha}$ by $\tilde{\alpha}$, (2.3) and (2.4) we obtain that

$$z(x_1,x_2) = (T^{\bar{\alpha}_1\tilde{\alpha}})(x_1,x_2) + \sum_{\substack{\alpha=(\tilde{\alpha}_1+1,\alpha_2)\\ \alpha_2=0,\dots,\tilde{\alpha}_2-1}} (T^{\alpha_1 (\tilde{\alpha}_1,\alpha_2)})(x_1) +$$

$$+ \Big(\sum_{\substack{\alpha=(\alpha_1,\tilde{\alpha}_2)\\ \alpha_1=1,\dots,\tilde{\alpha}_1}} (T^{\alpha_1 (\alpha_1-1,\tilde{\alpha}_2)})(x_2) + (T^{(0,\tilde{\alpha}_2)} h^{\tilde{\alpha}_2})(x_2) \Big) +$$

$$+ \Big(\sum_{\alpha=(1,0)}^{(\tilde{\alpha}_1,\tilde{\alpha}_2-1)} T^{\alpha} c^{\alpha} + \sum_{\alpha_2=0}^{\tilde{\alpha}_2-1} T^{(0,\alpha_2)} d^{\alpha_2} \Big)$$

for $(x_1,x_2) \in P$.

After appropriate renaming functions and constants appearing in the above formula we obtain (2.2) with $\bar{\alpha} = (\bar{\alpha}_1,\bar{\alpha}_2)$, $\bar{\alpha}_1 = \tilde{\alpha}_1 + 1$, $\bar{\alpha}_2 = \tilde{\alpha}_2$. On the other hand, if z r'tisfies (2.2) with $\bar{\alpha} = (\tilde{\alpha}_1+1,\tilde{\alpha}_2)$ then the properties of an

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

integral yield that (2.3) and (2.4) holds and $\partial z/\partial x_1$ is continuously extended onto P from Int P and satisfies (2.2) with $\bar{\alpha}$ replaced by $\tilde{\alpha}$. Thus, by inductive assumption $\partial z/\partial x_1$ belongs to $AC_m^{\tilde{\alpha}}(P)$ and by (2.3) and (2.4) $z \in AC_m^{\bar{\alpha}}(P)$.

The case $\bar{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2 + 1)$ is analogous.

Proposition 2.3 ([2, Lemma 2]).

If $u(x_1,x_2) = \int_0^{x_1} l(r,x_2)dr$, for $x_1 \in [0,1]$ and for a.e. (almost everywhere) $x_2 \in [0,1]$, $l \in L_m^1(P)$, then there exists $\partial u(x_1,x_2)/\partial x_1 = l(x_1,x_2)$ for a.e. $(x_1,x_2) \in P$.

Corollary 2.1. If $z \in AC_m^{\bar{\alpha}}(P)$, then for a.e. $(x_1,x_2) \in P$ there exists $\partial^{|\bar{\alpha}|} z(x_1,x_2)/\partial(x_1)^{\bar{\alpha}_1}\partial(x_2)^{\bar{\alpha}_2} = l^{\bar{\alpha}}(x_1,x_2)$ (see (2.2)).

Remark 2.1. It follows from Propositions 2.2 and 2.3 that any function $z \in AC_m^{\bar{\alpha}}(P)$ satisfying boundary conditions (1.2) has the unique representation

$$z(x_1,x_2) = (T^{\bar{\alpha}} l^{\bar{\alpha}})(x_1,x_2), \quad (x_1,x_2) \in P, \quad l^{\bar{\alpha}} \in L_m^1(P). \tag{2.3}$$

The set of all such functions we denote by $AC_{m,0}^{\bar{\alpha}}(P)$. It is a linear subspace of $AC_m^{\bar{\alpha}}(P)$ and it becomes a Banach space when endowed with the norm $\|z\| = |l^{\bar{\alpha}}|_{L^1}$, where $|\cdot|_{L^1}$ is the norm of $L_m^1(P)$ and $l^{\bar{\alpha}} \in L_m^1(P)$ is connected with z by (2.3).

Corollary 2.2. For any function $z \in AC_{m,0}^{\bar{\alpha}}(P)$ satisfying, by Remark 2.1, equation (2.3) and any multiindex $\alpha \le \bar{\alpha}$ there exist by Proposition 2.3, $\partial^{|\alpha|} z(x_1,x_2)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2} = (T^{\bar{\alpha}-\alpha} l^{\bar{\alpha}})(x_1,x_2)$ for a.e. $(x_1,x_2) \in P$, where $l^{\bar{\alpha}}$ is that from (2.3).

Remark 2.2. For every $(x_1,x_2) \in P$ the following inequality holds for $z \in A_{m,0}^{\bar{\alpha}}(P)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$|z(x_1,x_2)| = |(T^{\bar\alpha}1^{\bar\alpha})(x_1,x_2)| \le |1^{\bar\alpha}|_{L^1} = \|z\|. \tag{2.4}$$

The following theorem is proved in [1].

Theorem 2.1. For any function $g \in L^1_m(P)$ the hyperbolic p.d.e.

$$\sum_{\alpha=0}^{\bar\alpha} A_{\bar\alpha-\alpha}(x)\partial^{|\alpha|}z(x)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2} + g(x) = 0 \tag{2.5}$$

for a.e. $x = (x_1,x_2) \in P$ with boundary conditions

$$\begin{cases} \partial^{\bar\alpha_1-1+\alpha_2}z(x_1,0)/\partial(x_1)^{\bar\alpha_1-1}\partial(x_2)^{\alpha_2} = 0, & \alpha_2=0,\dots,\bar\alpha_2-1, \\ \partial^{\alpha_1}z(0,x_2)/\partial(x_1)^{\alpha_1} = 0, & \alpha_1=0,\dots,\bar\alpha_1-1, \end{cases} \tag{2.6}$$

$$x_1,x_2 \in [0,1],$$

has a unique solution in $AC^{\bar\alpha}_{m,0}(P)$ provided $A_{\bar\alpha-\alpha}(\cdot)$ are $m \times m$-matrix valued functions with components from $L^\infty(P)$ and A_0 is an identity matrix ($A_0 = \mathrm{id}$).

Remark 2.3. Let $F_g : AC^{\bar\alpha}_{m,0}(P) \longrightarrow L^1_m(P)$ be an operator (of $z(\cdot)$) defined by the left-hand side of the equation (2.5) for $g \in L^1_m(P)$. It is continuous, as a consequence of the norm introduced and Proposition 2.3, and affine. Therefore, it is continuously Frechet-differentiable and its Frechet-derivative is of the form

$$F'_g(z)(h) = F_0(h)$$

for all $h \in AC^{\bar\alpha}_{m,0}(P)$ and for every $z \in AC^{\bar\alpha}_{m,0}(P)$ (here F_0 stands for F_g with $g \equiv 0$). Moreover, by Theorem 2.1, for any $g \in L^1_m(P)$ and $z \in AC^{\bar\alpha}_{m,0}(P)$ the operator $F'_g(z)$ transforms $AC^{\bar\alpha}_{m,0}(P)$ onto $L^1_m(P)$.

The following lemma establishes conditions which guarantee the functional (1.4) is continuously Frechet-differentiable

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

with respect to z.

Lemma 2.1. Let M be a closed subset of $\mathbb{R}^l$. Let $f : P \times \mathbb{R}^m \times M \longrightarrow \mathbb{R}$ be such a function that

(i) f is $\mathcal{L} \times \mathcal{B}_{\mathbb{R}^m} \times \mathcal{B}_M$-measurable (i.e. with respect to the σ-algebra of subsets of $P \times \mathbb{R}^m \times M$ generated by Lebesgue sets in P and Borel sets in $\mathbb{R}^m$ and M),

(ii) for all $x \in P$, $u \in M$, $f(x,\cdot,u)$ belongs to $C^1(\mathbb{R}^m)$.

Assume in addition there exist $z_0(\cdot) \in AC^{\bar{\alpha}}_{m,0}(P)$ and $\tilde{u}_0 : P \longrightarrow M$ measurable such that

(iii) the function $x \longrightarrow f(x,z_0(x),\tilde{u}_0(x))$ is summable,

(iv) there exist $\varepsilon > 0$ and a summable function $p_{\tilde{u}_0}$ such that

$$|f_{|z}(x,z,\tilde{u}_0(x))| \leq p_{\tilde{u}_0}(x) \tag{2.7}$$

if

$$|z - z_0(x)| \leq \varepsilon.$$

Then the functional

$$J(z,\tilde{u}_0) = \int_P f(x,z(x),\tilde{u}_0(x))dx$$

$J(\cdot,\tilde{u}_0) : AC^{\bar{\alpha}}_{m,0}(P) \longrightarrow \mathbb{R}$, is continuously Frechet-differentiable in a neighbourhood V of $z_0(\cdot)$ and

$$J_{|z}(z,\tilde{u}_0)(h) = \int_P f_{|z}(x,z(x),\tilde{u}_0(x))h(x)dx \tag{2.8}$$

for all $h \in AC^{\bar{\alpha}}_{m,0}(P)$ and $z \in V$.

Proof. Let $V = \{ z \in AC^{\bar{\alpha}}_{m,0}(P) : \|z - z_0\| < \varepsilon$, ε defined in (iv)$\}$ be the needed neighbourhood of z_0. Because of Remark 2.2 and assumptions (iv) and (iii), we notice that the operator $L : AC^{\bar{\alpha}}_{m,0}(P) \longrightarrow (AC^{\bar{\alpha}}_{m,0}(P))^*$

$$L(z)(h) = \int_P f_{|z}(x,z(x),\tilde{u}_0(x))h(x)dx, \quad h \in AC^{\bar{\alpha}}_{m,0}(P),$$

is well defined for all $z \in V$. Now, by assumption (ii), the mean-value theorem [3, p. 38] and Remark 2.2, we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\left|\frac{J(z+h,\tilde{u}_0) - J(z,\tilde{u}_0) - L(z)(h)}{\|h\|}\right| \le$$

$$\le \int_P \frac{1}{\|h\|} |f(x,(z+h)(x),\tilde{u}_0(x)) - f(x,z(x),\tilde{u}_0(x)) -$$

$$- f_{|z}(x,z(x),\tilde{u}_0(x))h(x)|dx \le \int_P \sup_{0\le r\le 1} |f_{|z}(x,(z+rh)(x),\tilde{u}_0(x)) -$$

$$- f_{|z}(x,z(x),\tilde{u}_0(x))|dx.$$

Because of (iv) and Remark 2.2 for $\|h\| < \varepsilon$, the function under the last integral is bounded by the summable function $2p_{\tilde{u}_0}(\cdot)$.

Now, by Fatou lemma and assumption (ii) the above inequality gives that $L(z)$ is Frechet-derivative of $J(z,\tilde{u}_0)$ (with respect to z) for every $z \in V$. Moreover, by the same arguments we show that $L(z)$ is continuous in V, what ends the proof.

Lemma 2.2. Let M be a Borel subset of a metric space X, Q a Lebesgue-measurable subset of $\mathbb{R}^n$, $g : Q \times M \to \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$) an $\mathcal{L} \times \mathcal{B}_X$-measurable function such that for a.e. $x \in Q$ $g(x,\cdot)$ is lower semicontinuous and not identically equal to $+\infty$.

Let U be the set of measurable functions $u : Q \to M$ satisfying the following decomposability condition

(i) for any function $u \in U$ and any measurable and bounded function $u_0 : S \to M$ (S is an arbitrary measurable subset of Q with finite measure) the function

$$u(x) = \begin{cases} u_0(x) : x \in S \\ u(x) : x \in Q \setminus S \end{cases}$$

belongs to U.

Then, if there exists a function $u_1 \in U$ that

(ii) $\int_Q g(x,u_1(x))dx < +\infty$,

the function

$$m(x) = \inf_{v \in M} g(x,v)$$

is Lebesgue-measurable and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\inf_{u(\cdot)\in U} \int_Q g(x,u(x))dx = \int_Q (\inf_{v\in M} g(x,v))dx. \tag{2.9}$$

Proof. Measurability of $m(x)$ is a consequence of [5, Lemma III. 39]. Going to the second assertion we notice, by (ii), that there exist a summable function $a_1 : Q \to \overline{\mathbb{R}}$, for which $g(x,u_1(x)) \le a_1(x)$ for $x \in Q$. Hence

$$\int_Q m(x)dx < +\infty,$$

$$\int_Q m(x)dx \le \inf_{u(\cdot)\in U} \int_Q g(x,u(x))dx.$$

Let $b \in \mathbb{R}$ be such that

$$\int_Q m(x)dx < b. \tag{2.10}$$

We claim that there exist $u(\cdot) \in U$ that

$$\int_Q g(x,u(x))dx < b.$$

In fact, by (2.10) and the definition of Lebesgue integral there exists a summable function $a_0(\cdot)$ satisfying

$$m(x) < a_0(x) \text{ for a.e. } x \in Q, \tag{2.11}$$

$$\int_Q a_0(x)dx < b. \tag{2.12}$$

Let $\Gamma : Q \to 2^M$ be the multifunction defined

$$\Gamma(x) = \{v \in M : g(x,v) \le a_0(x)\}.$$

It is measurable because the function $(x,u) \to g(x,u) - a_0(x)$ is measurable. Moreover, for a.e. $x \in Q$ $\Gamma(x)$ is nonempty (by(2.11)) and closed. Therefore, [5, Thm III. 22] there exists a measurable selection $u_0(x)$ of $\Gamma(x)$. By (2.12) and Lusin theorem there exists $S \subset Q$ of finite measure, that $u_0|S$ is bounded and

$$\int_S a_0(x)dx + \int_{Q\setminus S} a_1(x)dx < b. \tag{2.13}$$

Taking

$$u(x) = \begin{cases} u_0(x) : x \in S \\ u_1(x) : x \in Q \setminus S \end{cases}$$

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we see, by (i), that $u(\cdot) \in U$, and by (2.13) that $\int_Q g(x,u(x))dx < b$. Because b satisfying (2.10) was arbitrary the equality (2.9) is proved.

In the sequel, we shall use the following corollary from Hölder inequality.

Lemma 2.3. For any function $l \in L^1_m(P)$ and multiindex $\alpha = (\alpha_1,\alpha_2) \geq 0$

$$|T^\alpha l|_{L^1} \leq \frac{1}{\alpha_1!\alpha_2!} |l|_{L^1}. \tag{2.14}$$

3. The main results

We give here necessary conditions for the optimization problem (1.1) - (1.4) where $A_{\bar\alpha-\alpha}$ are $m \times m$ - matrices with $L^\infty(P)$ components for $0 \leq \alpha \leq \bar\alpha$, $A_0 = id$, $U = \{u : P \to M, u(\cdot) \in L^p_m(P)\}$ and M is fixed convex and closed subset of $\mathbb{R}^l$, B is a $m \times 1$ - matrix with $L^q(P)$ $(\frac{1}{p} + \frac{1}{q} = 1,\ p \geq 1)$ components. We say that a pair $(z,u) \in AC^{\bar\alpha}_{m,0}(P) \times U$ is admissible if it satisfies (1.1) and (1.3). An admissible pair (z_0,u_0) is called a point of a local minimum in problem (1.1) - (1.4) if

$$J(z_0,u_0) \leq J(z,u)$$

for all z from some neighbourhood of z_0 in $AC^{\bar\alpha}_{m,0}(P)$ and all $u \in U$ such that (z,u) is an admissible pair.

Theorem 3.1. If U, M, $A_{\bar\alpha-\alpha}$, B are as above, $(z_0,u_0) \in AC^{\bar\alpha}_{m,0}(P) \times U$ is a point of a local minimum in problem (1.1)-(1.4) and $f : P \times \mathbb{R}^m \times M \to \mathbb{R}$ (appearing in (1.4)) is convex and lower semicontinuous with respect to the last variable and satisfies assumptions (i)-(iv) (in (z_0,u), $u \in U$) from Lemma 2.1 then there exists a unique element $y_0 \in L^\infty_m(P)$ satisfying

$$f(x,z_0(x),u_0(x)) + y_0(x)B(x)u_0(x) = \min_{v\in M} (f(x,z_0(x),v) + y_0(x)B(x)v) \quad \text{for a.e. } x \in P, \tag{3.1}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$(\tilde{T}^{\bar{\alpha}}f|_z)(x,z_0(x),u_0(x)) + \sum_{\alpha=\bar{0}}^{\bar{\alpha}} (\tilde{T}^{\alpha}y_0 A_{\alpha})(x) = 0 \tag{3.2}$$

for a.e. $x \in P$, where the integral operator $\tilde{T}^{\alpha}$ is a composition of operators $\tilde{T}^{(1,0)}$ and $\tilde{T}^{(0,1)}$,

$$(\tilde{T}^{(1,0)}l)(x_1,x_2) = \int_{x_1}^{1} l(r,x_2)\,dr,$$

$$(\tilde{T}^{(0,1)}l)(x_1,x_2) = \int_{x_2}^{1} l(x_1,s)\,ds,$$

$l \in L^1_m(P)$, in the same way as T^{α}.

Proof. Let $F : AC^{\bar{\alpha}}_{m,0}(P) \times U \longrightarrow L^1_m(P)$ be defined

$$F(z,u)(x) = \sum_{\alpha=\bar{0}}^{\bar{\alpha}} A_{\bar{\alpha}-\alpha}(x)\,\partial^{|\alpha|}z(x)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2} + B(x)u(x)$$

for $x \in P$. Taking into account assumptions, Remark 2.3 and Lemma 2.1, we see that all assumptions of Ioffe-Tikhomirov extremum principle [3] are satisfied and therefore there exist $y_0 \in L^{\infty}_m(P)$ $(= (L^1_m(P))^*)$ satisfying the following equalities

$$J(z_0,u_0) + \langle y_0, F(z_0,u_0)\rangle_{L^{\infty},L^1} = \min_{u(\cdot)\in U} (J(z_0,u) + \langle y_0, F(z_0,u)\rangle_{L^{\infty},L^1}) \tag{3.3}$$

and

$$J|_z(z_0,u_0) + y_0 F|_z(z_0,u_0) = 0 \tag{3.4}$$

where $\langle\cdot,\cdot\rangle_{L^{\infty},L^1}$ is the duality operation between $L^{\infty}_m(P)$ and $L^1_m(P)$. Because of the linearity of F (3.3) is equivalent to

$$\int_P (f(x,z_0(x),u_0(x)) + y_0(x)B(x)u_0(x))\,dx = \min_{u(\cdot)\in U} \int_P (f(x,z_0(x),u(x)) + y_0(x)B(x)u(x))\,dx. \tag{3.5}$$

Now, using Lemma 2.2 we get that the right-hand side of (3.5) is equal to $\int_P \inf_{v\in M} (f(x,z_0(x),v) + y_0(x)B(x)v)\,dx$. As the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

integral on the left-hand side of (3.5) is finite and $(f(x,z_0(x),u_0(x)) + y_0(x)B(x)u_0(x)) \geq \inf_{v\in M}(f(x,z_0(x),v) + y_0(x)B(x)v)$ for $x \in P$, we conclude that (3.1) holds.

To prove (3.2) let us notice that (3.4), by Remark 2.3 and Lemma 2.1, can be written in the form

$$0 = \int_P (f_{|z}(x,z_0(x),u_0(x))h(x) + \sum_{\alpha=0}^{\bar{\alpha}} y_0(x)A_{\bar{\alpha}-\alpha}(x)\partial^{|\alpha|}h(x)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2})dx \tag{3.6}$$

for all $h \in AC^{\bar{\alpha}}_{m,0}(P)$.

Let us investigate the following procedure. Take $g \in L^\infty_m(P)$, $k \in L^1_m(P)$ and let $j(x_1,x_2) = \int_0^{x_1} k(r,x_2)dr = (T^{(1.0)}k)(x_1,x_2)$ for all $x_1 \in [0,1]$ and a.e. $x_2 \in [0,1]$. Of course, $j \in L^1_m(P)$ and, by Proposition 2.3, for a.e. $(x_1,x_2) \in P$ $\partial j(x_1,x_2)/\partial x_1 = k(x_1,x_2)$. Analogously, $g_1 = \tilde{T}^{(1,0)}g \in L^\infty_m(P)$ and $\partial g_1(x_1,x_2)/\partial x_2 = g(x_1,x_2)$ for a.e. $(x_1,x_2) \in P$. Therefore, using the theorem of integrating by parts for absolutely continuous functions of one variable [7] we obtain that for a.e. $x_2 \in [0,1]$

$$\int_0^1 g(x_1,x_2)(\int_0^{x_1} k(r,x_2)dr)dx_1 = \int_0^1 (\int_{x_1}^1 g(r,x_2)dr)\, k(x_1,x_2)dx_1 -$$

$$- \Big[(\int_{x_1}^1 g(r,x_2)dr) \int_0^{x_1} k(r,x_2)dr\Big]_{x_1=0}^{x_1=1} =$$

$$\int_0^1 (\int_{x_1}^1 g(r,x_2)dr)k(x_1,x_2)dx_1.$$

Now, because of summability of function $g(T^{(1,0)}k)$ and $(\tilde{T}^{(1,0)}g)k$, and by Fubini theorem we get

$$\int_P g(x)(T^{(1,0)}k)(x)dx = \int_P (\tilde{T}^{(1,0)}g)(x)k(x)dx. \tag{3.7}$$

Analogously,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.8)\qquad \int_P g(x)(T^{(0,1)}k)(x)dx = \int_P (\tilde{T}^{(0,1)}g)(x)k(x)dx.$$

Taking into account integrals

$$(3.9)\qquad \int_P y_0(x)A_{\bar{\alpha}-\alpha}(x)\partial^{|\alpha|}h(x)/\partial(x_1)^{\alpha_1}\partial(x_2)^{\alpha_2}dx, \quad \alpha \le \bar{\alpha},$$

recalling the representation formula (2.3) ($h(x) = (T^{\bar{\alpha}}1^{\bar{\alpha}})(x)$) for $h \in AC^{\bar{\alpha}}_{m,0}(P)$, Corollary 2.2 and using equalities (3.7) and (3.8), we obtain that (3.9) is equal to

$$\int_P (\tilde{T}^{\bar{\alpha}-\alpha}(y_0A_{\bar{\alpha}-\alpha}))(x)1^{\bar{\alpha}}(x)dx, \quad \alpha \le \bar{\alpha}.$$

Analogously,

$$\int_P f_{|z}(x,z_0(x),u_0(x))h(x)dx =$$

$$= \int_P (\tilde{T}^{\bar{\alpha}}f_{|z})(x,z_0(x),u_0(x))\ 1^{\bar{\alpha}}(x)dx.$$

Hence (3.6) is equivalent to

$$(3.10)\qquad 0 = \int_P (\tilde{T}^{\bar{\alpha}}f_{|z})(x,z_0(x),u_0(x)) +$$

$$+ \sum_{\alpha=0}^{\bar{\alpha}} (\tilde{T}^{\alpha}(y_0A_{\alpha}))(x))1^{\bar{\alpha}}(x)dx \quad \text{for all } 1^{\bar{\alpha}} \in L^1_m(P),$$

and thus

$$(\tilde{T}^{\bar{\alpha}}f_{|z})(x,z_0(x),u_0(x)) + \sum_{\alpha=0}^{\bar{\alpha}} (\tilde{T}^{\alpha}(y_0A_{\alpha}))(x) = 0$$

for a.e. $x \in P$, so (3.2) is proved. Moreover, the above multidimensional Volterra-type equation (of y_0) possesses exactly one solution in $L^\infty_m(P)$ (see [8, Thm. 3]). This ends the proof.

4. Remarks

We show here, that making some additional assumptions on A_{α}, the conjugate equation (3.2) can be written in a classical form. To simplify consideration we limit ourselves to the case $\bar{\alpha} = (1,1)$, but for other $\bar{\alpha}$ the analogous result to the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

following theorem will take place, too.

Theorem 4.1. Suppose assumptions of Theorem 3.1 are satisfied for $\overline{\alpha} = (1,1)$, and additionally assume that there exist $\partial A_{(1,0)}(x)/\partial x_2$ and $\partial A_{(0,1)}(x)/\partial x_1$ for a.e. $x = (x_1,x_2) \in P$ and belong to $L^{\infty}_m(P)$.

Then, the necessary condition in order that $(z_0,u_0) \in AC_m(P) \times U$ be a point of a local minimum in (1.1) - (1.4) is the existence of a unique function $y_0 \in AC_m(P)$ satisfying (3.1) and

$$(4.1)\quad \partial^2 y_0(x)/\partial x_1\partial x_2 - \partial(y_0 A_{(0,1)})(x)/\partial x_1 - \partial(y_0 A_{(1,0)})(x)/\partial x_2 + (y_0 A_{(1,1)})(x) + f_{|z}(x,z_0(x),u_0(x)) = 0$$

for a.e. $x \in P$ with transversality conditions

$$(4.2)\quad y_0(x_1,1) = 0, \qquad y_0(1,x_2) = 0.$$

Proof. Let $AC_{m,1}(P) = \{y \in AC_m(P) : y(x_1,1) = y(1,x_2) = 0, x_1,x_2 \in [0,1]\}$. Notice that $AC_{m,1}(P) = \{y \in AC_m(P) : y(x_1,x_2) = z(1-x_1,1-x_2), \quad z \in AC_{m,0}(P)\}$ and any $y \in AC_{m,1}(P)$ possesses a representation formula

$$(4.3)\quad y \in AC_{m,1}(P) \iff y(x_1,x_2) = \int_{x_1}^{1}\int_{x_2}^{1} l^{(1,1)}(r,s)\,ds\,dr,$$

$$l^{(1,1)} \in L^1_m(P).$$

Therefore, by Theorem 2.1, we conclude that the equation

$$(4.4)\quad \partial^2 y(x)/\partial x_1\partial x_2 - (\partial y(x)/\partial x_1)A_{(0,1)}(x) - (\partial y(x)/\partial x_2)A_{(1,0)}(x) + y(x)(A_{(1,1)}(x) - \partial A_{(0,1)}(x)/\partial x_1 - \partial A_{(1,0)}(x)/\partial x_2) + f_{|z}(x,z_0(x),u_0(x)) = 0$$

for a.e. $x \in P$ with boundary conditions

$$(4.5)\quad y(x_1,1) = 0, \quad y(1,x_2) = 0, \quad x_1,x_2 \in [0,1],$$

has exactly one solution in $AC_m(P)$ (in fact, in $AC_{m,1}(P)$) and by representation formula (4.3) for $y \in AC_{m,1}(P)$ and Fubini theorem, we see that (4.4) and (4.5) are equivalent to the following equations

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$-y(x_1,x_2) = \int_{x_1}^{1}\int_{x_2}^{1} (-(\partial y(r,s)/\partial x_1)A_{(0,1)}(r,s) -$$

$$- y(r,s)\partial A_{(0,1)}(r,s)/\partial x_1)dsdr +$$

$$+ \int_{x_1}^{1}\int_{x_2}^{1} (-(\partial y(r,s)/\partial x_2)A_{(1,0)}(r,s) -$$

$$- y(r,s)\partial A_{(1,0)}(r,s)/\partial x_2)dsdr + \int_{x_1}^{1}\int_{x_2}^{1} (y(r,s)A_{(1,1)}(r,s) +$$

$$+ f_{|z}((r,s),z_0(r,s),u_0(r,s)))dsdr =$$

$$= \int_{x_2}^{1} [-y(r,s)A_{(0,1)}(r,s)]_{r=x_1}^{r=1} ds +$$

$$+ \int_{x_1}^{1} [-y(r,s)A_{(1,0)}(r,s)]_{s=x_2}^{s=1} dr +$$

$$+ \int_{x_1}^{1}\int_{x_2}^{1} (y(r,s)A_{(1,1)}(r,s) + f_{|z}((r,s),z_0(r,s),u_0(r,s)))dsdr =$$

$$= \int_{x_2}^{1} y(x_1,s)A_{(0,1)}(x_1,s)ds + \int_{x_1}^{1} y(r,x_2)A_{(1,0)}(r,x_2)dr +$$

$$+ \int_{x_1}^{1}\int_{x_2}^{1} (y(r,s)A_{(1,1)}(r,s) + f_{|z}((r,s),z_0(r,s),u_0(r,s)))dsdr.$$

Thus, because of the uniqueness of y_0 satisfying (3.2) and (3.1) in $L_m^\infty(P)$, equivalence of (4.1) and (4.4) and inclusion $AC_m(P) \subset L_m^\infty(P)$, we conclude that y_0 chosen in Theorem 3.1 belongs, in fact, to $AC_m(P)$ and satisfies (4.1) and (4.2) instead of (3.2).

Our Theorem 3.1 generalizes necessary conditions obtained in [4] and [6] to the case of higher order hyperbolic partial differential equations (1.1). In case $\overline{\alpha} = (1,1)$ Theorem 4.1 gives necessary conditions in the same form as Egorov in [6] (in similar case) but under much less assumptions on smoothness. On the other hand, Theorem 3.1, in case $\overline{\alpha} = (1,1)$, gives similar conditions to those of Suryanarayana [4] (in comparable case) but we do not assume the continuity of the cost function f and matrix coefficients A_α (comp. assumption

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

H_8 in [4]). Moreover, as proved in Theorem 4.1, a slight strengthening of the assumptions of A_α enables us to obtain necessary conditions in terms of absolutely continuous functions (of several variables) and classical differential form.

REFERENCES

[1] **S. Walczak:** Absolutely continuous functions of several variables and their application to differential equations, Bull. Pol. Acad. Sci. 35, (1987), 733-744.

[2] **S. Walczak:** On the differentiability of absolutely continuous functions of several variables. Remarks on the Rademacher theorem, Bull. Pol. Acad. Sci. 36 (1988) 513 - - 520.

[3] **A.D. Ioffe, V.M. Tikhomirov:** The theory of extremal problems, Moscow 1974.

[4] **M.B. Suryanarayana:** Necessary conditions for optimization problems with hyperbolic partial differential equations, SIAM J. Control 11, no. 1 (1973), 130-147.

[5] **C. Castaing, M. Valadier:** Convex analysis and measurable multifunctions, Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin (1977).

[6] **A.I. Egorov:** Necessary conditions for optimality for systems with distributed parameters, Mat. Sb. 69 (1966), 371-421.

[7] **S. Łojasiewicz:** Introduction to the theory of real functions, PWN, Warsaw (1976).

[8] **M.B. Suryanarayana:** On multidimensional integral equations of Volterra type, Pacific J. Math. 41, (1972), 809-828.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ,
90-238 ŁÓDŹ, POLAND

Received January 23, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Do Duc Hung and Jerzy Gawinecki

MATRIX OF FUNDAMENTAL SOLUTIONS FOR THE SYSTEM OF DYNAMIC EQUATIONS OF CLASSICAL THREE-DIMENSIONAL THERMOELASTICITY THEORY

1. Introduction

S. Klainerman proved in [5] that for sufficiently small nonlinear perturbations of linear dissipative equations and systems, such as the wave equations, heat equation, the corresponding solution of the initial-value problem behaves like the solution of the linear unperturbed equations and system. This proof was based on a special version of the Nash-Möser-Hörmander scheme and L^{∞}- L^{1} time decay, i.e. the behaviour of the solution of the initial-value problem for linear unperturbed equations and systems as $t \longrightarrow \infty$. One obtains this time decay of solution with the aid of the fundamental solution (cf. [5]).

In this paper we constructed the matrix of fundamental solutions for dynamic equations for linear classical thermoelasticity theory in three-dimensional Euclidean space E^3.

The dynamic system of equations describing the linear isotropic thermoelastic medium in three-dimensional space has the following form:

$$\partial_t^2 u - b^2 \Delta u - (a^2 - b^2)\,\mathrm{grad\ div}\ u - \gamma\,\mathrm{grad}\ T = \frac{1}{\rho} F \tag{1.1}$$

$$\eta \partial_t \mathrm{div}\ u + \frac{1}{\kappa}\partial_t T - \Delta T = \frac{Q}{\kappa}$$

where $\partial_t = \partial/\partial t$; $\partial_{tt} = \partial^2/\partial t^2$ $u = \left(u_1(x,t), u_2(x,t), u_3(x,t)\right)$ - is the displacement vector, $T(x,t)$ - the temperature

This work was supported by the Polish Grant CPBP 01.02.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$a^2 = \frac{2\mu+\lambda}{\rho}$; $b^2 = \frac{\mu}{\rho}$ - are the velocities of longitudinal and transversal waves respectively; λ, μ - Lame constants, $\gamma = \frac{(2\mu + 3\lambda)\alpha}{\rho}$, α - coefficient of linear thermal expansion; $\eta = \frac{\gamma T_0}{k}$ - k - conductivity coefficient, κ - temperature conductivity coefficient, T_0 - temperature in the natural strata. Moreover, $Q = \frac{q}{k}$ where q is the quantity of heat generated in unit volume per unit time, and $F = (F_1(x,t), F_2(x,t), F_3(x,t))$ is the vector of mass forces.

Now, we introduce the notations

$$(1.2) \qquad L(\partial_t, \partial_x) = \begin{pmatrix} \delta_{jk}(\partial_t^2 - b^2\Delta) - (a^2-b^2)\partial_{jk}^2, & \gamma\partial_j \\ \eta\partial_{kt}^2, & \frac{1}{\kappa}\partial_t - \Delta \end{pmatrix},$$

where δ_{jk} - denotes the Kronecker delta, $j,k = 1,2,3$, $\partial_{jk} = \frac{\partial^2}{\partial x_j \partial x_k}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial_{kt} = \frac{\partial^2}{\partial x_k \partial t}$. Obviously, $L(\partial_t, \partial_x)$ is a 4x4 matrix.

Introducing the four-dimensional vector

$V = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ T \end{bmatrix}$ of the displacement u and temperature T, we can write the system of equations (1,1) in the following matrix form[1]

$$(1.3) \qquad L(\partial_t, \partial_x)V = F, \quad \text{where} \quad F = \begin{bmatrix} F_1/\rho \\ F_2/\rho \\ F_3/\rho \\ Q/\kappa \end{bmatrix}.$$

There is extensive literature devoted to the investigation of

---

[1]The vector is regarded as a one - column matrix, and the matrix product follows the rule row x column.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the system (1.1). The basic results concerning the solvability and properties are presented in monographs [10], [7] containing also further references. In the papers [8] a closed solution was constructed for the most important problems for system (1.1), assuming the periodic time dependence.

The aim of our paper is to construct the matrix of fundamental solutions for the system of equations (1.1). We did this in two steps. At first, using the Hilbert-Levy method [9], [2] we obtained the matrix of fundamental solutions for the principal part (1.1), which can be written in the form

$$L_0(\partial_t,\partial_x)V = F, \tag{1.4}$$

where

$$L_0(\partial_t,\partial_k) = \begin{bmatrix} & & & 0 \\ \delta_{jk}(\partial_t^2-b^2\Delta) - (a^2-b^2)\partial_{jk}^2 & & & 0 \\ & & & 0 \\ \eta\partial_t\partial_1, & \eta\partial_t\partial_2, & \eta\partial_t\partial_3, & \frac{1}{\kappa}\partial_t-\Delta \end{bmatrix}$$

where $k = 1,2,3$.

On the base of the constructed matrix and by the method of potential theory, the solution of the Cauchy problem for the Eqs (1.4) was obtained. Next applying the Hilbert-Levy method once more we get the matrix of fundamental solutions for the entire system of equations (1.1). We will construct the matrices of fundamental solutions for the principal part and entire system of equations of classical theory of thermoelasticity by means of the Fourier transform [12].

2. Basic notations and formulae

Let $x = (x^1,x^2,\ldots,x^n)$ be a point of $\mathbb{R}^n$ $(n=1,2,3)$, $b = (b_1,\ldots,b_n)$ be a multiindex and let

$$|b| = \sum_{j=1}^{n} b_j, \quad \partial^b = \partial_1^{b_1}\ldots\partial_n^{b_n}, \quad \xi^b = \xi_1^{b_1}\ldots\xi_n^{b_n}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$x\cdot\xi = \sum_{j=1}^{n} x^j \xi_j$$

where $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1,2,\ldots,n$ and $\xi = (\xi_1,\ldots,\xi_n)$ is any n-dimensional vector. Let $C_0^\infty(\mathbb{R}^n)$ be the space of all infinitely differentiable complex functions on $\mathbb{R}^n$ consisting of those elements with compact support contained in $\mathbb{R}^n$. When $f \in C_0^\infty(\mathbb{R}^n)$, let

$$|f|_N = \sup_x (1 + |x|)^N |(D^\beta f)(x)|, \qquad |\beta| \le N$$

where $|\cdot|$ denotes some norm on $\mathbb{R}^n$.

Let $S = S(\mathbb{R}^n)$ be all $f \in C_0^\infty(\mathbb{R}^n)$ for which all $|f|_N$ are finite. S topologized by the semi-norms $|\cdot|_N$ is a Fréchet space, called the space of test-functions, and its dual space S' is the space of tempered distributions [12] in $\mathbb{R}^n$.

The Fourier transform

$$S \ni f \longrightarrow Ff = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-ix\xi)\, f(x)dx$$

is a linear homeomorphism $S \longrightarrow S$, as is the inverse Fourier transform F^{-1}

$$S \ni f \longrightarrow F^{-1} f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(ix\xi)\, f(\xi)d\xi$$

the Fourier transform extends to linear homeomorphism $S' \longrightarrow S'$ which is also denoted by F.

Then we have the formulae for $\forall f \in S(\mathbb{R}^n)$, $\forall b$

$$(2.1) \qquad (\partial_\xi^b \hat{f})(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-ix\xi)^b\, f(x)dx,$$

$$(2.2) \qquad \widehat{(\partial^b f)}(\xi) = (i\xi)^b\, (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-ix\xi)\, f(x)dx,$$

$$(2.3) \qquad (f * g)(\xi) = (2\pi)^{n/2}\, \hat{f}(\xi)\hat{g}(\xi),$$

where * denotes the convolution of the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-z)g(z)dz.$$

We shall, use here only some fundamental facts of this theory; in particular we shall employ the Dirac distribution δ. It is known that [12]

$$\hat{\delta}(\xi) = \frac{1}{(2\pi)^{n/2}} . \tag{2.4}$$

Moreover, the Dirac distribution $\delta(x,t)$ can be written in the form

$$\delta(x,t) = \delta(x)\delta(t). \tag{2.5}$$

3. The matrix of fundamental solutions for the principal part of dynamic equations of thermoelasticity theory

The matrix $\mathbb{E}_0(x,t)$ of the fundamental solutions for the operator $L_0(\partial_t, \partial_x)$ (cf (1.4) satisfies in the space $\mathbb{R}^4$ the equation

$$L_0(\partial_t, \partial_x)\mathbb{E}_0(x,t) = \delta(x,t)\mathbb{I}, \tag{3.1}$$

where $\mathbb{I}$ is the 4×4 unit matrix.

We seek the required matrix $\mathbb{E}_0$ in accordance with the Hilbert-Levy method in the form

$$\mathbb{E}_0(x,t) = \tag{3.2}$$

$$= \begin{bmatrix} & & & 0 \\ & \mathbb{H}(x,t) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \Gamma(x,t) \end{bmatrix} + \int_{\mathbb{R}^4} \begin{bmatrix} & & & 0 \\ & \mathbb{H}(x-y,t-s) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \Gamma(x-y,t-s) \end{bmatrix} V(y,s)dyds$$

where $\mathbb{H}(x,t)$ is the 3×3 matrix of the fundamental solutions for the operator $\partial_t^2 - b^2\Delta - (a^2 - b^2)\text{grad div}$ (cf. [1], [11])

$$\mathbb{H}(x,t) = \left[H_{jk}(x,t)\right] \tag{3.3}$$

where $j,k = 1,2,3$, with the elements

$$H_{jk}(x,t) = (4\Pi\rho)^{-1}\left[b^{-2} \frac{\delta_{jk}}{|x|} \delta\left(t - \frac{|x|}{b} \right) + \right. \tag{3.3a}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ \frac{x_j x_k}{|x|^3}\left(a^{-2}\delta\left(t - \frac{|x|}{a}\right) - b^{-2}\delta\left(t - \frac{|x|}{b}\right)\right) -$$

$$- t\left(\frac{\delta_{jk}}{|x|^3} - \frac{3x_j x_k}{|x|^5}\right)\left(H\left(t - \frac{|x|}{a}\right) - H\left(t - \frac{|x|}{b}\right)\right)\Bigg];$$

$H(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$ is the Heaviside's function,

$\Gamma(x,t)$ - is the fundamental solution (cf. [12]) for the heat conduction operator $\left(\frac{1}{\kappa}\partial_t - \Delta\right)$ of the form

$$\Gamma(x,t) = \frac{H(t)}{(4\Pi\kappa t)^{3/2}} e^{-\frac{|x|^2}{4\kappa t}} \tag{3.3b}$$

and $V(x,t)$ is an unknown 4×4 matrix.

Introducing (3.2) into Eqs. (3.1) we obtain the following integral equation:

$$V(x,t) + \int_{\mathbb{R}^4} \begin{bmatrix} \mathbb{O} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \eta\partial_t \sum_{j=1}^{3} \partial_j H_{jk}(x-y,t-s) & 0 \end{bmatrix} V(y,s)\,dyds = \tag{3.4}$$

$$= - \begin{bmatrix} \mathbb{O} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \eta\left(\partial_t \sum_{j=1}^{3} \partial_j H_{jk}\right)(x,t) & 0 \end{bmatrix}$$

where $\mathbb{O}$ - denotes the null 3×3 matrix.

Using the method of successive approximations we find that matrix V has the following form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$V(x,t) = -\begin{bmatrix} & \mathbb{O} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \eta\partial_t\Big(\sum_{j=1}^{3}\partial_j H_{jk}\Big)(x,t) & & \end{bmatrix}. \tag{3.5}$$

In accordance with (3.5), (3,2) the matrix $\mathbb{E}_0(x,t)$ of the fundamental solutions for (3.1) can be written in the explicit form

$$\mathbb{E}_0(x,t) = \begin{bmatrix} \mathbb{H}(x,t) & \begin{matrix}0\\0\\0\end{matrix} \\ -\eta\Big(\sum_{j=1}^{3}\partial_j H_{jk} *_4 \partial_t\Gamma\Big)(x,t) & \Gamma(x,t) \end{bmatrix}, \tag{3.6}$$

where: $*_4$ denotes the four-dimensional convolution.

In order to investigate the behaviour of the terms $-\eta\Big(\sum_{j=1}^{3}\partial_j H_{jk} *_4 \partial_t\Gamma\Big)(x,t)$ $(j = 1,2,3;\ k = 1,2,3)$ occurring in the matrix $\mathbb{E}_0(x,t)$ given by (3.6), we use the Fourier transform with respect to x:

$$F_{x\to\xi}\Big[\Big(\eta\sum_{j=1}^{3}\partial_j H_{jk} *_4 \partial_t\Gamma\Big)\Big](\xi,t) = \tag{3.7}$$

$$= \eta\int_0^t \widehat{\sum_{j=1}^{3}\partial_j H_{jk}}(\xi,t)\ \widehat{\partial_t\Gamma}(\xi,t-s)\,ds.$$

To calculate the term $\widehat{\sum_{j=1}^{3}\partial_j H_{jk}}(\xi,t) = \widehat{\operatorname{div} H_k}(\xi,t)$ we must use the matrix $\mathbb{H}(x,t)$ of fundamental solution of the system of dynamic equations of pure elasticity theory.

It is easy to see that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.8) $\hat{H}(\xi,t) =$

$$= \begin{bmatrix} A_{11}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{12}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{13}(\partial_t, i\xi)\hat{U}(\xi,t) \\ A_{21}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{22}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{23}(\partial_t, i\xi)\hat{U}(\xi,t) \\ A_{31}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{32}(\partial_t, i\xi)\hat{U}(\xi,t), & A_{33}(\partial_t, i\xi)\hat{U}(\xi,t) \end{bmatrix} =$$

$$= \left[\hat{H}_1(\xi,t), \hat{H}_2(\xi,t), \hat{H}_3(\xi,t) \right] \quad \text{where}$$

$$A_{11}(\partial_t, i\xi) = (\partial_t^2 + b^2|\xi|^2)\,[(\partial_t^2 + b^2|\xi|^2) + (1-b^2)(\xi_2^2 + \xi_3^2)],$$

$$A_{12}(\partial_t, i\xi) = A_{21}(\partial_t, i\xi) = -(1-b^2)\xi_1^2\xi_2(\partial_t^2 + b^2|\xi|^2),$$

$$A_{13}(\partial_t, i\xi) = A_{31}(\partial_t, i\xi) = -(1-b^2)\xi_1\xi_3(\partial_t^2 + b^2|\xi|^2),$$

$$A_{22}(\partial_t, i\xi) = (\partial_t^2 + b^2|\xi|^2)[(\partial_t^2 + b^2|\xi|^2) + (1-b^2)(\xi_1^2 + \xi_3^2)],$$

$$A_{23}(\partial_t, i\xi) = A_{32}(\partial_t, i\xi) = -(1-b^2)\xi_2\xi_3(\partial_t^2 + b^2|\xi|^2),$$

$$A_{33}(\partial_t, i\xi) = (\partial_t^2 + b^2|\xi|^2)[(\partial_t^2 + b^2|\xi|^2) + (1-b^2)(\xi_1^2 + \xi_2^2)]$$

and $\hat{U}(\xi,t)$ is the solution of the following Cauchy problem

$$(\partial_t^2 + |\xi|^2)(\partial_t^2 + b^2|\xi|^2)^2\,\hat{U}(\xi,t) = 0$$

with initial conditions

$$\hat{U}(\xi,0+) = \partial_t\hat{U}(\xi,0+) = \partial_t^2\hat{U}(\xi,0+) = \partial_t^3\hat{U}(\xi,0+) = \partial_t^4\hat{U}(\xi,0+) = 0;$$

$$\partial_t^5\hat{U}(\xi,0+) = 1.$$

After some calculations we get

$$(3.9) \qquad \hat{U}(\xi,t) = a_1 \frac{\sin|\xi|t}{|\xi|^5} + a_2 \frac{\sin b|\xi|t}{|\xi|^5} + a_3 t \frac{\cos b|\xi|t}{|\xi|^4}$$

where

$$a_1 = \frac{1}{(b^2-1)^2}\,; \quad a_2 = \frac{1-3b^2}{2b^3(b^2-1)^2}\,; \quad a_3 = \frac{1}{2b^2(b^2-1)}\,.$$

It is easy to verify that

---

[2] Without loss of generality we assume that $a = 1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.10)\quad \begin{cases} \widehat{\operatorname{div} H_1}(\xi,t) = i\xi_1(\partial_t^2 + b^2|\xi|^2)\ \hat{U}(\xi,t), \\ \widehat{\operatorname{div} H_2}(\xi,t) = i\xi_2(\partial_t^2 + b^2|\xi|^2)\ \hat{U}(\xi,t), \\ \widehat{\operatorname{div} H_3}(\xi,t) = i\xi_3(\partial_t^2 + b^2|\xi|^2)\ \hat{U}(\xi,t). \end{cases}$$

Taking into account the formulae (3.9) we obtain

$$(3.11)\quad \begin{cases} \widehat{\operatorname{div} H_1}(\xi,t) = i\xi_1 \dfrac{\sin|\xi|t}{|\xi|}, \\ \widehat{\operatorname{div} H_2}(\xi,t) = i\xi_2 \dfrac{\sin|\xi|t}{|\xi|}, \\ \widehat{\operatorname{div} H_3}(\xi,t) = i\xi_3 \dfrac{\sin|\xi|t}{|\xi|}. \end{cases}$$

Now, in view of (3.7) and (3.11) we have

$$(3.12)\quad F_{x\to\xi}\,[(\operatorname{div} H_k *_4 \partial_t\Gamma)](\xi,t) =$$

$$= \int_0^t \widehat{\operatorname{div} H_k}(\xi,s)\ \widehat{\partial_t\Gamma}(\xi,t-s)\,ds =$$

$$= -i\xi_k|\xi|\ e^{-t|\xi|^2} \int_0^t \sin|\xi|s\ e^{|\xi|^2 s}\,ds.$$

Denoting by I the integral of the form

$$(3.13)\quad I = \int_0^t \sin|\xi|s\ \ e^{|\xi|^2 s}\,ds$$

and integrating by parts we get

$$(3.14)\quad I = \frac{1}{1+|\xi|^2}\left(\frac{1-\cos|\xi|t\ e^{|\xi|^2 t}}{|\xi|} + \sin|\xi|t\ \ e^{|\xi|^2 t}\right).$$

So, from (3.12) and (3.14) it follows that

$$(3.15)\quad F_{x\to\xi}\,[(\operatorname{div} H_k *_4 \partial_t\Gamma)](\xi,t) = -i\xi_k \frac{1}{1+|\xi|^2} e^{-|\xi|^2 t} + \frac{i\xi_k \cos|\xi|t}{1+|\xi|^2} - \frac{i\xi_k \sin|\xi|t}{1+|\xi|^2},\quad k = 1,2,3.$$

Applying the inverse Fourier transform to the terms on the right-hand side of the formulae (3.15) we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.16)\qquad \Big(\sum_{j=1}^{3}\partial_j H_{jk} *_4 \partial_t\Gamma\Big)(x,t) = (\operatorname{div} H_k *_4 \partial_t\Gamma)(x,t) =$$

$$= -\frac{\partial}{\partial x_k}\left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \frac{e^{-\frac{|\cdot|^2}{4t}}}{(2\sqrt{\pi t})^3}\right)(x,t)\right] +$$

$$+\frac{\partial}{\partial x_k}(\partial_t + \Delta)\left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \frac{\delta_{S_t(\cdot)}}{4\pi t}\right)(x,t)\right],\quad k = 1,2,3.$$

where $\delta_{S_t(\cdot)}$ - denotes Dirac distribution concentrated on the sphere with radius t and $*_3$ denotes the three-dimensional convolution with respect to variable x.

Denoting by I_k the first term on the right hand side of (3.16) we have

$$(3.17)\qquad I_k = -\frac{\partial}{\partial x_k}\left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \frac{e^{-\frac{|\cdot|^2}{4t}}}{(2\sqrt{\pi t})^3}\right)(x,t)\right] =$$

$$= -\frac{1}{8\Pi^{3/2}t^{3/2}}\cdot\frac{1}{2t}\left[\left(\frac{e^{-|x|}}{|x|}\right) *_3 \left(x_k e - \frac{|x|^2}{4t}\right)\right].$$

Taking into account the fact that

$$(3.18)\qquad \frac{e^{-|x|}}{|x|} \in L^q(\mathbb{R}^3) \quad \text{where} \quad 1 \le q < 3$$

and applying the Young theorem: we obtain

$$(3.19)\qquad \|I_k\|_{L^r} \le C_{p,q}\, t^{\left(\frac{3}{2p} - 2\right)}$$

where $1 \le p \le 3/2$; $1 \le q < 3$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

It is easy to see that the expression of the form

$$(3.20)\qquad I_4 = \left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \frac{\delta_{S_t(\cdot)}}{4\Pi t}\right)(x,t)\right]$$

we can represent as follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$I_4 = \frac{1}{2|x|}\left[e^{-|t-|x||} - e^{-(t+|x|)} \right]. \tag{3.21}$$

Taking these results into account, we get the matrix $E_0(x,t)$, of the fundamental solutions for the principal part of (1.1) in the form

$$E_0(x,t) = \tag{3.22}$$

$$= \begin{bmatrix} & & 0 \\ & [H_{jk}(x,t)] & 0 \\ & & 0 \\ -\eta\partial_k\left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \Gamma\right)(x,t)\right] & -\eta\partial_k(\partial_t+\Delta)\left[\frac{1}{2|x|}(e^{-|t-|x||} - e^{-(t+|x|)})\right] & \Gamma(x,t) \end{bmatrix}$$

where : $H_{jk}(x,t)$ and $\Gamma(x,t)$ are given by formulae (3.3a) and (3.3b) respectively and the term

$I_k = -\partial_k\left[\left(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \Gamma \right)(x,t) \right]$ has the estimate given by (3.19), $j,k = 1,2,3$.

4. The solution of the Cauchy problem for linear system of equations of classical thermoelasticity

In this section we obtain the explicit formulae for the solution of the Cauchy problem for the system of equations

$$\partial_t^2 u - b^2\Delta u - (a^2 - b^2)\,\text{grad div } u = 0, \tag{4.1}$$

$$\eta\partial_t \text{ div } u + \frac{1}{\kappa}\partial_t T - \Delta T = 0 \tag{4.2}$$

with initial conditions

$$u(x,+0) = f(x), \quad (\partial_t u)(x,+0) = g(x), \tag{4.3}$$

$$T(x,+0) = T^0(x), \tag{4.4}$$

where

$f(x) = (f_1(x), f_2(x), f_3(x))^t$, $g(x) = (g_1(x), g_2(x), g_3(x))^t$, $T^0(x)$ are given functions belonging to the space $C_0^\infty(\mathbb{R}^3)$, $C_0^\infty(\mathbb{R}^3)$, $C_0^\infty(\mathbb{R}^3)$, respectively.

Applying the method of potential theory and basing on the matrix $E_0(x,t)$ of fundamental solutions for the system (4.1), (4.2) we get the solution of the Cauchy problem (4.1) - (4.4)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4.5)\qquad \begin{bmatrix} u \\ T \end{bmatrix}(x,t) = \mathbb{E}_0(\cdot,t) *_3 \begin{bmatrix} g \\ T^0 \end{bmatrix} + \partial_t \mathbb{E}_0(\cdot,t) *_3 \begin{bmatrix} f \\ 0 \end{bmatrix},$$

where $\mathbb{E}_0(x,t)$ denotes the matrix of fundamental solutions given by the formulae (3.22).

The solution of the Cauchy problem given by the formulae (4.5) has the following explicit form

$$(4.6)\qquad u_j = \frac{1}{4\Pi\rho}\sum_{l=1}^{3}\Bigg\{ \int_{r=bt} (br^2)^{-1}\Big[t\Big(\delta_{jl} - \eta_j\eta_l\Big)\Big(g_l + b\,\frac{\partial f_l}{\partial n}\Big) +$$

$$+ \Big(2\delta_{jl} - 4\eta_j\eta_l\Big)f_l\Big]dS_y + \int_{r=at} (ar^2)^{-1}\Big[t\eta_j\eta_l\Big(g_l + a\,\frac{\partial f_l}{\partial n}\Big) +$$

$$+ \Big(-\delta_{jl} + 4\eta_j\eta_l\Big)f_l\Big]dS_y - \int_{bt<r<at} r^{-3}\Big(\delta_{jl} - 3\eta_j\eta_l\Big)\Big(tg_l + f_l\Big)\,dy\Bigg\}$$

$$(4.7)\qquad T = \Gamma(\cdot,t) *_3 T^0 + \sum_{j=1}^{3}\Big(\partial_k H_{jk} *_4 \partial_t \Gamma\Big) *_3 g_j +$$

$$+ \sum_{j=1}^{3}\partial_t\Big[\partial_k H_{jk} *_4 \partial_t \Gamma\Big] *_3 f_j,$$

where $\eta_j = \dfrac{x_j - y_j}{|x-y|}$, $r = |x-y|$, dS_y is the area element of the sphere $r = at$ or $r = bt$ respectively, and

$$\partial_k H_{jk} *_4 \partial_t \Gamma = -\eta\,\partial_j\Big[\Big(\frac{e^{-|\cdot|}}{|\cdot|} *_3 \Gamma\Big)(\cdot,t) +$$

$$- (\partial_t + \Delta)\Big[\frac{1}{2|x|}\Big(e^{-|t-|x||} - e^{-(t+|x|)}\Big)\Big].$$

Remark 4.1. Applying the formulae (4.6), (4.7) we can deduce $L^\infty - L^1$ and $L^p - L^q$ estimates for the solution of the Cauchy problem (cf. [6]) for (4.1), (4.2).

Remark 4.2. In order to obtain the matrix of fundamental solutions for (1.1) we once more used the Hilbert-Levy method.

So, we will be seeking the matrix of fundamental solutions $\mathbb{E}(x,t)$ for (1.1) in the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\mathbb{E}(x,t) = \mathbb{E}_0(x,t) + \mathbb{E}_0(x,t) *_4 W(x,t), \tag{4.8}$$

where $\mathbb{E}_0(x,t)$ - is the matrix of fundamental solutions for the principal part of the Eqs (1.1) given by formulae (3.22), and W is the required 4×4 matrix.

Let us represent the operator $L(\partial_t, \partial_x)$ as the sum

$$L(\partial_t, \partial_x) = L_0(\partial_t, \partial_x) + L_1(\partial_t, \partial_x) \tag{4.9}$$

where

$$L_1(\partial_t, \partial_x) = \begin{bmatrix} & & & \gamma\partial_1 \\ & \mathbb{O} & & \gamma\partial_2 \\ & & & \gamma\partial_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking into account (4.9), (4.8) and (1.1), we arrive at the integral equation for unknown matrix $W(x,t)$

$$W(x,t) = (-L_1\mathbb{E}_0) *_4 W + (-L_1\mathbb{E}_0) \tag{4.10}$$

where

$$-L_1\mathbb{E}_0 = \begin{bmatrix} & & & -\frac{1}{\eta}\partial_1\Gamma \\ \left[\gamma\eta \operatorname{div} H_k *_4 \partial_t\partial_j\Gamma\right] & & & -\frac{1}{\eta}\partial_2\Gamma \\ & & & -\frac{1}{\eta}\partial_3\Gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.11}$$

where: $j,k = 1,2,3$.

Using the method of successive approximations we find that matrix $W(x,t)$ has the following form

$$W(x,t) = \sum_{n=1}^{\infty} (\gamma\eta)^n K^{*(n-1)} *_4 \begin{bmatrix} & & & -\frac{1}{\eta}\partial_1\Gamma \\ \left[\operatorname{div} H_k *_4 \partial_t\partial_j\Gamma\right] & & & -\frac{1}{\eta}\partial_2\Gamma \\ & & & -\frac{1}{\eta}\partial_3\Gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.12}$$

where: $j,k = 1,2,3$;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$K = \operatorname{div}(\operatorname{div} H_k *_4 \partial_t \Gamma) = \Big[\sum_{k=1}^{3} \partial_k \Big(\sum_{j=1}^{3} \partial_j H_{jk} \Big) \Big] *_4 \partial_t \Gamma$$

is a scalar, and

$$K^{*(0)} * A = A,$$

$$K^{*(1)} * A = K * A,$$

$$K^{*(j)} * A = \underbrace{K * K * K * \ldots * K}_{j\text{-times}} * A.$$

Remark 4.3. The series $W(x,t)$ given by formulae (4.12) is convergent in the space of generalized functions (in weak sense) (cf. [4] p. 102 th. 12. 5. 3).

Remark 4.4. Using the matrix $\mathbb{E}(x,t)$ of fundamental solutions for the system of equations (1.1) we obtain the solution for this system, which can express by the convolutions

$$\begin{bmatrix} u \\ T \end{bmatrix}(x,t) = \mathbb{E}(\cdot,t) *_3 \begin{bmatrix} u^1 \\ T^0 \end{bmatrix} + \partial_t \mathbb{E}(\cdot,t) *_3 \begin{bmatrix} u^0 \\ 0 \end{bmatrix}, \tag{4.13}$$

where u^0, u^1, T^0 - denote the initial conditions for (1.1).

REFERENCES

[1] **V. M. Babich and others:** Linear equations of mathematical physics (in Russian), Moskva, (1976).

[2] **Z. Domański, A. Piskorek:** Matrices of fundamental solutions for the system of quasi-static equations of thermoelasticity and the system of dynamic equations of thermal stresses, Arch. Mech. 23 (1971) 265-270.

[3] **J. Gawinecki:** Matrix of fundamental solutions for the system of equations of hyperbolic thermoelasticity with two relaxation times and solution of the Cauchy problem, Bull. Acad. Polon. Sci. Sér. Sci. Tech., 36, No. 7-9 (1988) 449-466.

[4] **L. Hörmander:** The Analysis of Linear Partial Differential Operators, Vol 2. Differential Operators with Constant

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Coefficients, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1983), (Russian Edition - Mir, Moskva, 1986).

[5] S. Klainerman: Long-time behaviour of solutions to nonlinear evolution equations, Arch. Rational Mech. Anal., 78 (1982) 73-98.

[6] S. Klainerman, G. Ponce: Global, small amplitude solutions to nonlinear evolution equations, Comm. Pure Appl. Math., 36 (1983) 133-141.

[7] V. D. Kupradze and others: Three-dimensional problems of mathematical theory of elasticity and thermoelasticity (in Russian), Moskva, (1976).

[8] V. D. Kupradze, T.V. Burchuladze: The boundary value problem of the thermoelasticity, Differential Equations, 5 (1), (1969), 3-43.

- Some boundary value problems of the thermoelasticity solving in the quadratures I, 5 (10), (1969) 1735-1761.
- Some boundary value problems of the thermoelasticity solving in the quadratures II, 5(11), (1969) 1923-1939.

[9] E. E. Levy: Sulle equazione lineari totalmente ellittiche alle derivate parziali. Rend. Circ. Mat. Palermo 24 (1907), 275-317.

[10] W. Nowacki: Dynamic problems of thermoelasticity (in Polish), Warsaw, 1966.

[11] A. Piskorek: Radon-Transformation und hyperbolische Differential-gleichungen der Mathematischen Physik, Methoden und Verfahren der Mathematischen Physik, Band 10, Bibliographisches Institut AG, Mannheim, Wien, Zürich 1973.

[12] V. S. Vladimirov: Generalized functions in mathematical physics, (in Russian), Moskva 1976.

DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY, HANOI, VIETNAM;
DEPARTMENT OF MATHEMATICS, MILITARY TECHNICAL ACADEMY, 00-908 WARSZAWA

Received January 26, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Grażyna Kozłowska

SOME REMARKS ON MODULAR APPROXIMATION BY A FILTERED FAMILY OF LINEAR OPERATORS

1. Introduction

In the paper [2] some approximation theorems are proved in which the φ - function $\varphi = \varphi(t,u)$ generating the modular, is convex with respect to u.

In this paper it is proved that the same results hold, if the convexity of φ - function φ is replaced by the other conditions.

We begin by recalling notation and some definitions used by J. Musielak in [2] and [3].

Let (Ω,Σ,μ) be a measure space with a nonnegative, non - trivial, σ - finite and complete measure μ, and let X be a space of all extended real-valued, Σ - measurable functions $x = x(\cdot)$ over Ω, finite μ - almost everywhere.

Let φ be a φ - function with parameter, i.e. $\varphi : \Omega \times R_+ \longrightarrow$ $R_+ = \langle 0,\infty)$, φ is a nondecreasing, continuous function of u, equal to zero if $u = 0$, and $\varphi(t,u) > 0$ for $u > 0$ for every $t \in \Omega$, and is a Σ - measurable function of $t \in \Omega$, for every $u \in R_+$.

Now, we denote

$$\rho(x) = \int_{\Omega} \varphi(t,|x(t)|)\,d\mu, \tag{1}$$

ρ is a modular on X. The respective modular space generated by ρ is denoted by $L^{\varphi}(\Omega,\Sigma,\mu)$ (or briefly, L^{φ}) and is called a generalized Orlicz space.

Let $\Omega = \langle 0,b) \subset R$, $0 < b < \infty$, μ - the Lebesgue measure in the σ - algebra Σ of all Lebesgue measurable subsets of

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$<0,b)$.

The translation operator $\tau_v : X \to X$ will be defined by the equality $\tau_v x(t) = x(t+v)$, where x is extended to the whole R, b-periodically.

Also, the φ - function with parameter generating the modular ρ by formula (1) will be extended periodically with respect to the variable $t \in <0,b)$ to the whole R, i.e. $\varphi(t+b,u) = \varphi(t,u)$ for $u \in R_+$, $t \in R$.

We shall say that the function φ is τ - bounded, if there exist positive constants k_1, k_2 such that

$$(2) \qquad \varphi(t-v,u) \le k_1\varphi(t,k_2u) + f(t,v) \quad \text{for} \quad u \in R_+, \quad v,t \in R$$

where the function $f : R \times R \to R_+$ is measurable and b-periodic with respect to the first variable, and such that writing

$$h(v) = \int_0^b f(t,v)\,dt \quad \text{for every} \quad v \in R, \quad \text{we have} \quad H = \sup_{v \in R} h(v) < \infty$$

and $h(v) \to 0$ as $v \to 0^+$ and $v \to b^-$.

Let V be an abstract nonempty set and let $\mathcal{V}$ be a filter of subset of V.

A function $g : V \to R$ tends to zero with respect to $\mathcal{V}$, $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\varepsilon > 0$ there is a set $V_0 \in \mathcal{V}$ such that $|g(v)| < \varepsilon$ for all $v \in V_0$.

A family $T = (T_v)_{v \in V}$ of linear operators $T_v : X_\rho \to X$ will be called $\mathcal{V}$ - bounded, if there exist positive numbers k_1, k_2 and a function $g : V \to R_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$, and for every $x \in X_\rho$ there is a set $V_x \in \mathcal{V}$ for which $\rho(T_v x) \le k_1\rho(k_2 x) + g(v)$ for all $x \in V_x$.

Moreover, we shall use the following definitions.

We say that the φ - function φ satisfies the condition (A), if there exists a function $\psi = \psi(t,u)$ continuous and convex with respect to $u \in R_+$ for all $t \in \Omega$, and two constants $\alpha > 0$ and $\lambda > 1$ such that $\forall_{u \in R_+} \forall_{t \in \Omega} \psi(t,u) \le \varphi^\alpha(t,u) \le \lambda\psi(t,u)$.

Let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $\rho_\alpha(x) = \int_\Omega \varphi^\alpha(t,|x(t)|)dt.$

since, for $\alpha \geq 1$, φ^α is φ - function too, then $\rho_\alpha(x)$ is a modular on X.

It is easily seen that if $0 < \alpha < 1$, then $\rho_\alpha(x)$ is a modular on X, too.

Remark 1. Using the Hölder inequality we can prove that, if φ is τ - bounded then for $0 < \alpha < 1$ φ^α is τ - bounded too.

Remark 2. If φ is τ - bounded, and f (see (2)) satisfies the condition (B) i.e. f is continuous and concave function of t, and not identically equal to zero for $t \in \langle 0,b\rangle$, then φ^α is τ - bounded for $\alpha > 1$, too.

For the proof of Remark 2 we may use the inequality:

(4) $$\left\{\frac{\beta+1}{b-a}\int_a^b [f(x)]^\beta dx\right\}^{1/\beta} \leq \left\{\frac{\alpha+1}{b-a}\int_a^b [f(x)]^\alpha dx\right\}^{1/\alpha},$$

where $0 < \alpha < \beta$ and f is a nonnegative function, continuous, concave, and not identically equal to zero for $x \in \langle a,b\rangle$ (see [1] p.67, (10)).

2. The results

In this section we shall prove some theorems from [2] § 3, in which the convexity of φ - function φ will be replaced by the others conditions.

Let

$$\omega_{\tau,\alpha}(x,\delta) = \sup_{|v|\leq\delta} \int_0^b \varphi^\alpha[t,|x(t+v) - x(t)|]dt \quad \text{for } \delta>0,\ \alpha>0.$$

By the Remark 2 and Theorem 3 [2] p. 104, there holds the following

Lemma 1. Let φ be τ - bounded and f (see (2)) satisfies the condition (B). Let $\int_0^b \varphi^\alpha(t,c)\,dt < \infty$, for every $c > 0$ and $\alpha > 1$. Then for every $x \in L^{\varphi^\alpha}$ there exists $c > 0$ such that $\omega_{\tau,\alpha}(cx,\delta) \to 0$ as $\delta \to 0^+$ for $\alpha > 1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By the Hölder inequality and Theorem 3 [2] p. 104, we get

Lemma 2. If φ is τ - bounded and $\int_0^b \varphi(t,c)\,dt < \infty$, for every $c > 0$, then for every $x \in L^\varphi$ there exists $c > 0$ such that $\omega_{\tau,\alpha}(cx,\delta) \to 0$ as $\delta \to 0^+$ for $0 < \alpha < 1$.

Let W be an abstract set and $\mathcal{W}$ - a filter in W. Let $K_w : \langle 0,b) \to R_+$ for $w \in W$ be integrable in $\langle 0,b)$ and singular, i.e.

$$\sigma(w) = \int_0^b K_w(t)\,dt \xrightarrow{\mathcal{W}} 1, \quad \sigma_\delta(w) = \int_\delta^{b-\delta} K_w(t)\,dt \xrightarrow{\mathcal{W}} 0$$

for every $0 < \delta < \frac{b}{2}$, $\quad \sigma = \sup_{w\in W} \int_0^b K_w(t)\,dt < \infty$,

and let us extend K_w b - periodically to the whole R.

Let

$$T_w x(s) = \int_0^b K_w(t-s)\,x(t)\,dt. \tag{5}$$

We shall prove the following

Lemma 3. Let φ be a τ - bounded φ - function with parameter satisfying the condition (A) for $\alpha > 1$, and f (see (2)) satisfies the condition (B). Let $(K_w)_{w\in W}$ be singular. Then $T_w : L^{\varphi^\alpha} \to L^{\varphi^\alpha}$ for every $w \in W$.

Proof. Since φ satisfied the condition (A) and is τ-bounded, we get by Jensen inequality and Remark 2

$$\rho_\alpha(T_w x) = \int_0^b \varphi^\alpha(s, |\int_0^b K_w(t-s)\,x(t)\,dt|)\,ds =$$

$$= \int_0^b \varphi^\alpha(s, \int_0^b K_w(t)\,|x(s+t)|\,dt)\,ds \le$$

$$\le \lambda \int_0^b \psi(s, \frac{1}{\sigma(w)} \int_0^b K_w(t)\,\sigma(w)\,|x(s+t)|\,dt)\,ds \le$$

$$\le \lambda \int_0^b \frac{1}{\sigma(w)} \int_0^b K_w(t)\,\psi(s,\sigma(w)\,|x(s+t)|)\,dt\,ds \le$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq \frac{\lambda}{\sigma(w)} \int_0^b \int_0^b K_w(t)\ \varphi^{\alpha}(s,\sigma|x(s+t)|)\ dt\ ds =$$

$$= \frac{\lambda}{\sigma(w)} \int_0^b K_w(t) \int_0^b \varphi^{\alpha}(u-t,\sigma|x(u)|)\ du\ dt \leq$$

$$= \frac{\lambda}{\sigma(w)} \int_0^b K_w(t) \int_0^b [k_1\varphi^{\alpha}(u,k_2\sigma|x(u)| + f(u,t)]\ du\ dt =$$

$$= \frac{\lambda}{\sigma(w)} \int_0^b K_w(t)\ [k_1\rho_{\alpha}(k_2\sigma x) + h(t)]\ dt = \lambda k_1\rho_{\alpha}(k_2\sigma x) + \lambda g_1(w)$$

where $g_1(w) = \frac{1}{\sigma(w)} \int_0^b K_w(t)h(t)\ dt.$

Similarly as in [2] p. 105 we obtain $g(w) \xrightarrow{W} 0$.

Lemma 4. Let φ be a τ - bounded φ - function with parameter satisfying the condition (A) for $0 < \alpha < 1$.

Let $(K_w)_{w\in W}$ be singular. Then $T_w : L^{\varphi} \longrightarrow L^{\varphi}$ for every $w \in W$ and $T = (T_w)_{w\in W}$ is W - bounded.

Proof. If $0 < \alpha < 1$ then $\psi^{1/\alpha}$ is a convex function of u and

$$\psi^{1/\alpha}(t,u) \leq \varphi(t,u) \leq \lambda^{1/\alpha}\psi^{1/\alpha}(t,u).$$

From this similarly as in Lemma 3 we obtain

$$\rho(T_w x) = \int_0^b \varphi(s,\ \int_0^b K_w(t)|x(s+t)|\ dt\ ds \leq$$

$$\leq \lambda^{1/\alpha} \int_0^b \psi^{1/\alpha}\left(s, \frac{1}{\sigma(w)} \int_0^b K_w(t)\ \sigma(w)|x(s+t)|\ dt\right) ds \leq$$

$$\leq \frac{\lambda^{1/\alpha}}{\sigma(w)} \int_0^b K_w(t) \int_0^b \varphi(u-t,\sigma|x(u)|)\ du\ dt \leq$$

$$\leq \frac{\lambda^{1/\alpha}}{\sigma(w)} \int_0^b K_w(t)\ [\ k_1\rho(k_2\sigma x) + h(t)\ ]\ dt =$$

$$= k_1\lambda^{1/\alpha}\rho(k_2\sigma x) + \lambda^{1/\alpha}g(w)$$

where $g(w) = \frac{1}{\sigma(w)} \int_0^b K_w(t)h(t)\ dt.$

Similarly as in [2] p. 105 we obtain $g(w) \xrightarrow{W} 0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, we shall prove the following

Theorem 1. Let φ be a τ - bounded φ - function with parameter satisfying the condition (A) for $\alpha > 1$ and let

$$\int_0^b \varphi^\alpha(t,c)\,dt < \infty \quad \text{for every} \quad c > 0.$$

Let f (see (2)) satisfy the condition (B). and $(K_w)_{w\in W}$ be singular. Then operators T_w defined by (5) satisfy the condition $\rho(a(T_w x - x)) \xrightarrow{W} 0$ for every $x \in L^{\varphi^\alpha}$ and for some $a > 0$

Proof. By Lemma 3 $T_w x \in L^{\varphi^\alpha}$ for $x \in L^{\varphi^\alpha}$. Let $x \in L^{\varphi^\alpha}$, then by Lemma 1 $\omega_{\tau,\alpha}(cx,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0^+$ for sufficiently small $c > 0$. Now, let us choose $a > 0$ so small that $2a\sigma \le c$ and $\rho_\alpha(4\sigma k a x) < \infty$ where $k = \max(k_2,1)$ and k_2 is equal to the constant from the definition of τ - boundedness of φ^α.

Because ψ is convex, we get by condition (A) and Jensen inequality

$$\varphi^\alpha\left(s,\left|\frac{1}{\sigma(w)}\int_0^b K_w(t)\sigma(w)ax(s+t)\,dt - ax(s)\right|\right) \le$$

$$\le \lambda\psi\left(s,\frac{1}{\sigma(w)}\left|\int_0^b K_w(t)\sigma(w)ax(s+t)\,dt - ax(s)\right|\right) \le$$

$$\le \lambda\left\{\frac{1}{2}\frac{1}{\sigma(w)}\int_0^b K_w(t)\psi[s,2\sigma(w)a|x(s+t) - x(s)|]\,dt + \right.$$

$$\left. + \frac{1}{2}\psi[s,2a|(\sigma(w) - 1)x(s)|]\right\} \le$$

$$\le \lambda\left\{\frac{1}{2}\frac{1}{\sigma(w)}\int_0^b K_w(t)\varphi^\alpha[s,2\sigma a|x(s+t) - x(s)|]\,dt + \right.$$

$$\left. + \frac{1}{2}\varphi^\alpha[s,2a|(\sigma(w) - 1)x(s)|]\right\}.$$

From this

$$\rho_\alpha[a(T_w x - x)] \le \lambda\left[\frac{1}{2\sigma(w)}\int_0^b\int_0^b K_w(t)\varphi^\alpha(s,2\sigma a|x(s+t) -$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$- x(s)|)\,dt\,ds + \frac{1}{2}\int_0^b \varphi^\alpha(s,2a|(\sigma(w)-1)x(s)|)\,ds\Big] =$$

$$= \lambda\left\{\frac{1}{2\sigma(w)}\int_0^b K_w(t)\int_0^b \varphi^\alpha[s,2\sigma a|x(s+t)-x(s)|]\,ds\,dt + \right.$$

$$\left. + \frac{1}{2}\rho_\alpha(2a(\sigma(w)-1)x)\right\}.$$

Let $0 < \delta < \frac{b}{2}$, then

$$\int_0^\delta K_w(t)\int_0^b \varphi^\alpha[s,2\sigma a|x(s+t)-x(s)|]\,ds\,dt \le$$

$$\le \int_0^\delta K_w(t)\,\omega_{\tau,\alpha}(2\sigma ax,\delta)\,dt \le \sigma(w)\omega_{\tau,\alpha}(2\sigma ax,\delta).$$

$$\int_{b-\delta}^\delta K_w(t)\int_0^b \varphi^\alpha[s,2\sigma a|x(s+t)-x(s)|]\,ds\,dt \le$$

$$\le \int_0^\delta K_w(b-u)\omega_{\tau,\alpha}(2\sigma ax,\delta)\,du = \sigma(w)\omega_{\tau,\alpha}(2\sigma ax,\delta).$$

Since ψ is convex and by Remark 2 φ^α is τ-bounded, we get by the condition (A) and Proposition 1 ([2] p. 103) the following

$$\int_\delta^{b-\delta} K_w(t)\int_0^b \varphi^\alpha[s,2\sigma a|x(s+t)-x(s)|]\,ds\,dt \le$$

$$\le \lambda\int_\delta^{b-\delta} K_w(t)\int_0^b \psi(s,2\sigma a|x(s+t)-x(s)|]\,ds\,dt \le$$

$$\le \frac{\lambda}{2}\int_\delta^{b-\delta} K_w(t)\int_0^b [\varphi^\alpha[s,4\sigma a|x(s+t)|) + \varphi^\alpha(s,4\sigma a|x(s)|)]\,ds\,dt =$$

$$= \frac{\lambda}{2}\int_\delta^{b-\delta} K_w(t)\,[\rho_\alpha(4\sigma a\tau_t x) + \rho_\alpha(4\sigma ax)]dt \le$$

$$\le \frac{\lambda}{2}\int_\delta^{b-\delta} K_w(t)\,[k_1\rho_\alpha(k_2 4\sigma ax) + h(t) + \rho_\alpha(4\sigma ax)]\,dt \le$$

$$\le \frac{\lambda}{2}\,[2k_3\rho_\alpha(4k\sigma ax) + H]\int_\delta^{b-\delta} K_w(t)\,dt,$$

where $k_3 = \max(k_1,1)$, $k = \max(k_2,1)$. From this

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\rho_\alpha(a(T_w x - x)) \le \lambda\Big\{2\omega_{\tau,\alpha}(2\sigma a x,\delta) + \frac{\lambda}{2}\,\frac{2k_3\rho_\alpha(4k\sigma a x)+H}{\sigma(w)}$$

$$\int_\delta^{b-\delta} K_w(t) + \frac{1}{2}\,\rho_\alpha(2a(\sigma(w) - 1)x)\Big\}.$$

By Lemma 1, taking $\delta > 0$ sufficiently small, we may make the first term on the right-hand side of the above inequality smaller than $\varepsilon/3$. By singularity of K_w, the second term may be made less than $\varepsilon/3$ taking $w \in W_1$, with an appropriate $W_1 \in \mathcal{W}$. Since $\sigma(w) \xrightarrow{\mathcal{W}} 1$ the third term becomes less than $\varepsilon/3$ for $w \in W_2$ with some $W_2 \in \mathcal{W}$. From this $\rho_\alpha(a(T_w x - x)) < \varepsilon$ for $w \in W_1 \cap W_2$. Then, by Hölder inequality

$$\rho(a(T_w x - x)) \le b^{1-1/\alpha}\{\rho_\alpha[a(T_w x - x)]\}^{1/\alpha} \xrightarrow{\mathcal{W}} 0.$$

For $0 < \alpha < 1$ we can prove

Theorem 2. Let φ be a τ - bounded φ - function with parameter satisfying the condition (A) for $0 < \alpha < 1$ and let $\int_0^b \varphi(t,c)\,dt < \infty$ for every $c > 0$. Let $(K_w)_{w\in W}$ be singular. Then the operators T_w defined by (5) satisfy the condition $\rho(a(T_w x - x)) \xrightarrow{\mathcal{W}} 0$ for every $x \in L^\varphi$ and for some $a > 0$.

Proof. By Lemma 4 $T_w x \in L^\varphi$ for $x \in L^\varphi$. Let $x \in L^\varphi$ then by Lemma 2 $\omega_{\tau,1}(cx,\delta) \to 0$ as $\delta \to 0^+$ for sufficiently small $c > 0$. Now, let us choose $a > 0$ so small that $2a\sigma \le c$ and $\rho(4\sigma k x) < \infty$ where $k = \max(k_2, 1)$ and k_2 is equal to the constant from the definition of τ - boundedness of φ.

Similarly as in Theorem 1 we obtain

$$\varphi\Big(s, \frac{1}{\sigma(w)}\Big|\int_0^b K_w(t)\sigma(w)ax(s+t)\,dt - ax(s)\Big|\Big) \le$$

$$\le \lambda^{1/\alpha}\psi^{1/\alpha}\Big(s, \frac{1}{\sigma(w)}\Big|\int_0^b K_w(t)\sigma(w)ax(s+t)\,dt - ax(s)\Big|\Big) \le$$

$$\le \lambda^{1/\alpha}\Big\{\frac{1}{2\sigma(w)}\int_0^b K_w(t)\varphi(s, 2\sigma a|x(s+t) - x(s)|)dt +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ \frac{1}{2}\,\varphi[s,2a|(\sigma(w) - 1)x(s)|]\Big\}$$

and

$$\rho(a(T_w x - x)) \le \lambda^{1/\alpha}\Big\{2\omega_{\tau,1}(2\sigma a x,\delta) +$$

$$+ \frac{\lambda^{1/\alpha}}{2}\,\frac{2k_3\rho\,(4k\sigma a x)+H}{\sigma(w)}\int_{\delta}^{b-\delta} K_w(t)\,dt + \frac{1}{2}\,\rho(2a(\sigma(w) - 1)x)\Big\}.$$

From this by Lemma 2 and singularity of K_w $\rho(a(T_w x-x)) \xrightarrow{w} 0$.

REFERENCES

[1] E. Beckenbach, R. Bellman: The inequalities, Moscow 1965, p. 67 (in Russian).

[2] J. Musielak: Modular approximation by the filtered family of linear operators, Functional Analysis and Approximation, Internat. Confer. Oberwolfach, 1981, Birkhäuser Verlag Basel, p. 99-110.

[3] J. Musielak: Orlicz spaces and modular spaces. Lecture Notes in Mathematics 1034. Springer-Verlag 1983.

INSTITUTE OF MATHEMATICS, SILESIAN TECHNICAL UNIVERSITY,
44-100 GLIWICE, POLAND

Received January 26, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Roman Taberski

PARTIAL EXPONENTIAL APPROXIMATION

1. Preliminaries

Introduce the (closed) intervals $I_n := \langle -n,n \rangle$, $n \in N$, $J := \langle \gamma,\eta \rangle$ with fixed $\gamma,\eta \in R$, and the Cartesian product $\Omega := R \times J$.

Let $\varphi \equiv \varphi(\cdot)$ and $f \equiv f(\cdot,\cdot)$ be the (Lebesgue) measurable functions of one and two variables, defined almost everywhere on R and Ω, respectively. Write, for any positive number p,

$$\|\varphi(\cdot)\|_p := \left\{ \int_{-\infty}^{+\infty} |\varphi(x)|^p dx \right\}^{1/p},$$

$$\|f\|_p := \left\{ \int\int_{\Omega} |f(x,y)|^p dx\, dy \right\}^{1/p};$$

both these Lebesgue integrals may be finite or not. In case $|f|_p < \infty$, the function f is said to be L^p-integrable over Ω, in symbols $f(\cdot,\cdot) \in L^p$ or $f \in L^p(\Omega)$.

Denote by $\varphi'(x)$, $\varphi''(x)$, ... ,$\varphi^{(r)}(x)$ the successive derivatives of the real- or complex-valued function $\varphi(\cdot)$. Moreover, let $f^{r,0}(x,y)$ mean the r-th partial derivative $\frac{\partial^r}{\partial x^r} f(x,y)$.

Introduce the class AC_{loc} consisting of all (complex-valued) functions $\varphi(\cdot)$ absolutely continuous on each finite interval I_n $(n = 1,2,\ldots)$. Denote by B_σ the Bernstein class of all entire functions $\Phi(\cdot)$ of exponential type σ, bounded on R. Denote by L^p_σ the set of (complex-valued) functions $F(\cdot,\cdot)$ L^p - integrable over Ω, with diagrams $F(\cdot,v) \in B_\sigma$ for a.e. $v \in J$.

Consider a (complex-valued) function $f = f(\cdot,\cdot)$ with

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

values $f(x,y)$ defined for every $x \in R$ and almost every $y \in J$. Suppose that $f(\cdot,y)$ has the finite (Jordan) variation on I_n, signified by

$$V(f(\cdot,y);I_n) =: \psi_n(y),$$

for every $n \in N$ and a.e. $y \in J$. If all $\psi_n(\cdot)$ are (Lebesgue) measurable, the (Jordan) variation

$$V(f(\cdot,y);R) := \lim_{n\to\infty} \psi_n(y)$$

determines a non-negative function of y measurable on J, too. Hence, for any positive number p, the non-negative quantity

$$\|f\|_p^* := \left\{ \int_\gamma^\eta |V(f(\cdot,y);R)|^p \, dy \right\}^{1/p}$$

is either finite or infinite. In case $\|f\|_p^* < \infty$, we say that $f(\cdot,\cdot)$ belongs to the class BV_p^*.

Suppose now that almost all diagrams $f(\cdot,y)$ $(y \in J)$ are continuous on R, and that all diagrams $f(x,\cdot)$ $(x \in R)$ are measurable on J. Then the function $\psi(\cdot)$ defined by

$$\psi(y) := V(f(\cdot,y);R) \equiv \lim_{n\to\infty} V(f(\cdot,y);I_n)$$

is also measurable on J. Indeed, taking an arbitrary normal sequence of partitions

$$-n = x_0^{(\nu)} < x_1^{(\nu)} < \ldots x_{m_\nu}^{(\nu)} = n, \quad n \in N, \quad \nu = 1,2,\ldots,$$

we can write

$$\psi_n(y) \equiv V(f(\cdot,y);I_n) = \lim_{\nu\to\infty} \sum_{j=1}^{m_\nu} |f(x_j^{(\nu)},y) - f(x_{j-1}^{(\nu)},y)| \quad \text{for}$$

a.e. $y \in J$.

Therefore, by the well-known theorems, all functions $\psi_n(\cdot)$ $(n = 1,2,\ldots)$ and their limit $\psi(\cdot)$ are measurable on J. All these (complex-valued) functions $f(\cdot,\cdot)$ form a basic linear class CM.

Denote by $\overline{BV}_p^*$ $(1 \le p \le \infty)$ the intersection $CM \cap BV_p^*$. In this linear class of functions $f(\cdot,\cdot)$, the non-negative functional $\|f\|_p^*$ can be treated as a certain seminorm. Let A_σ^p be the set of functions $F(\cdot,\cdot) \in \overline{BV}_p^*$ such that $F(\cdot,v) \in B_\sigma$ for a.e. $v \in J$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

If $f(\cdot,\cdot) \in L^p$, $1 \le p < \infty$, the non-negative number

$$A^{\cdot}_{\sigma}(f)_p := \inf_{S \in L^p_{\sigma}} \|f - S\|_p, \quad \sigma > 0,$$

is called the best partial exponential approximation of f of order σ in L^p- norm. Assuming that $f(\cdot,\cdot) \in \overline{BV}^*_p$, $1 \le p < \infty$, we can also introduce the related quantity

$$\nabla^*_{\sigma}(f)_p := \inf_{T \in \Lambda^p_{\sigma}} \|f - T\|^*_p, \quad \sigma > 0.$$

It may be named the best partial exponential approximation of f in $\overline{BV}^*_p$ - seminorm.

The partial integral modulus of continuity of a function f belonging to the class $L^p(\Omega)$, $p \ge 1$, with respect to the first variable, is given by

$$\omega^{\cdot}(\delta;f)_p := \sup_{h \in <0,\delta>} \left\{ \int\int_{\Omega} |f(x+h,y) - f(x,y)|^p dx\, dy \right\}^{1/p}, \quad \delta \ge 0.$$

As in one-dimensional case, the following Jackson type estimate

(1) $$A^{\cdot}_{\sigma}(f)_p \le 3\omega^{\cdot}(\tfrac{1}{\sigma};f)_p, \quad 0 < \sigma < \infty,$$

can be easily deduced (see e.g. [1], Sect. 105).

Let $f(\cdot,\cdot)$ be a (complex-valued) function defined on the product $R \times E$, where $E \subset J$ and E is equivalent to J (i.e., meas $(J \setminus E) = 0$). Suppose that $f(\cdot,v)$ is bounded on every interval I_n, $n = 1,2,\ldots$, for every fixed $v \in E$. Write

$$I_{\delta}(x) := <x - \delta/2, x + \delta/2>, \quad x \in R, \quad 0 \le \delta < \infty.$$

Introduce the local partial modulus of continuity of $f(\cdot,\cdot)$ at $(x,y) \in R \times E$, defined by

$$M^{\cdot}_{\delta} f(x,y) := \sup_{t, t+h \in I_{\delta}(x)} |f(t+h,y) - f(t,y)|, \quad \delta \ge 0.$$

Assuming that the function $M^{\cdot}_{\delta} f(\cdot,\cdot)$ is (plane) measurable on Ω for every fixed $\delta > 0$, and choosing a number $p \ge 1$, we can also define the partial averaged modulus of continuity of $f(\cdot,\cdot)$ by

$$\tau^{\cdot}(\delta;f)_p := \|M^{\cdot}_{\delta} f\|_p, \quad \delta \ge 0.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Evidently, if δ denotes a positive number, $\tau^*(\delta;f)_p$ may be finite or infinite. Furthermore,

$$0 = \tau^*(0;f)_p \le \tau^*(\delta;f)_p \le \tau^*(\lambda;f)_p \text{ , if } \lambda > \delta,$$

and

$$\tau^*(n\delta;f)_p \le n\tau^*(\delta;f)_p \text{ , if } n \in N.$$

Consider next a (complex-valued) function $f(\cdot,\cdot)$ (plane) measurable on Ω, with values $f(x,y)$ defined and bounded in x (on R) for a.e. $y \in J$. Suppose that all diagrams $f(x,\cdot)$ are continuous on a (line) set Γ equivalent to J, and that $V(f(\cdot,y);R) < \infty$ a.e. on J. Then the last variation is measurable on J, and $M^*_\delta f(\cdot,\cdot)$ is (plane) measurable on Ω for each finite $\delta \ge 0$. Moreover,

$$\tau^*(\delta;f)_p \le \delta^{1/p}\|f\|^*_p \,, \tag{2}$$

whenever $1 \le p \le \infty$ and $0 \le \delta < \infty$ (see Sects. 2, 3 of [5]).

The aim of this paper is to obtain, for (real-valued) functions of classes L^p and $\overline{BV}^*_p$, the two-dimensional analogues of Theorems 2, 3 presented in [2]. It will be constructed here the suitable functions of classes L^p_σ and Λ^p_σ, with "good" approximation properties.

The symbols c_j, $c_k(r,\nu)$, $j,k \in N$, will mean positive absolute constants and positive numbers depending on the indicated parameters r, ν, only.

2. Auxiliary results

We begin with the Bernstein type inequalities.

Lemma 1. (i) Suppose that $G(\cdot,\cdot) \in L^p_\sigma$, $1 \le p < \infty$, $\sigma > 0$. Then $G^{1,0}(\cdot,\cdot) \in L^p_\sigma$ and $\|G^{1,0}\|_p \le \sigma\|G\|_p$.

(ii) The condition $G(\cdot,\cdot) \in \Lambda^p_\sigma$ $1 \le p < \infty$, $\sigma > 0$, ensures that $G^{1,0}(\cdot,\cdot) \in \Lambda^p_\sigma$ and $\|G^{1,0}\|^*_p \le \sigma\|G\|^*_p$.

Proof of (i). Write

$$u_k := \frac{2k+1}{2\sigma}\pi, \quad k = 0, \mp1, \mp2, \dots .$$

Then, for every $x \in R$ and a.e. $v \in J$, we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$G^{1,0}(x,v) = \frac{1}{\sigma}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{u_k^2}\,G(x+u_k,v). \tag{3}$$

Consequently, $G^{1,0}(\cdot,\cdot) \in L_\sigma^p$ and

$$\|G^{1,0}\|_p^p = \int_J \left\{ \int_R |G^{1,0}(x,v)|^p\,dx \right\} dv \le$$

$$\le \int_J \left\{\sigma^p \int_R |G(x,v)|^p\,dx\right\} dv = \sigma^p \|G\|_p^p ,$$

by the Fubini and Bernstein theorems.

Proof of (ii). From (3) it follows that $G^{1,0}(\cdot,\cdot) \in CM$ and

$$V(G^{1,0}(\cdot,v);R) \le \sigma V(G(\cdot,v);R) \quad \text{for a.e. } v \in J.$$

This, together with $G^{1,0}(\cdot,v) \in B_\sigma$ for a.e. $v \in J$, gives immediately our thesis.

Consider the complex-valued function $K_\sigma(\cdot)$ continuous on the complex plane, with values

$$K_\sigma(\zeta) := (\cos\sigma\zeta - \cos 2\sigma\zeta)/(\sigma\zeta^2),$$

σ being a positive fixed number. Clearly, $K_\sigma(\cdot) \in B_{2\sigma}$. Moreover,

$$\|K_\sigma(\cdot)\|_1 \le c_1\pi, \quad c_1 \le (4 + 2\log 3)/\pi \tag{4}$$

(see [1], Sect. 106).

Lemma 2. (i) Let $\varphi(\cdot,\cdot)$ be of class L^α, $1 \le \alpha < \infty$. Then the singular integral

$$W[\varphi](z,v) := \frac{1}{\pi}\int_{-\infty}^{+\infty} \varphi(t,v)K_\sigma(z-t)\,dt \quad (z = x + iy,\ x,y \in R) \tag{5}$$

determines a (complex-valued) function such that $W[\varphi](\cdot,v) \in B_{2\sigma}$ for a.e. $v \in J$ and $W[\varphi](x,\cdot)$ is measurable on J for every $x \in R$. Moreover, $W[\varphi](\cdot,\cdot) \in L^\alpha$ and

$$\|W[\varphi]\|_\alpha \le c_1\|\varphi\|_\alpha. \tag{6}$$

(ii) Under the additional assumption $\varphi(\cdot,\cdot) \in BV_p^*$, $1 \le p < \infty$, we have $W[\varphi](\cdot,\cdot) \in \overline{BV}_p^*$ and

$$\|W[\varphi]\|_p^* \le c_1\|\varphi\|_p^* . \tag{7}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof of (i). For a.e. $v \in J$, the integrals

$$\psi_n(z;v) := \int_{-n}^{n} \varphi(t,v) K_\sigma(z-t)\,dt, \quad n = 1,2,\ldots$$

define some entire functions of z. Moreover, by Hölder's inequality,

$$|\psi_n(z;v)| \le \left\{ \int_{-\infty}^{\infty} |\varphi(t,v)|^\alpha dt \right\}^{1/\alpha} \left\{ \int_{-\infty}^{\infty} |K_\sigma(z-t)|^\beta dt \right\}^{1/\beta}$$

with $1/\beta = 1 - 1/\alpha$. Writing

$$K_\sigma(z-t) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} K_\sigma^{(\nu)}(x-t)(iy)^\nu, \quad z = x + iy,$$

and applying the Bernstein and Minkowski inequalities, we obtain

$$|\psi_n(z;v)| \le \|\varphi(\cdot,v)\|_\alpha \|K_\sigma(\cdot)\|_\beta e^{2\sigma|y|}, \quad n = 1,2,\ldots . \tag{8}$$

Also, it is easy to see that the sequence $\{\psi_n(z;v)\}$ satisfies the Cauchy condition uniformly in z on each bounded domain D. Hence, for a.e. $v \in J$,

$$\frac{1}{\pi} \lim_{n\to\infty} \psi_n(z;v) = W[\varphi](z;v) \quad \text{uniformly in } z \in D.$$

In view of the well-known Weierstrass theorem and estimate (8), $W[\varphi](\cdot,v) \in B_{2\sigma}$.

By Fubini's theorem, for every fixed $x \in R$, the functions $\psi_n(x;\cdot)$, $n = 1,2,\ldots$, defined a.e. on J are (Lebesgue) integrable over this interval. Therefore, $W[\varphi](x,\cdot)$ as a limit of $\pi^{-1}\psi_n(x;\cdot)$ is measurable on J.

Analogously, we can show that the functions $\psi_n(\cdot;\cdot)$, $n = 1,2,\ldots$, defined a.e. on Ω, are (Lebesgue) integrable over all strips $I_m \times J$. Consequently, $W[\varphi](\cdot,\cdot)$ is measurable on Ω. Estimate (6) follows at once from the generalized Minkowski inequality and (4).

Proof of (ii). Since $W[\varphi](\cdot,\cdot) \in CM$, by (i), the variation $V(W[\varphi](\cdot,v);R)$ is measurable (in v) on J.

In view of (4),

$$V(W[\varphi](\cdot,v);R) \le c_1 V(\varphi(\cdot,v);R) \quad \text{for a.e. } v \in J.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This immediately implies (7) and $W[\varphi](\cdot,\cdot) \in BV_p^*$.

Lemma 3. Let $f(\cdot,\cdot)$ be of class $\overline{BV}_p^*$, $1 \le p < \infty$, and let

(9) $V_\sigma^*(f)_p = 0$ for some finite $\sigma > 0$.

Then, for every $x \in R$ and almost every $v \in J$, we have

(10) $f(x,v) = F(x,v)$,

where $F(\cdot,v) \in B_\sigma$ for a.e. $v \in J$.

Proof. Condition (9) implies the equality

$$\inf_{T \in \Lambda_\sigma^p} V(f(\cdot,v) - T(\cdot,v);R) = 0 \quad \text{for a.e. } v \in J.$$

Hence, there exist functions $T_\nu(\cdot,\cdot) \in \Lambda_\sigma^p$ such that

$$V(f(\cdot,v) - T_\nu(\cdot,v);R) < \frac{1}{\nu}, \quad \nu = 1,2,\dots,$$

for every $v \in \tilde{J}$, where $\tilde{J}$ is equivalent to J. Consequently,

$$|f(u,v) - T_\nu(u,v) - f(0,v) + T_\nu(0,v)| < \frac{1}{\nu}, \quad \nu = 1,2,\dots,$$

uniformly in $u \in R$. Clearly, it may be assumed that $f(0,v) = T_\nu(0,v)$. Then, for all $(u,v) \in R \times \tilde{J}$,

(11) $$|f(u,v) - T_\nu(u,v)| < \frac{1}{\nu}, \quad \nu = 1,2,\dots.$$

In particular, $|T_\nu(u,v)| < M(v)$, $u \in R$, $\nu = 1,2,\dots$, where $M(v) = 1 + \sup_{s \in R} |f(s,v)| < \infty$, $v \in \tilde{J}$.

Given any complex number $z = x + iy$, we have

$$T_\nu(z,v) = \sum_{k=0}^{\infty} \frac{T_\nu^{k,0}(x,v)}{k!}(iy)^k, \quad v \in \tilde{J}, \quad \nu = 1,2,\dots;$$

whence

(12) $$|T_\nu(z,v)| \le M(v) \sum_{k=0}^{\infty} \frac{\sigma^k |y|^k}{k!} = M(v) e^{\sigma|y|},$$

by Bernstein's inequality. Analogously, if $\mu,\nu \in N$,

$$|T_\nu(z,v) - T_\mu(z,v)| \le e^{\sigma|y|} \sup_{u \in R} |T_\nu(u,v) - T_\mu(u,v)|.$$

Therefore, for any fixed $v \in \tilde{J}$, the sequence $\{T_\nu(z,v)\}$ converges uniformly on every bounded domain of the complex plane. In view of the Weierstrass theorem,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(13) $F(z,v) := \lim_{\nu\to\infty} T_\nu(z,v)$

determines some entire function of z. Estimate (12) ensures that $T_\nu(\cdot,v) \in B_\sigma$, $\nu = 1,2,\dots$. Combining (11) and (13), we get identity (10) in which $F(\cdot,v) \in B_\sigma$ for every $v \in \tilde{J}$.

Proposition 1. Suppose that $g(\cdot,\cdot) \in L^p$, $1 \le p < \infty$, $g^{1,0}(\cdot,\cdot) \in L^p \cap BV_p^*$, $U(\cdot,\cdot) \in L^p_\sigma$, $0 < \sigma < \infty$, and

(14) $\|g - U\|_p \le c_2 \sigma^{-1-1/p} \|g^{1,0}\|_p^*$.

Then

(15) $\|g^{1,0} - U^{1,0}\|_p \le c_3 \sigma^{-1/p} \|g^{1,0}\|_p^*$.

Proof. Given arbitrary $\lambda > 1$ and $\varphi(\cdot,\cdot) \in L^p$, there is a function $S[\varphi](\cdot,\cdot) \in L^p_\sigma$ such that

(16) $\|\varphi - S[\varphi]\|_p \le \lambda A_\sigma^*(\varphi)_p$.

Let W be the linear operator defined by (5).

In view of Lemma 1 (i), $S^{1,0}[\varphi](\cdot,\cdot) \in L^p_\sigma$, $U^{1,0}(\cdot,\cdot) \in L^p_\sigma$. By Lemma 2, $W[g](\cdot,\cdot)$ and $W[g^{1,0}](\cdot,\cdot)$ belong to L^p. Applying the Minkowski inequality, we obtain

$$\|g^{1,0} - U^{1,0}\|_p \le \|g^{1,0} - W[g^{1,0}]\|_p + \|S^{1,0}[W[g]] - U^{1,0}\|_p + \|W[g^{1,0}] - S^{1,0}[W[g]]\|_p =: N_1 + N_2 + N_3.$$

In case of $\varphi(\cdot,\cdot) \in L^p_\sigma$, we have

(17) $W[\varphi](x,y) = \varphi(x,v)$ for a.e. $v \in J$, $x \in R$

(see [1], Sect. 106). Hence

$$N_1 \le \|g^{1,0} - S[g^{1,0}]\|_p + \|W[S[g^{1,0}] - g^{1,0}]\|_p \le (1 + c_1)\lambda A_\sigma^*(g^{1,0})_p,$$

by (16) and Lemma 2. Applying (1) and the estimate of Marcinkiewicz type (see Sect. 3 of [5]), we conclude that

$$N_1 \le (1 + c_1)\lambda \cdot 3\omega^*(\tfrac{1}{\sigma}; g^{1,0})_p \le 3(1 + c_1)\lambda\sigma^{-1/p}\|g^{1,0}\|_p^*.$$

Further, Lemma 1 together with (16) and (17) gives

$$N_2 \le \sigma\|S[W[g]] - U\|_p \le \sigma\{\|S[W[g]] - W[g]\|_p +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ \|W[g] - U\|_p\} \le \sigma\{\lambda A^*_\sigma(W[g])_p + \|W[g] - U\|_p\} \le$$

$$\le \sigma(\lambda + 1)\|W[g - U]\|_p .$$

Consequently (see Lemma 2 and (14)), we get

$$N_2 \le \sigma(\lambda + 1)c_1\|g - U\|_p \le (\lambda + 1)c_1c_2\sigma^{-1/p}\|g^{1,0}\|^*_p .$$

Analogously, since $W[g^{1,0}](x,y) = W^{1,0}[g](x,y)$ for every $x \in R$ and a.e. $y \in J$, we have

$$N_3 \le 2\sigma\|W[g] - S[W[g]]\|_p \le 2\sigma\lambda A^*_\sigma(W[g])_p \le$$

$$\le 2\sigma\lambda c_1\|g - U\|_p \le 2\lambda c_1c_2\sigma^{-1/p}\|g^{1,0}\|^*_p .$$

Thus,

$$\|g^{1,0} - U^{1,0}\|_p \le \{3(1 + c_1)\lambda\sigma^{-1/p} +$$

$$+(\lambda + 1)c_1c_2\sigma^{-1/p} + 2\lambda c_1c_2\sigma^{-1/p}\}\|g^{1,0}\|^*_p ,$$

and estimate (15) with $c_3 = 3(1 + c_1) + 4c_1c_2$ is proved.

Proposition 2. Suppose that $g(\cdot,\cdot) \in L^\alpha \cap \overline{BV}^*_p$, $1 \le \alpha, p < \infty$, $g^{1,0}(\cdot,\cdot) \in L^\beta \cap \overline{BV}^*_p$, $1 \le \beta < \infty$, $\tilde{U}(\cdot,\cdot) \in \Lambda^p_\sigma$, $0 < \sigma < \infty$ and

(18) $$\|g - \tilde{U}\|^*_p \le c_4\sigma^{-1}\|g^{1,0}\|^*_p ,$$

then

(19) $$\|g^{1,0} - \tilde{U}^{1,0}\|^*_p \le c_5\|g^{1,0}\|^*_p .$$

Proof. Let λ be an arbitrary fixed number from (1,2), and let $\varphi(\cdot,\cdot)$ belong to $\overline{BV}^*_p$. Choose a function $T[\varphi](\cdot,\cdot) \in \Lambda^p_\sigma$ such that $\|\varphi - T[\varphi]\|^*_p \le \lambda\nabla^*_\sigma(\varphi)_p$ (see Lemma 3). Consider the Akhiezer and Levitan operator (5). In view of Lemma 1(ii), $T^{1,0}[\varphi](\cdot,\cdot) \in \Lambda^p_\sigma$ and $\tilde{U}^{1,0}(\cdot,\cdot) \in \Lambda^p_\sigma$.

By Lemma 2, $W[g](\cdot,\cdot) \in L^\alpha \cap \overline{BV}^*_p$ and $W[g^{1,0}](\cdot,\cdot) \in L^\beta \cap \overline{BV}^*_p$.

Further,

$$\|g^{1,0} - \tilde{U}^{1,0}\|^*_p \le \|g^{1,0} - W[g^{1,0}]\|^*_p + \|T^{1,0}[W[g]] - \tilde{U}^{1,0}\|^*_p +$$

$$+ \|W[g^{1,0}] - T^{1,0}[W[g]]\|^*_p =: N^*_1 + N^*_2 + N^*_3$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and $N_1^* \le \|g^{1,0}\|_p^* + \|W[g^{1,0}]\|_p^* \le (1 + c_1)\|g^{1,0}\|_p^*$.

Lemma 1(ii), Lemma 2(ii) and (18) yield

$$N_2^* \le \sigma\|T[W[g]] - \tilde{U}\|_p^* \le \sigma\{ \|T[W[g]] - W[g]\|_p^* + \|W[g] - \tilde{U}\|_p^* \} \le \sigma\{ \lambda\nabla_\sigma^*(W[g])_p + \|W[g] - \tilde{U}\|_p^* \} \le$$
$$\le \sigma(\lambda + 1)\|W[g - \tilde{U}]\|_p^* \le \sigma(\lambda + 1)c_1\|g - \tilde{U}\|_p^* \le$$
$$\le (\lambda + 1)c_1c_4\|g^{1,0}\|_p^* .$$

Analogously,

$$N_3^* \le 2\sigma\|W[g] - T[W[g]]\|_p^* \le 2\sigma\lambda\nabla_\sigma^*(W[g])_p \le$$
$$\le 2\sigma\lambda\|W[g] - \tilde{U}\|_p^* \le 2\sigma\lambda c_1\|g - \tilde{U}\|_p^* \le 2\lambda c_1c_4\|g^{1,0}\|_p^* .$$

Hence, we get estimate (19) with $c_5 := 1 + c_1 + 4c_1c_4$.

Given a positive number c and a positive integer r, let $\rho(\cdot)$ be an even real-valued function from $C^3(R)$, satisfying the conditions:

1° $\rho(0) = \rho'(0) = \rho''(0) = 0$,

2° $\rho''(t) = o(t^{r+1})$ and $\rho'''(t) = O(t^r)$, as $t \to 0+$,

3° $\rho(t) = 1$ for all $t \ge c$.

Introduce the Bernoulli type functions $\Phi_m(\cdot)$, $m = 1,2,\ldots,r$, defined by

$$\Phi_m(u) := \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \frac{\rho(t)}{(it)^m} e^{itu} dt \quad \text{for all } u \in R. \tag{20}$$

As is well-known, all these $\Phi_m(\cdot)$ are real-valued, bounded and Lebesgue-integrable on R. In case of $m \ge 2$, we have

$$\Phi_m(x) = \int_{-\infty}^{x} \Phi_{m-1}(s)ds, \quad -\infty < x < \infty$$

(see e.g. [1], Sect. 101).

Proposition 3. If $\sigma \ge c$, then there exist entire functions $P_r(\cdot)$, $Q_r(\cdot) \in B_\sigma$ such that

$$P_r(u) \ge \Phi_r(u), \quad Q_r(u) \le \Phi_r(u) \quad \text{for all } u \in R \tag{21}$$

and

$$\|P_r^{(\nu)}(\cdot) - \Phi_r^{(\nu)}(\cdot)\|_1 + \|\Phi_r^{(\nu)}(\cdot) - Q_r^{(\nu)}(\cdot)\|_1 \le \frac{c_6(r,\nu)}{\sigma^{r-\nu}} \tag{22}$$

for $\nu = 0,1,\ldots,r$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The proof can be found in [4], Sect. 2 (for the special case $r = 1$ see also [3], Sect. 2).

3. Approximation in the norms and seminorms

Now we give our main results.

Theorem 1. Let $f(\cdot,\cdot)$ be a real-valued function of class L^p $(1 \le p < \infty)$, defined on the product $R \times E$, where E is a (line) set equivalent to J $(E \subset J)$. Suppose, further, that $f(\cdot,v)$ is bounded on R for every fixed $v \in E$, and that all diagrams $f(u,\cdot)$, $u \in R$, are continuous on a certain (line) set Γ equivalent to E. Then, to any number $\sigma > 0$ there exists a function $G_\sigma(\cdot,\cdot) \in L^p_\sigma$ with measurable diagrams $G_\sigma(x,\cdot)$ such that

$$G_\sigma(x,v) \ge f(x,v) \quad \text{for all } (x,v) \in R \times E, \tag{23}$$

$$\|G_\sigma - f\|_p \le c_7 \tau^*(\sigma^{-1};f)_p. \tag{24}$$

If, in addition, $f(\cdot,\cdot) \in BV^*_p$, then

$$\|G_\sigma - f\|_p \le c_8 \sigma^{-1/p} \|f\|^*_p. \tag{25}$$

Moreover, in the special case $f(\cdot,\cdot) \in \overline{BV}^*_p$, $G_\sigma(\cdot,\cdot) \in \Lambda^p_\sigma$ and

$$\|G_\sigma - f\|^*_p \le c_9 \|f\|^*_p. \tag{26}$$

Proof. Given any $\sigma > 0$, let us put

$$x_j := \frac{\pi}{\sigma} j, \quad \xi_j := x_j - \frac{\pi}{2\sigma}, \quad j = 0,\pm1,\pm2,\dots .$$

Consider the function $S(\cdot,\cdot)$ defined on $R \times E$ by the formulae

$$S(\xi_k,v) := \sup_{u \in \langle x_{k-1}, x_k \rangle} f(u,v), \quad k = 0,\pm1,\pm2,\dots,$$

and

$$S(x_k,v) := \max\{ S(\xi_k,v), S(\xi_{k+1},v) \},$$

$$S(x,v) := \begin{cases} S(\xi_k,v) + \dfrac{S(x_k,v) - S(\xi_k,v)}{x_k - \xi_k}(x - \xi_k), \\ \qquad \text{if } x \in (\xi_k, x_k), \\ S(\xi_{k+1},v) + \dfrac{S(\xi_{k+1},v) - S(x_k,v)}{\xi_{k+1} - x_k}(x - \xi_{k+1}), \\ \qquad \text{if } x \in (x_k, \xi_{k+1}). \end{cases} \tag{27}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Clearly, the functions $S(\cdot,v)$, $v \in E$ determine some bounded polygonal lines. Therefore they are of class AC_{loc}. Moreover,

(28) $\quad S(x,y) \geq f(x,y)$ for all $(x,y) \in R \times E$.

We first prove that every diagram $S(\xi_k,\cdot)$ is measurable on J. Let us introduce the set

$$H_\lambda := \{ v: v \in E,\ S(\xi_k,v) \leq \lambda \} =$$
$$= \{ v: v \in E,\ f(u,v) \leq \lambda \text{ for all } u \in \langle x_{k-1}, x_k \rangle \},$$

where λ is an arbitrary fixed (real) number.

Suppose that v_0 denotes an accumulation point of H_λ. Consider a sequence of points $v_\nu \in H_\lambda$ $(\nu = 1,2,\ldots)$, tending to v_0. Then $f(u,v_\nu) \leq \lambda$ for all $u \in \langle x_{k-1}, x_k \rangle$ $(\nu = 1,2,\ldots)$ and, in case of $v_0 \in \Gamma$ the last inequalities lead to

$$f(u,v_0) \leq \lambda \text{ for all } u \in \langle x_{k-1}, x_k \rangle,$$

i.e., $v_0 \in H_\lambda$. Consequently, H_λ is equivalent to a sum of closed sets lying on J. Thus $S(\xi_k,\cdot)$ is (line) measurable, and whence $S(\cdot,\cdot) \in CM$.

Applying now formula (27) we conclude that the function $S(\cdot,\cdot)$ is (plane) measurable on Ω.

In view of (27), for every $v \in E$,

$$(29) \qquad S^{1,0}(x,v) := \begin{cases} \dfrac{S(x_k,v) - S(\xi_k,v)}{x_k - \xi_k}, & \text{if } x \in (\xi_k, x_k), \\ \dfrac{S(\xi_{k+1},v) - S(x_k,v)}{\xi_{k+1} - x_k}, & \text{if } x \in (x_k, \xi_{k+1}). \end{cases}$$

Therefore, $S^{1,0}(\cdot,\cdot)$ is (plane) measurable on Ω. Moreover,

$$|S^{1,0}(x,v)| \leq \frac{2\sigma}{\pi} M^*_\delta f(x,v),$$

where $\delta = 3\pi\sigma^{-1}$. Consequently,

$$(30) \qquad \|S^{1,0}\|_p \leq \frac{2\sigma}{\pi} \tau^*(\delta;f)_p.$$

If $\xi_k \leq x < x_k$ or $x_k \leq x < \xi_{k+1}$ and $v \in E$, then

$$0 \leq S(x,v) - f(x,v) \leq S(x_k,v) - f(x,v) =$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= \max \{ S(\xi_k,v) - f(x,v), S(\xi_{k+1},v) - f(x,v) \} \le$$
$$\le \sup_{s,t \in \langle x_{k-1}, x_{k+1} \rangle} |f(s,v) - f(t,v)| \le M^*_\delta f(x,v)$$

with δ as in (30). This immediately implies the estimate

$$\|S - f\|_p \le \tau^*(\delta;f)_p. \tag{31}$$

Evidently, we may further suppose that $\tau^*(\delta;f)_p < \infty$.

According to the well-known representation formula ([1], Sect. 101), for every real x and every $v \in E$, we get

$$S(x,v) = F_c(x,v) + \int_{-\infty}^{\infty} S^{1,0}(t,v)\Phi_1(x-t)dt, \quad 0 < c \le \sigma,$$

where $\Phi_1(u)$ is defined by (20) (case $m = r = 1$), $F_c(\cdot,v) \in B_c$ and $F_c(x,\cdot)$ is measurable on J (by the generalized Minkowski inequality).

Putting $h^{\pm}(t,v) := \frac{1}{2}\left\{|S^{1,0}(t,v)| \pm S^{1,0}(t,v)\right\}$, we obtain

$$S(x,v) = F_c(x,v) + \int_{-\infty}^{\infty} h^+(t,v)\Phi_1(x-t)dt - \int_{-\infty}^{\infty} h^-(t,v)\Phi_1(x-t)dt$$

for all $(x,v) \in R \times E$.

Consider the entire functions $P_1(\cdot)$, $Q_1(\cdot)$ satisfying conditions (21), (22). Write, for complex numbers $z = x + iy$ and $v \in E$,

$$G_\sigma(z,v) := F_c(z,v) + \int_{-\infty}^{\infty} h^+(t,v)P_1(z-t)dt - \int_{-\infty}^{\infty} h^-(t,v)Q_1(z-t)dt.$$

It is easily seen that $G_\sigma(\cdot,v) \in B_\sigma$, $G_\sigma(x,\cdot)$ is (line) measurable for every $x \in R$ and $G_\sigma(\cdot,\cdot) \in L^p$ (by the generalized Minkowski inequality). Moreover, when $(x,v) \in R \times E$, we have

$$G_\sigma(x,v) - S(x,v) = \int_{-\infty}^{\infty} h^+(t,v)\{P_1(x-t) - \Phi_1(x-t)\}dt +$$
$$+ \int_{-\infty}^{\infty} h^-(t,v)\{\Phi_1(x-t) - Q_1(x-t)\}dt.$$

These facts and (28) give the initial assertions together with estimate (23). In view of Minkowski's inequalities,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|G_\sigma - S\|_p \le \int_R \Big\{ \int_\Omega \int |h^+(x-u,v)(P_1(u) - \Phi_1(u))|^p dx\, dv \Big\}^{1/p} du +$$

$$+ \int_R \Big\{ \int_\Omega \int |h^-(x-u,v)(\Phi_1(u) - Q_1(u))|^p dx\, dv \Big\}^{1/p} du \le$$

$$\le \|S^{1,0}\|_p \Big\{ \|P_1(\cdot) - \Phi_1(\cdot)\|_1 + \|\Phi_1(\cdot) - Q_1(\cdot)\|_1 \Big\}.$$

Hence, by (30) and (22), we get

$$(32) \qquad \|G_\sigma - S\|_p \le \frac{2}{\pi} c_6(1,0)\tau^*(\delta;f)_p , \quad 0 < c \le \sigma.$$

Observing that

$$\|G_\sigma - f\|_p \le \|G_\sigma - S\|_p + \|S - f\|_p$$

and applying (31) and (32) we get at once estimate (24).

Under the additional assumption $f \in BV_p^*$, $1 \le p < \infty$, $\tau^*(3\pi\sigma^{-1};f)_p \le 10\tau^*(\sigma^{-1};f)_p \le 10\sigma^{-1/p}\|f\|_p^*$, $0 < \sigma < \infty$ (see (2)). This, together with (24), leads to (25).

In particular, if $f \in \overline{BV}_p^*$, then

$$V(S(\cdot,v);R) = \sum_{k=-\infty}^{\infty} |S(\xi_{k+1},v) - S(\xi_k,v)| \le 2V(f(\cdot,v);R), \quad v \in E.$$

Hence $S(\cdot,\cdot) \in \overline{BV}_p^*$ and $\|S\|_p^* \le 2\|f\|_p^* < \infty$.

In view of (29), for every $v \in E$, we have

$$V(S^{1,0}(\cdot,v);R) = \frac{2\sigma}{\pi} \sum_{k=-\infty}^{\infty} |S(\xi_{k+1},v) - S(\xi_k,v)| \le \frac{4\sigma}{\pi} (S(\cdot,v);R).$$

Consequently, $S^{1,0}(\cdot,\cdot) \in BV_p^*$ and

$$\|S^{1,0}\|_p^* \le \frac{4\sigma}{\pi}\|S\|_p^* \le \frac{8\sigma}{\pi}\|f\|_p^* .$$

Further, $G_\sigma(\cdot,\cdot) - S(\cdot,\cdot) \in CM$ and, by (22),

$$\|G_\sigma - S\|_p^* \le c_6(1,0)\frac{1}{\sigma}\|S^{1,0}\|_p^* \le \frac{8}{\pi}c_6(1,0)\|f\|_p^* .$$

Thus, $G_\sigma(\cdot,\cdot) \in \overline{BV}_p^*$. Observing that

$$\|G_\sigma - f\|_p^* \le \|G_\sigma - S\|_p^* + \|S\|_p^* + \|f\|_p^*$$

and collecting the results, we obtain inequality (26).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 2. Let $f(\cdot,\cdot)$ be a real-valued function of class $L^p \cap \overline{BV}_p^*$, $1 \le p < \infty$, having the partial derivative $f^{r,0}(u,v)$, $r \in N$, at every point $(u,v) \in R \times E_r$, where E_r is a (line) set equivalent to $J(E_r \subset J)$. Suppose that $f^{r,0}(\cdot,v)$ is continuous and bounded on R for every $v \in E_r$, all diagrams $f^{r,0}(u,\cdot)$, $u \in R$, are continuous on a (line) set Γ_r equivalent to E_r and $f^{r,0}(\cdot,\cdot) \in L^p \cap BV_p^*$. Then $f^{\mu,0}(\cdot,v) \in AC_{loc}$ for every $v \in E_r$ and $f^{\mu,0}(\cdot,\cdot) \in L^p \cap \overline{BV}_p^*$, if $\mu = 0,1,\ldots,r-1$. Moreover, to any positive number σ, there is a function $T_\sigma(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$ such that

(33) $T_\sigma(x,y) \ge f(x,v)$ for all $(x,v) \in R \times E_r$,

(34) $$\|T_\sigma^{\nu,0} - f^{\nu,0}\|_p \le \frac{c_{10}(r,\nu)}{\sigma^{r-\nu+1/p}} \|f^{r,0}\|_p^*,$$

(35) $$\|T_\sigma^{\nu,0} - f^{\nu,0}\|_p^* \le \frac{c_{11}(r,\nu)}{\sigma^{r-\nu}} \|f^{r,0}\|_p^*,$$

for each non-negative integer $\nu \le r$.

Proof. Clearly, the continuity of $f^{r,0}(\cdot,v)$, $v \in E_r$, implies $f^{\mu,0}(\cdot,v) \in AC_{loc}$ for every non-negative integer $\mu \le r-1$. Further, if $(x,v) \in R \times E_r$, then

(36) $$f(x,v) = F_{c,r}(x,v) + \int_{-\infty}^{\infty} f^{r,0}(t,v)\Phi_r(x-t)\,dt, \quad 0<c\le\sigma<\infty,$$

where $\Phi_r(u)$ is defined by (20) and $F_{c,r}(\cdot,v) \in B_\sigma$ for every $v \in E_r$. Moreover, $F_{c,r}(\cdot,\cdot) \in L^p \cap \overline{BV}_p^*$ (by the generalized Minkowski inequality) and

$$f^{\mu,0}(x,v) = F_{c,r}^{\mu,0}(x,v) + \int_{-\infty}^{\infty} f^{r,0}(t,v)\Phi_r^{(\mu)}(x-t)\,dt, \quad (x,v) \in R\times E_r,$$

whenever $1 \le \mu \le r-1$. Thus, in this case, $F_{c,r}^{\mu,0}(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$ (by Lemma 1) and $f^{\mu,0}(\cdot,\cdot) - F_{c,r}^{\mu,0}(\cdot,\cdot) \in L^p \cap \overline{BV}_p^*$ (by the generalized Minkowski inequality). Consequently, $f^{\mu,0}(\cdot,\cdot) \in L^p \cap \overline{BV}_p^*$ $(\mu = 0,1,\ldots,r-1)$. Also, in view of the assumptions, $f^{r,0}(\cdot,\cdot) \in L^p \cap \overline{BV}_p^*$.

By Theorem 1, there exists a function $G_{\sigma,r}(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$, with real values on $R \times E_r$, satisfying the conditions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|G_{\sigma,r} - f^{r,0}\|_p \le c_8 \sigma^{-1/p} \|f^{r,0}\|_p^* , \tag{37}$$

$$\|G_{\sigma,r} - f^{r,0}\|_p^* \le c_9 \|f^{r,0}\|_p^* . \tag{38}$$

In view of (36), for all $(x,v) \in R \times E_r$, we get

$$f(x,v) = F_{c,r}(x,v) + J_\sigma(x,v) +$$

$$+ \int_{-\infty}^{\infty} \left\{ f^{r,0}(t,v) - G_{\sigma,r}(t,v) \right\} \Phi_r(x-t)\,dt,$$

where

$$J_\sigma(z,v) := \int_{-\infty}^{\infty} \Phi_r(u) G_{\sigma,r}(z-u,v)\,du, \quad z = x + iy, \ y \in R.$$

It is easily seen that $J_\sigma^{\mu,0}(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$ for each non-negative integer μ (see Lemma 1).

Introduce the auxiliary function g with values

$$g(x,v) := f(x,v) - F_{c,r}(x,v) - J_\sigma(x,v) =$$

$$= \int_{-\infty}^{\infty} \left\{ f^{r,0}(t,v) - G_{\sigma,r}(t,v) \right\} \Phi_r(x-t)\,dt,$$

defined for all $(x,v) \in R \times E_r$ and

$$h_r^{\mp}(t,v) := \frac{1}{2} \left\{ |f^{r,0}(t,v) - G_{\sigma,r}(t,v)| \pm f^{r,0}(t,v) \mp G_{\sigma,r}(t,v) \right\}.$$

Then

$$g(x,v) = \int_{-\infty}^{\infty} h_r^+(t,v) \Phi_r(x-t)\,dt - \int_{-\infty}^{\infty} h_r^-(t,v) \Phi_r(x-t)\,dt .$$

Putting

$$Y_\sigma(z,v) := \int_{-\infty}^{\infty} h_r^+(t,v) P_r(z-t)\,dt - \int_{-\infty}^{\infty} h_r^-(t,v) Q_r(z-t)\,dt ,$$

where $P_r(\cdot)$, $Q_r(\cdot)$ are as in Proposition 3, we have

$$Y_\sigma(x,v) - g(x,v) = \int_{-\infty}^{\infty} h_r^+(t,v) \left\{ P_r(x-t) - \Phi_r(x-t) \right\} dt + \tag{39}$$

$$+ \int_{-\infty}^{\infty} h_r^-(t,v) \left\{ \Phi_r(x-t) - Q_r(x-t) \right\} dt .$$

Therefore, $Y_\sigma(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$ and

$$Y_\sigma(x,v) \ge g(x,v) \text{ for all } (x,v) \in R \times E_r . \tag{40}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Taking the function T_σ with values

$$T_\sigma(z,v) := F_{c,r}(z,v) + J_\sigma(z,v) + Y_\sigma(z,v), \quad z = x + iy,$$

we observe that

(41) $T_\sigma(x,v) - f(x,v) = Y_\sigma(x,v) - g(x,v)$ for all $(x,v) \in R\times E_r$ and $T_\sigma(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$.

Identity (41) and inequality (40) lead to (33), immediately.

In view of Lemma 1, $T^{\mu,0}(\cdot,\cdot) \in L_\sigma^p \cap \Lambda_\sigma^p$ for $\mu = 1,2,\ldots$ From (41) it follows that, for every $(x,v) \in R \times E_r$ and $\nu = 0,1,\ldots,r-1$,

(42) $T_\sigma^{\nu,0}(x,v) - f^{\nu,0}(x,v) = Y_\sigma^{\nu,0}(x,v) - g^{\nu,0}(x,v)$.

By (39),

$$Y_\sigma^{\nu,0}(x,v) - g^{\nu,0}(x,v) = \int_{-\infty}^{\infty} h_r^+(t,v)\left\{P_r^{(\nu)}(x-t) - \Phi_r^{(\nu)}(x-t)\right\}dt +$$

$$+ \int_{-\infty}^{\infty} h_r^-(t,v)\left\{\Phi_r^{(\nu)}(x-t) - Q_r^{(\nu)}(x-t)\right\}dt .$$

Further, the generalized Minkowski inequality gives

$$\|Y_\sigma^{\nu,0} - g^{\nu,0}\|_p \le \|h_r^+\|_p\|P_r^{(\nu)}(\cdot) - \Phi_r^{(\nu)}(\cdot)\|_1 +$$

$$+ \|h_r^-\|_p\|\Phi_r^{(\nu)}(\cdot) - Q_r^{(\nu)}(\cdot)\|_1 \le \|f^{r,0} - G_{\sigma,r}\|_p \frac{c_6(r,\nu)}{\sigma^{r-\nu}} .$$

Consequently (see (42) and (37)),

$$\|T_\sigma^{\nu,0} - f^{\nu,0}\|_p \le c_6(r,\nu)c_8\sigma^{\nu-r-1/p}\|f^{r,0}\|_p^* ,$$

and, in case of $\nu \le r - 1$, the assertion (34) is established.

A similar calculation shows that

$$\|T_\sigma^{\nu,0} - f^{\nu,0}\|_p^* = \|Y_\sigma^{\nu,0} - g^{\nu,0}\|_p^* \le \|f^{r,0} - G_{\sigma,r}\|_p^* \frac{c_6(r,\nu)}{\sigma^{r-\nu}} .$$

Thus, with the help of (38), we get (35) for each non-negative integer $\nu \le r - 1$.

Finally, the above results and Propositions 1, 2 lead to (34) and (35) for $\nu = r$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] Н. И. Ахиезер : Лекции по теории аппроксимации, Москва 1965.

[2] **P. Pych-Taberska, R. Taberski** : Exponential approximation in the norms and seminorms; in Analysis and related mathematical fields, Pliska (in print).

[3] **R. Taberski** : One-sided approximation in metrics of the Banach spaces $L^p(-\infty,\infty)$, Functiones et Approximatio, 12 (1982) 113-125.

[4] **R. Taberski** : Exponential approximation on the real line; in Approximation and function spaces, Banach Center Publications, vol. 22, PWN, Warsaw 1989, 449-464.

[5] **R. Taberski** : On partial power variations and averaged moduli of continuity, Fasc. Math. 19(1990) (in print).

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY,
60-769 POZNAŃ, POLAND
Received January 26, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Aleksei Tralle

ONE NEW EXISTENCE THEOREM FOR THE GENERALIZED SYMMETRIC RIEMANNIAN SPACES WITH SOLVABLE ISOMETRY GROUP

1. Introduction

The first existence theorem on the generalized symmetric spaces of solvable type was proved by M. Bożek [1]. His result can be formulated in the following way:

Theorem 1. (Bożek). For every integer $m \geq 4$ there is an irreducible generalized symmetric Riemannian space M of order a diffeomorphic to $\mathbb{R}^{m-1}$ and such that the identity component $I(M)_0$ of its full isometry group is solvable.

The method of constructing of this space is worth analyzing. Firstly, the notion of the generalized symmetric space with a finite isometry group at a point is introduced:

Definition 1. Generalized symmetric space with finite group $I(M,o)$ is called the generalized symmetric space with a finite isometry group at a point $(I(M,o) = \{ g \in I(M) \mid g(o) = o \}$ for a fixed initial point $o \in M$ $\})$.

Secondly, it is shown in [1] that any space of that kind has solvable identity component of its full isometry group. The last step of this proof is the constructing of the explicit example of the space with finite isometry group at a point. So one can reformulate the above theorem in the following way:

Theorem 1'. (i) For any generalized symmetric Riemannian space with a finite isometry group at a point the identity component of its full isometry group is solvable.
(ii) For every integer $m \geq 4$ there exists an irreducible

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

generalized symmetric space with finite isometry group at a point.

The attempt to classify generalized symmetric spaces of solvable type leads to the question: does the space with solvable isometry group at a point exist? What is the method of constructing of such a space?

This note is devoted to the answer on the above questions. Everywhere in this paper we denote by $G_1 * G_2$ the semidirect product of Lie groups, σ - the automorphism of Lie group G. We use the notation G^σ for the subgroup of stable elements of σ. We also use the notion of "torus" in the same sense of Lie group theory. Everywhere the symbol G_0 means the connected component of Lie group G.

Theorem 2. Let G be a Lie group of the form $G = T*U$ where U is a complex unipotent algebraic group of Chevalley type and T is a torus. Let $\sigma \in \mathrm{Aut}\, G$ be a periodical authomorphism of G such that $G^\sigma = T$. Let $M = G/T$ and also suppose that G acts on M effectively. Then, if $\dim U > 6$ over $\mathbb{R}$:

(i) there exists G-invariant Riemannian metric g and regular Riemannian s-structure $\{s_x, x \in M\}$ such that $\{M,g,s_x\}$ is a generalized symmetric Riemannian space with solvable group $I(M)_0$;

(ii) the isotropy group $I(M,o)$ is infinite.

2. Preliminaries

We shall use the abbreviation g.s.r.s. for the generalized symmetric Riemannian space.

All the algebraic group theory terminology is taken from [2] all [6]. Any complex algebraic group is a complex Lie group [6]. It is evident that any complex Lie group is a real Lie group of doubled dimension. We use these facts in the present paper where necessary without additional explanations.

For a given authomorphism $\sigma : G \longrightarrow G$ of a finite order and a closed subgroup H of the kind $G_0^\sigma \subset H \subset G^\sigma$ there exists a canonical regular s-structure on $M = G/H$ defined by the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

formulas:

$$(1)\qquad \begin{cases} \pi\cdot\sigma = s_o\cdot\pi, \\ s_{\tau(a)\cdot o} = \tau(a)\cdot s_o\cdot\tau(a^{-1}), \quad a\in G. \end{cases}$$

Here π is a canonical projection $\pi : G \longrightarrow G/H$, $\tau(a)$ is a left translation $\tau(a)(bH) = abH$ (see [3]).

Definition 2. Unipotent algebraic group over a field K is said to be of Chevalley type if it is maximal unipotent K-decomposable subgroup in K-decomposable simple algebraic group.

3. Proof of Theorem 2

Everywhere in the paper Lie algebras corresponding to Lie groups G,T,U are denoted by $\mathcal{G},\mathcal{T},\mathcal{U}$ respectively.

Consider the homogeneous space M = G/T as in the Theorem 2. Let

$$(2)\qquad \mathcal{G} = \mathcal{T} \circledcirc \mathcal{U}$$

be a semidirect product (or a sum) of Lie algbras $\mathcal{T}$ and $\mathcal{U}$. It should be mentioned that (2) also corresponds to the reductive decomposition of $\mathcal{G}$ (it is evident because of $\mathcal{U}$ being an ideal in $\mathcal{G}$ and so $[\mathcal{T},\mathcal{U}] \subset \mathcal{U}$). It is also evident from the direct computation that for any inner automorphism $I_t \in \mathrm{Aut}\, G$, $t \in G^{\sigma}$, $I_t(a) = tat^{-1}$ the following equality holds

$$(3)\qquad \sigma\cdot I_t = I_t\cdot\sigma .$$

Then

$$(4)\qquad d\sigma_e\cdot \mathrm{Ad}t = \mathrm{Ad}t\, d\sigma_e .$$

Consider the group Γ generated by all $\mathrm{Ad}t|_{\mathcal{U}}$, $t \in T$ and $d\sigma_e|_{\mathcal{U}}$

Then $$\Gamma = \langle \mathrm{Ad}|_{\mathcal{U}},\ d\sigma_e|_{\mathcal{U}},\ t \in T\rangle \le GL(\mathcal{U}).$$

$$(5)\qquad \Gamma = \bigcup_{i=1}^{k} (d\sigma_e)^i \mathrm{Ad}\, T ,$$

because (4) implies $\mathrm{Ad}t_1\cdot(d\sigma_e)^{l_1}\cdot\mathrm{Ad}t_2(d\sigma_e)^{l_2}\cdot\ldots\cdot =$ $= (d\sigma_e)^s\cdot\mathrm{Ad}t$, $s \in Z$, $t \in T$. The formula (5) shows that Γ is a compact subgroup in $GL(\mathcal{U})$. It is a well known fact that there

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

exists on $\mathcal{U}\times\mathcal{U}$ a definite Γ-invariant symmetric bilinear form B. Define G-invariant Riemannian metric on M by the usual procedure:

$$g_p(X_p,Y_p) = g_{\tau(a)\cdot o}\,(d\tau(a)_o X_o,\ d\tau(a)_o Y_o) =$$
$$= g_o(X_o,Y_o) = B(X,Y)\ .$$

Here $\mathcal{U}$ is identified with the tangent space T_oM, $X_o \in T_oM$ corresponds to $X \in \mathcal{U}$, $o = T$ is an initial point, $p \in M$ an arbitrary point.

Γ-invariance of B gives the correctness of this definition and besides that the equality $B(X,Y) = B(d\sigma_e X, d\sigma_e Y)$ holds for all $X,Y \in \mathcal{U}$. Now consider the regular s-structure on M defined by (1). Then

$$g_o((ds_o)_o X_o,(ds_o)_o Y_o) = B(d\sigma_e X,d\sigma_e Y) = B(X,Y) =$$
$$= g_o(X_o,Y_o)\ .$$

This equality shows that $(ds_o)_o : T_oM \longrightarrow T_oM$ is isometry. Then it follows easily from the equality

$$d(s_o\cdot\tau(b)\cdot s_o^{-1})_o = (ds_o)_x\cdot d\tau(b)_o\cdot(ds_o^{-1})_o : T_oM \longrightarrow T_xM$$

that $d(s_o\cdot\tau(b)\cdot s_o^{-1})_o : T_oM \longrightarrow T_xM$ is an isometry (it is sufficient to obtain by the direct computation the formula:

$$s_o\cdot\tau(b)\cdot s_o^{-1} = \tau(\sigma(b)),\quad b \in G\ .$$

Thus we have shown that s_x is an isometry for any $x \in M$ and s-structure $\{s_x,\ x \in M\}$ is Riemannian regular s-structure.

Consider the homomorphism $a \longrightarrow \tau(a)$, $G \longrightarrow I(M)$. The conditions of Theorem 2 require it to be injective and G becomes a subgroup of I(M).

On the other hand one can easily notice that U acts transitively on M (note that G is a semidirect product of T and U and any element $bT \in G/T$ can be represented in the form $bT = tuT = ut'T = uT$ as far as U is a normal subgroup in G). Therefore the effective U-action on M implies the evident diffeomorphism $M \cong U$. Define the Riemannian metric g_U on U by the last diffeomorphism. It is evident that left translations $L_v : U \longrightarrow U, L_v(u) = vu$ become the isometries of

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Riemannian manifold (U,g_U). Analogously define Riemannian s-structure on U by the above mentioned diffeomorphism. Thus $(U,g_U,\{s_v\},v \in U)$ is a g.s.r.s.

On the other hand U is a nilpotent Lie group equipped with left-invariant Riemannian metric. For the nilpotent Lie group of this kind the following imbedding is known [5]:

$$I(U,g) \hookrightarrow U*Aut\ U\ .$$

Since we have supposed U being an algbraic group of Chevalley type one theorem of Fauntleroy [4] implies $Aut^{alg}(U)$ being solvable algbraic group (the symbol $Aut^{alg}(U)$ denotes the group of all automorphisms of U as an algebraic group). Since U is a unipotent complex algebraic group, the exponential map $exp : \mathcal{U} \longrightarrow U$ is the isomorphism of algebraic manifolds [6]. Therefore the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & U \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathcal{U} & \xrightarrow{d\sigma_e} & \mathcal{U} \end{array}$$

shows that $Aut^{alg}(U) = Aut(U)$. Thus Aut(U) is solvable and the semidirect product U*Aut(U) is also solvable. Then $I(M,g) \cong I(U,g_U)$ is solvable and so is $I(M,g)_0$. Since $I(M,g,o) \supset T$ it is infinite. The proof of Theorem 2 is completed.

4. Explicite example

Let $\tilde{G}$ be a group of complex triangular matrices of the form

$$\tilde{G} = \left\{ \begin{pmatrix} e^{\sqrt{-1}\,t_1} & & & * \\ & e^{\sqrt{-1}\,t_2} & & \\ & & \ddots & \\ 0 & & & e^{\sqrt{-1}\,t_n} \end{pmatrix} \right\}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $t_1, t_2, \ldots, t_n$ are arbitrary real numbers. Denote by D the subgroup of scalar matrices of the form

$$D = \{ d = \mathrm{diag}(e^{\sqrt{-1}\, r}, e^{\sqrt{-1}\, r}, \ldots, e^{\sqrt{-1}\, r}),\ r \in \mathbb{R} \}.$$

Put $G = \bar{G}/D$ and $T = \bar{T}/D$ where

$$\bar{T} = \{ \bar{t} = \mathrm{diag}(e^{\sqrt{-1}\, t_1}, \ldots, e^{\sqrt{-1}\, t_n}),\ t_1, \ldots, t_n \}.$$

Then T is a torus and $G = T * UT(n)$, where $UT(n)$ denotes a unipotent group of all triangular complex matrices with units on the diagonal. The direct computation shows that G acts effectively on $M = G/T$. It is sufficient to prove that for any $t \in T$ there exists $a \in G$ such that $ata^{-1} \in \bar{T}$. If $t \in T$, consider its representative $\bar{t} \in \bar{T}$,

$$\bar{t} = \mathrm{diag}(e^{\sqrt{-1}\, t_1}, \ldots, e^{\sqrt{-1}\, t_n}) \text{ with } t_i \neq t_j \text{ for at least}$$

one pair of (i,j). Then the direct computation shows that for transvection $b = t_{ij}(\alpha)$ the matrix btb^{-1} is not diagonal and thus $btb^{-1} \in T$. Here $t_{ij}(\alpha)$ is a usual transvection e.a. a unipotent triangular matrix with only one nonzero element $\alpha \in \mathbb{C}$ on the position (i,j).

Now choose the sufficiently large positive integer k and such complex $\varepsilon_1, \ldots, \varepsilon_n$ that $\varepsilon_1^k = \varepsilon_2^k = \ldots = \varepsilon_n^k = 1$ and $\varepsilon_i \neq \varepsilon_j$ for all the pairs $i \neq j$. Consider the authomorphism

$$\sigma : G \longrightarrow G, \quad \sigma = \mathrm{Int}\, a, \quad a = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n).$$

The direct computation shows that $G^\sigma = T$, $U = UT(n)$ and therefore the homogeneous space $M = G/T$ satisfies the conditions of Theorem 2.

Acknowledgement. This work has been done during the author's research visit to Szczecin Technical University. The author is grateful to prof. A. Jakubowicz for his interest to this work.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] M. Bożek: Existence of generalized symmetric Riemannian spaces with solvable isometry group, Časopis Pèst. Mat. 105(1980), 368-384.

[2] J. Humphreys: Linear Algebraic Groups, New York, 1975.

[3] O. Kowalski: Generalized Symmetric Spaces, Berlin, 1980.

[4] A. Fauntleroy: Authomorphism groups of unipotent groups of Chevally type, Pacific. J. Math. 66 (1976), 373-390.

[5] M. Schubert: Die geometrie nilpotenter Liegruppen mit linksinvarianter Metrik, Bonner Math. Shr. 149(1983),1-74.

[6] E. Vinberg, A. Oniscik: Seminar po gruppam Li i algebraiceskim gruppam, Moscow, 1988.

BYELORUSSIAN UNIVERSITY, MIŃSK, USSR

Received January 31, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

S.L. Singh, U.C. Gairola

A GENERAL COINCIDENCE THEOREM

Jungck's fixed point theorem [7] proved in 1976 and improved among others by Conserva [1], Das-Naik [4], Fisher [5], Jungck [8], Park [11], Rhoades et al. [13], Sessa [14] and Tivari-Singh [18] is generally called as Jungck contraction principle (Jcp) (cf. Remark 3). On the other hand, with a view to generalizing the Banach contraction principle, Matkowski [10] proved a fixed point theorem (see Remark 1) for a system of n transformations on a product of n metric spaces. This result has been extended and generalized by Czerwik [2]-[3], Kominek [9], Reddy-Subrahmanyam [12], Singh--Kulshrestha-Mehndiratta [15]-[17]. In this paper, we establish a coincidence theorem which extends and unify the results of [5], [6], [7], [9], [10] and [15]-[17].

We generally follow the notations of Matkowski [10] (see also [3], [15]-[17]):

$$(1)\qquad c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for } i\neq k,\ i,k=1,\dots,n, \\ 1-a_{ik} & \text{for } i=k, \end{cases}$$

and $c_{ik}^{(t)}$ are defined recursively by

$$(2)\qquad c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)}\, c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)}\, c_{1,k+1}^{(t)}, & \text{for } i\neq k \\ c_{11}^{(t)}\, c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)}\, c_{i,k+1}^{(t)}, & \text{for } i=k \end{cases}$$

$i,k = 1,\dots,n-t-1,\ t = 0,1,\dots,n-2;$

AMS (MOS) Subject Classifications (1980): 54H25. 47H19.
Key words and phrases: Coincidence fixed point, metric space and contraction.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $(v_1, v_2, \ldots, v_n) = v(1,n)$ and $(v_{i1}, \ldots, v_{in}) = v_i(1,n)$.

The following lemma is essentially due to Matkowski [10, page 9] (see also [17]).

Lemma. Let $c_{ik}^{(0)} > 0$, $i,k = 1,\ldots,n$, $n \geq 2$ then the system of inequalities

$$(4) \qquad \sum_{k=1}^{n} a_{ik} r_k < r_i, \quad i = 1,\ldots,n$$

has a positive solution $r_1, r_2, \ldots, r_n$ if and only if the following inequalities hold:

$$(5) \qquad c_{ii}^{(t)} > 0, \quad i = 1,\ldots,n-1, \quad t = 0,1,\ldots,n-1.$$

Let $A_1, A_2, \ldots, A_n$ be n arbitrary non-empty sets, assume that $A := A_1 \times A_2 \times \ldots \times A_n$ and $(X_1,d_1),\ldots,(X_n,d_n)$ metric spaces.

(6) P_i, Q_i, S_i, T_i are transformations defined on A with values in X_i, $i = 1,\ldots,n$;

(7) $P_i(A) \cup Q_i(A) \subset T_i(A) \cap S_i(A)$, $i = 1,\ldots,n$;

(8) $T_i(A) \cap S_i(A)$ is a complete subspace of X_i, $i=1,\ldots,n$;

(9) $d_i(P_i(x(1,n)),Q_i(y,(1,n))) \leq$

$$\leq \sum_{k=1}^{n} a_{ik} d_k(S_k(x(1,n)),T_k(y(1,n))) +$$

$$+b[d_i(S_i(x(1,n)),P_i(x(1,n)))+d_i(T_i(y(1,n)),Q_i(y(1,n)))]+$$

$$+c[d_i(S_i(x(1,n)),Q_i(y(1,n)))+d_i(T_i(y(1,n)),P_i(x(1,n)))]$$

for all $x(1,n)$ $y(1,n)$ in A, where b and c are some non-negative numbers.

Suppose that $r_i > 0$, $i = 1,\ldots,n$ is the solution of the system of inequalities (4). Define

$$h = \max_i (r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k).$$

In view of the system of inequalities (4), the definition of h is proper and $0 < h < 1$ (cf. [3]).

Let b, c be real numbers such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(10) $0 \le 2b + 2c < 1 - h.$

Theorem. Let a_{ik}, $i,k = 1,\ldots,n$ and b, c be non-negative numbers defined in (1), (2) and (10) such that (5)-(9) hold. Then the systems $P_i(x(1,n))=S_i(x(1,n))$ and $Q_i(x(1,n))=T_i(x(1,n))$, $i=1,\ldots,n$, have solution in A.

Proof. First we note the systems (4) and (5) are equivalent. From the lemma we may choose a system of positive numbers $r_1,\ldots,r_n$ such that

(11) $$\sum_{k=1}^{n} a_{ik}r_k \le hr_i , \quad i = 1,\ldots,n.$$

Pick $x_i^{(0)} \in X_i$, $i = 1,\ldots,n$. We, in view of (7), construct sequences $\{x_i^m\}$ in X_i and $\{z_i^m\}$ in $S_i(A) \cap T_i(A)$ such that

$$P_i(x^{2m}(1,n)) = T_i(x^{2m+1}(1,n)) := z_i^{2m+1}$$

and

$$Q_i(x^{2m+1}(1,n)) = S_i(x_i^{2m+2}(1,n)) := z_i^{2m+2}, \quad m=0,1,2,\ldots.$$

From the homogeneity of the system (11), we may assume (indeed, if necessary, increasing the value of r_i, $i=1,\ldots,n$) that

From (9), $$d_i(z_i^1,z_i^2) \le r_i , \quad i = 1,\ldots,n.$$

$$d_i(z_i^2,z_i^3) = d_i(P_i(x^2(1,n)), Q_i(x^1(1,n))) \le$$

$$\le \sum_{k=1}^{n} a_{ik}d_k(z_k^2,z_k^1) + b[d_i(z_i^2,z_i^3) + d_i(z_i^1,z_i^2)] +$$

$$+ c[d_i(z_i^2,z_i^2) + d_i(z_i^1,z_i^3)] \le$$

$$\le \sum_{k=1}^{n} a_{ik}r_k + b[d_i(z_i^2,z_i^3) + r_i] + c[r_i + d_i(z_i^2,z_i^3)],$$

that is $d_i(z_i^2,z_i^3) \le qr_i$, where $q = (h+b+c)/(1-b-c)$. Similarly $d_i(z_i^3,z_i^4) \le q^2r_i$. It follows by induction that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$d_i(z_i^{m+1}, z_i^{m+2}) \le q^m r_i, \quad m = 1,2,\ldots.$$

Therefore $\{z_i^m\}$ is a Cauchy sequence, and has a limit in $S_i(A) \cap T_i(A)$ say u_i, $i = 1,\ldots,n$.

Now let $v_i(1,n)$ be a point in $S_i^{-1}(u_i)$, that is

$$S_i(v_i(1,n)) = u_i, \quad i = 1,\ldots,n.$$

From (9), we have

$$d_i(S_i(v_i(1,n)), P_i(v_i(1,n))) \le$$
$$\le d_i(S_i(v_i(1,n)), Q_i(x^{2m+1}(1,n))) +$$
$$+ d_i(P_i(v_i(1,n)), Q_i(x^{2m+1}(1,n))) \le d_i(S_i(v_i(1,n)), z_i^{2m+2}) +$$
$$+ \sum_{k=1}^{n} a_{ik} d_k(S_k(v_k(1,n)), z_k^{2m+1}) +$$
$$+ b[d_i(S_i(v_i(1,n)), P_i(v_i(1,n))) + d_i(z_i^{2m+1}, z_i^{2m+2})] +$$
$$+ c[d_i(S_i(v_i(1,n)), z_i^{2m+2}), d_i(z_i^{2m+1}, P_i(v_i(1,n)))].$$

Putting $m \to \infty$, we get

$$d_i(S_i(v_i(1,n)), P_i(v_i(1,n))) \le (b+c) d_i(S_i(v_i(1,n)), P_i(v_i(1,n)))$$

which yields

$$S_i(v_i(1,n)), P_i(v_i(1,n)).$$

Similarly there exists a point $w_i(1,n)$ in $T^{-1}(u_i)$ such that

$$T_i(w_i(1,n) = Q_i(w_i(1,n)), \quad i=1,\ldots,n.$$

This completes the proof.

Remarks

1. Matkowski's fixed point theorem [10, page 11] is obtained as corollary from our theorem when $b = c = 0$, $A_i = X_i$, $P_i = Q_i$ and $T_i(x(1,n)) = S_i(x(1,n)) = x_i$ for each x_i in X_i, $i = 1,\ldots,n$, for in such a situation X_i, $i = 1,\ldots,n$, become complete (cf. (8)).

2. A slightly different version of Kominek's coincidence theorem [9, Th. 1] generalizing the coincidence theorem of K. Goebel [7] and Matkowski's fixed point theorem [op. cit.] is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

obtained from our theorem when $b = c = 0$, $P_i = Q_i$ and $S_i = T_i$, $i = 1,\ldots,n$.

3. The Jcp (see [18]) is obtained from our theorem when $b = c = 0$, $A_i = X_i$, $P_i = Q_i$ and $S_i = T_i$, $i = 1$ with an additional condition that P_1 commutes with T_1.

4. Fisher's contractive condition for three mappings (see [5]) on a metric space is obtained from (6)-(8) when $b,c=0$, $A_i = X_i$, $P_i = Q_i$, $i = 1$.

5. A fixed point theorem for two systems of transformations due to Singh-Kulshrestha [17] is obtained from obtained from our theorem when $A_i = X_i$, $c = 0$ and $S_i(x(1,n)) = X_i = T_i(x(1,n))$ for each x_i in X_i, $i=1,\ldots,n$.

6. The coincidence theorem of Singh-Gairola [15] is obtained from our theorem when $c = 0$ and $S_i = T_i$, $i = 1,\ldots n$.

7. The main result of Singh-Gairola-Mehndirata [16] is obtained from our theorem when $c = 0$ and $P_i = Q_i$, $i = 1,\ldots,n$.

Acknowledgment. The authors thank the referee for his appreciation and suggestion and suggestions to improve upon the original typescript.

REFERENCES

[1] **V. Conserva:** Common fixed point theorems for commuting maps on a metric space, Publ. Inst. Math. 32 (46)(1982), 37-43.

[2] **S. Czerwik:** Generalization of Edelstein's fixed point theorem, Demonstratio Math. 9 (1976), 281-285.

[3] **S. Czerwik:** A fixed point theorem for a system of multivalued transformations, Proc. Amer. Math. Soc. 55 (1976), 136-139.

[4] **K.M. Das, K.V. Naik:** Common fixed point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc. 77 (1979), 369-373.

[5] **B. Fisher:** Mappings with a common fixed point, Math. Sem. Notes Kobe Univ. 7 (1979) 81-84; addendum 8 (1980), 513-514.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[6] **K. Goebel:** A coincidence theorem, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 733-735.

[7] **G. Jungck:** Commuting maps and fixed points, Amer. Math. Monthly 83 (1976), 261-263.

[8] **G. Jungck:** Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103 (1988), 977-983.

[9] **Z. Kominek:** A generalization of K. Goebel's and J. Matkowski's theorems, Univ. Śląski, Katowice, Prace Nauk. - Prace Mat. 12 (1982), 30-33.

[10] **J. Matkowski:** Integrable solutions of functional equations, Dissertationes Mat. 77 (Rozprawy), Warszawa, 1975.

[11] **S. Park:** Fixed points of f-contractive maps, Rocky Mountain J. Math. 8 (1978), 743-750.

[12] **K.B. Reddy, P.V. Subrahmanyam:** Extensions of Krasnoselskii's and Matkowski's fixed point theorems, Funcial. Ekv. 24 (1981), 67-83.

[13] **B.E. Rhoades, S. Sessa, M.S. Khan and M. Swaleh:** On fixed points of assymptotically regular mappings. J.Austral. Math. Soc. (Series A) 43 (1987), 328-346.

[14] **S. Sessa:** On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. 32 (46) (1982), 149-153.

[15] **S.L. Singh, U.C. Gairola:** A coincidence theorem for three systems of transformations, Demonstratio Math. 23 (1990).

[16] **S.L. Singh, U.C. Gairola and R. Mehndiratta:** A coincidence theorem for three systems of transformations.

[17] **S.L. Singh, C.Kulshrestha:** A common fixed point theorem for two systems of transformations, Pusan, Kyo, Math. J. 2(1986), 1-8.

[18] **B.M.L. Tivari, S.L. Singh:** A note on recent generalizations of Jungck contraction principle, J. UPGC, Acad. Soc. 3(1) (1986) 13-18.

DEPARTMENT OF MATHEMATICS, GURUKUL KANGRI UNIVERSITY,
HARDWAR, 249 404 INDIA

Received February 2nd, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Eugeniusz Chrzanowski

ON SOME PROBLEMS FOR THE SYSTEM OF DIFFUSION EQUATIONS WITH OVERSPECIFIED BOUNDARY DATA

1. Introduction

We shall deal with some problems concerned with the following system of diffusion equations. These problems are known in the theory of diffusion in three component metal alloys,

$$(*)\quad P_\alpha(u_1,u_2) = \sum_{j=1}^{2} d_{\alpha j} \frac{\partial^2 u_j}{\partial x^2} - \frac{\partial u_\alpha}{\partial t} = \varphi_\alpha(x,t), \qquad \alpha = 1,2,$$

either in $\Omega = \{(x,t): x>0,\ t>0\}$,

$\Omega_1 = \{(x,t): 0<x<1, 0<t\le T\}$.

The unknown functions u_1, u_2 denote concentrations of two components in the three component metal system and φ_α, $\alpha = 1,2$, are connected with the thermal treatment, $d_{\alpha j}$ are the coefficients of diffusion. The system (*) was deduced by Kirkaldy in [3],[4]. In general case $d_{\alpha j}$ depend on (x,t) and on the unknown concentrations u_α. In this paper we admit coefficients $d_{\alpha j}$ constant and so obtain rather explicit results. The unknown functions φ_α, $\alpha = 1,2$ to be determined from the overspecified data, and either $\varphi_\alpha = \varphi_\alpha(t)$, or $\varphi_\alpha(x)$.

Problem I: Determine the pairs of functions $(u_\alpha,\varphi_\alpha)$, $u_\alpha = u_\alpha(x,t)$, $\varphi_\alpha = \varphi_\alpha(t)$ satisfying the following initial-boundary value problem:

(1) $P_\alpha(u_1,u_2) = \varphi_\alpha(t)$, $\alpha=1,2$ in Ω,

(2) $u_\alpha(x,0) = p_\alpha(x)$, $x>0$,

(3) $u_\alpha(0,t) = q_\alpha(t)$, $t>0$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(4)\qquad \sum_{j=1}^{2} d_{\alpha j}\,\frac{\partial u_j(0,t)}{\partial x} = m_\alpha(t),\ t>0,$$

where the functions p_α, q_α, m_α and the constants $d_{\alpha j}$ are given. The additional condition (4) is connected with the law of conservation of mass. The problem of similar kind for the heat equation was examined in [2].

To solve the problem (1) - (4) we divide it as usually in two others. Namely we look for the pairs of functions $(u_\alpha,\varphi_\alpha)$ in the form

$$(5)\qquad (u_\alpha,\varphi_\alpha)=(v_\alpha,0)+(z_\alpha,\varphi_\alpha),\ \alpha=1,2,$$

where the functions v_α, $\alpha = 1,2$ are solutions of the problem

$$(1')\qquad P_\alpha(v_1,v_2) = 0 \quad \text{in } \Omega,$$

$$(2')\qquad v_\alpha(x,0) = p_\alpha(x),\ x>0,$$

$$(3')\qquad v_\alpha(0,t) = q_\alpha(t),\ t>0,$$

and $(z_\alpha,\varphi_\alpha)$, $\alpha = 1,2$, are solutions of the problem

$$(1")\qquad P_\alpha(z_1,z_2)= \varphi_\alpha(t) \quad \text{in } \Omega,$$

$$(2")\qquad z_\alpha(x,0)= 0,\ x>0,$$

$$(3")\qquad z_\alpha(0,t)= 0,\ t>0,$$

$$(4")\qquad \sum_{j=1}^{2} d_{\alpha j}\,\frac{\partial z_j(0,t)}{\partial x} = m_\alpha(t)-\sum_{j=1}^{2} d_{\alpha j}\,\frac{\partial v_j(0,t)}{\partial x} \equiv S_\alpha(t),\ t>0.$$

The problem $(1')$-$(3')$ is a classical one and is well posed in the sense of Hadamard. Its solution may be expressed explicitely. If we assume that $d_{\alpha j}> 0$, $\alpha,j = 1,2$, $\det[d_{\alpha j}]_{2\times 2}> 0$, then the eigenvalues λ_1,λ_2 of the matrix $[d_{\alpha j}]_{2\times 2}$ are real, positive and single. Thus there exists the matrix K such that

$$K\,D\,K^{-1} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \Lambda$$

and

$$K = \begin{bmatrix}1 & k_{12}\\ k_{21} & 1\end{bmatrix},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where k_{12}, k_{21} are expressed by $d_{\alpha j}$ and λ_j, $(\alpha, j=1,2)$, [5].

Hence if we introduce the vector

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & k_{12} \\ k_{21} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{6}$$

the problem (1'')-(4'') is transformed into

$$\Lambda \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial t} = K \Phi \text{ in } \Omega, \tag{7}$$

$$W(x,0) = \Theta, \quad x>0, \tag{8}$$

$$W(0,t) = \Theta, \quad t>0, \tag{9}$$

$$\Lambda \frac{\partial W(0,t)}{\partial x} = KM - \Lambda \frac{\partial v(0,t)}{\partial x} \equiv S(t), \quad t>0, \tag{10}$$

where

$$\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \Theta' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \tag{11}$$

Introducing the matrix of fundamental solution

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}, \tag{12}$$

where

$$\Gamma_\alpha(x,t-\tau) = \frac{1}{\sqrt{2\pi(t-\tau)}} \exp\left[- \frac{x^2}{4\lambda_\alpha(t-\tau)} \right], \tag{13}$$

we get

$$W(x,t) = \int_0^t K \Phi(\eta)\, d\eta + 2\int_0^t \Gamma_x(x,t-\tau) \left[\int_0^\tau K\Phi(\eta)\, d\eta \right] d\tau \tag{14}$$

and

$$W_x(x,t) = 2 \int_0^t \Lambda^{-1} \Gamma_\tau \left[\int_0^\tau K\Phi(\eta)\, d\eta \right] d\tau. \tag{15}$$

Assuming Φ continuous and integrating (15) by parts, we obtain

$(x \longrightarrow 0)$

$$\Gamma_x(x,t) = 2\,\Lambda^{-1} \int_0^t \Gamma(x,t-\tau) K\Phi(\tau)\, d\tau$$

and from (10),

(15)

$$2\,\Lambda^{-1} \int_0^t \Gamma(0,t-\tau)\, K\Phi(\tau)\, d\tau = KM - \Lambda \frac{\partial v}{\partial x},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

whence

$$\int_0^t \frac{K\Phi(\tau)}{\sqrt{t-\tau}}\,d\tau = \sqrt{\frac{\pi}{2}}\,(KM-\Lambda\,\frac{\partial v}{\partial x}) = \sqrt{\frac{\pi}{2}}\,S(t). \tag{17}$$

Thus (17) is equivalent to two Abel's equations of the form

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{\varphi_1+k_{12}\varphi_2}{\sqrt{t-\tau}}\,d\tau = S_1(t), \tag{18}$$

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{k_{21}\varphi_1+\varphi_2}{\sqrt{t-\tau}}\,d\tau = S_2(t). \tag{19}$$

After simple calculation, assuming that S is differentiable, one gets the explicit solution for φ_1,φ_2 in the form

$$\varphi_\alpha(t) = \frac{1}{\sqrt{\pi}}\left[\frac{S_\alpha(0)}{\sqrt{t}} + \int_0^t \frac{S'_\alpha(\eta)\,d\eta}{\sqrt{t-\tau}}\right]. \tag{20}$$

The function (14) with (20) presents the solution of (7)-(10), This solution depends continuously on the data.

Problem II. Determine pairs of the functions $(u_\alpha,\varphi_\alpha)$, $\alpha = 1,2$, satisfying

$$P_\alpha(u_1,u_2) = \varphi_\alpha(x) \quad \text{in } \Omega_1, \tag{21}$$

$$u_\alpha(x,0) = p_\alpha(x),\ 0<x<1, \tag{22}$$

$$u_\alpha(0,t) = q_\alpha(t),\ 0<t\le T, \tag{23}$$

$$u_\alpha(1,t) = r_\alpha(t),\ 0<t\le T, \tag{24}$$

$$\sum_{j=1}^{2} d_{\alpha j}\,\frac{\partial u_j(0,\tau)}{\partial x} = m_\alpha(t),\ 0\le\tau\le t\le T. \tag{25}$$

As before

$$(u_\alpha,\varphi_\alpha) = (v_\alpha,0)+(z_\alpha,\varphi_\alpha),$$

where the solutions $(v_\alpha,0)$ are well known.

The problem, whose solutions are $(z_\alpha,\varphi_\alpha)$ is similar to (1)-(4) with two boundary conditions (3") i.e. $z_\alpha(0,t) = z_\alpha(1,t) = 0$ and (4") for $0 \le \tau \le t \le T$.

We shall study the problem by the method of separation of variables, assuming that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(a) the functions φ_α are twice continuously differentiable in $0 \le x \le 1$,

(b) $\varphi_\alpha(0) = \varphi_\alpha(1) = 0$,

(c) there exists a positive constant κ such that $|\varphi^{(j)}(x)| \le \kappa$, $j = 0,1,2$.

Theorem: If assumptions (a)-(c) hold, then for each $N>0$ there exist families of constants A_N^α, B_N^α, C_N^α, D_N^α, $\alpha = 1,2$, and constant ν, $0<\nu<1$, such that

$$\|\varphi_\alpha\|_2^2 \le A_N^\alpha \eta^\nu + B_N^\alpha, \quad \alpha = 1,2,$$

$$\|z_\alpha\|_2^2 \le C_N^\alpha \eta^\nu + D_N^\alpha,$$

where

(26) $\eta = \|S_\alpha\|_{<t,\tau>} = \sup_{<\tau,t>} |S(t)|$, $\alpha = 1,2$, S_α being defined by (4") and $(z_\alpha, \varphi_\alpha)$ is a solution of (21)-(25).

Proof. By (a),(b),(c) we can expand the functions φ_α into the series

$$\varphi_\alpha(x) = \sum_{k=1}^{\infty} C_k^\alpha \psi_k(x), \quad \alpha = 1,2, \tag{27}$$

where

$$C_k^\alpha = \int_0^1 \varphi_\alpha(x)\psi_k(x)\,dx \tag{28}$$

and ψ_k are the normalized eigenfunctions corresponding to the eigenvalues α_k for the problem

$\psi_k'' + \alpha\psi_k = 0$,

$\psi_k(0) = \psi_k(1) = 0$.

Evidently

$\alpha_k = \pi^2 k^2$, $\psi_k(x) = \sqrt{2}\sin k\pi x$.

An estimate for φ_k can be obtained from the estimate for C_k^α. Standard separation of variables yields the formal representation

$$z_\alpha(x,t) = \sum_{k=1}^{\infty} \frac{\sqrt{2}\sin k\pi x}{k^2\pi^2\sqrt{\pi}} \left\{ \sum_{j=1}^{2} C_k^j \left(p_{j\alpha}^1 \exp(-k^2\pi^2\rho_1 t) + \right.\right. \tag{29}$$

$$\left.\left. + p_{j\alpha}^2 \exp(-k^2\pi^2\rho_2 t) + p_\alpha^3 \right) \right\},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $p^1_{j\alpha}$, (j=1,2, l=1,2,3, α=1,2) are constants and depend on $d_{j\alpha}$, ρ_1, ρ_2, are explicitely expressed and depend also on $d_{\alpha j}$.

From the Hilbert-Schmidt theorem we can deduce that

$$|c^\alpha_k| \le \frac{\text{cont}}{\pi^2 k^2}, \quad \alpha = 1,2, \tag{30}$$

and the formal representation for z_α converges absolutely and uniformly and their partial derivatives with respect to x can be obtained by differentiating term by term. This yields

$$F_\alpha(t) = \sum_{j=1}^{2} d_{\alpha j} \frac{\partial z_j(0,t)}{\partial x} = \tag{31}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k\pi\sqrt{\pi}} \sum_{j=1}^{2} c^j_k \left[\tilde{p}^1_{j\alpha} \exp(-k^2\pi^2\rho_1 t) + \tilde{p}^2_{j\alpha} \exp(-k^2\pi^2\rho_2 t) + \tilde{p}^3_\alpha\right],$$

where $\tilde{p}^1_{j\alpha}$, $\tilde{p}^2_{j\alpha}$, $\tilde{p}^3_\alpha$ depend on $p^1_{j\alpha}$, $p^2_{j\alpha}$, p^3_α and $d_{\alpha j}$.

From (26) we have

$$|F_\alpha(t)| \le \eta, \quad 0 \le \tau \le t \le T. \tag{32}$$

Let $\xi = t + i\beta$. Clearly, $F_\alpha(\xi)$ are analytic functions in the complex domain $\mathrm{Re}\,\xi \ge 0$. There exists a constant κ_1 such that

$$|F_\alpha(\xi)| \le \kappa_1. \tag{33}$$

Basing on the argument, similar to these of Cannon and Ewing [1], one can prove that there exist the constants K_1, K_2 and $\nu \in (0,1)$ such that

$$|F_\alpha(\xi)| \le K_1 \eta^\nu \quad \text{for } \frac{3}{4}\tau + \frac{1}{4}T \le \mathrm{Re}\,\xi \le \frac{1}{4}\tau + \frac{3}{4}T \tag{34}$$

and

$$|F'_\alpha\left(\frac{\tau+T}{2} + i\beta\right)| \le K_2 \eta^\nu. \tag{35}$$

From (15) we see, that

$$F'_\alpha\left(\frac{t+T}{2} + i\beta\right) = \sum_{k=1}^{\infty} \left[a^1_{k\alpha} \exp(-ik^2\pi^2\rho_1\beta) + a^2_{k\alpha} \exp(-ik^2\pi^2\rho_2\beta)\right], \tag{36}$$

where

$$a^1_{k\alpha} = \left(c^1_k \tilde{p}^1_{1\alpha} + c^2_k \tilde{p}^1_{2\alpha}\right) \exp\left(-k^2\pi^2 \frac{\tau+T}{2}\right), \tag{37}$$

$$a^2_{k\alpha} = \left(c^1_k \tilde{p}^2_{1\alpha} + c^2_k \tilde{p}^2_{2\alpha}\right) \exp\left(-k^2\pi^2 \frac{\tau+T}{2}\right). \tag{38}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

To the formula (20) one can apply the lemma of Binmore [1]. So we obtain $|a^j_{k\alpha}| \le \text{const}\ \eta^\nu$, j=1,2, and then from (21),(22) we find

$$|c^\alpha_k| \le \text{const}\cdot\exp\left(\frac{\tau+T}{2}\,k^2\pi^2\right)\eta^\nu. \tag{39}$$

In view of

$$\|\varphi_\alpha\|^2_2 \le \sum_{n=1}^{\infty} |c^\alpha_k|^2 \tag{40}$$

we obtain for any N>0

$$\|\varphi_\alpha\|^2_2 \le \sum_{n=1}^{[N]} c^2_n + \sum_{n=[N]+1}^{\infty} c^2_n. \tag{41}$$

Recalling (30) and (39) we come to the inequality

$$\|\varphi_\alpha\|^2_2 \le A^\alpha_N\eta^\nu + B_N, \text{ where } A_N \longrightarrow \infty,\ B_N \longrightarrow 0.$$

Thus, by (29), we have

$$\|z_\alpha(\cdot,t)\|^2_2 \le \text{const}\ \|\varphi_\alpha\|^2_2$$

and the theorem is valid.

REFERENCES

[1] J.R. Cannon, R.E. Ewing: Determination of the source term in a linear parabolic partial differential eqation, J.Appl. Math. Phys. 27 (1976) 215-224.

[2] J.R. Cannon, D.Zachmann: Parametr determination in partial differential equation from overspecified boundary data, J. Engin. Sci. 20 (1984) 127-135.

[3] J.S. Kirkaldy: Diffusion in multicomponent metalic system, Canadian J.Phys. 35 (1957) 435-440.

[4] J.S. Kirkaldy: Diffusion in multicomponent metalic system, Canadian J.Phys. 36 (1958) 899-925.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] J. Wolska-Bochenek, L.v Wolfersdorf: On some generalized free boundary value problem for a system of one-dementional difffusion equation, Math. Zeit. 61 (1981) 541-545.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received February 8, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Vladimír Janiš

NOTES ON SUMS OF BOOLEAN ALGEBRAS AND LOGICS

1. Introduction

This paper is closely connected with [2] where sum of a Boolean algebra and a logic is proposed. We study how properties of Boolean algebras and logics are reflected in properties of their sums. It is shown that some important algebraic and topologic properties are inherited in the sum.

Section 2 includes basic notions and definitions, most of them generally known, and a brief review of some results of [2]. Properties concerning algebraic structure, states and topologic structure are studied in sect. 3,4 and 5.2.

2. Preliminaries

By a logic we understand a lattice $\underline{L}$ with the zero 0, the unity element 1, with orthocomplementation (the ortho-complement of an element $\underline{a}$ is denoted by $\underline{a}'$) and fulfilling the orthomodular law, i.e. $b = a \vee (b \wedge a')$ whenever $a, b \in L$ and $a \leq b$. (The term "orthomodular lattice" is used for logics as well.) The elements $\underline{a}$, $\underline{b}$ of a logic are called orthogonal iff $a \leq b'$.

If L_1, L_2 are logics then the mapping $f: L_1 \longrightarrow L_2$ is called a homomorphism if for each $a, b \in L_1$ there is $f(a \vee b) = f(a) \vee f(b)$ and $f(a') = (f(a))'$.

If $\underline{f}$ is bijective and both $\underline{f}$ and $\underline{f}^{-1}$ are homomorphisms then $\underline{f}$ is called an isomorphism. The mapping $f: L_1 \longrightarrow L_2$ is called an embedding if $f: L_1 \longrightarrow f(L_1)$ is an isomorphism.

The sum proposed in [2] in a certain way "respects" states. Recall that a function $s: L \longrightarrow [0;1]$ is a state on the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

logic $\underline{L}$ if for each orthogonal pair $a, b \in L$ there is

$$s(a \vee b) = s(a) + s(b) \text{ and } s(1) = 1.$$

Now we summarize the basic results of [2]:

Let $\underline{B}$ be a Boolean algebra and let $\underline{L}_1$ be a logic. The logic $\underline{L}$ is called a sum of $\underline{B}$ and $\underline{L}_1$ if

(i) there exist embeddings $f: B \longrightarrow L$, $f_1: L_1 \longrightarrow L$ such that $f(a) \wedge f_1(b) = 0$ if and only if $a = 0$ or $b = 0$,

(ii) there is no proper sublogic of $\underline{L}$ containing

$$f(B) \cup f_1(L_1),$$

(iii) if $\underline{s}$ is a state on $\underline{B}$, $\underline{s}_1$ is a state on $\underline{L}_1$ then there exists a state $\tilde{\underline{s}}$ on $\underline{L}$ such that $\tilde{s}(f(a)) = s(a)$ for each $a \in B$ and $\tilde{s}(f_1(b)) = s_1(b)$ for each $b \in L_1$.

It is proved in [2] that for any Boolean algebra and any logic there exists their sum. It is necessary for our purposes to present its structure.

Let us first consider the set $\underline{S}$ of all the elements of the form $\langle (a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)\rangle$, where n is a natural number, $a_i \in B$, $b_i \in L_1$, $i = 1,2,\ldots,n$, $\{a_i\}_{i=1}^n$ is a set of mutually orthogonal elements such that $\bigvee_{i=1}^n a_i = 1$.

The relation $\leq$ in $\underline{S}$ is defined in the following way: $\langle (a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)\rangle \leq \langle (c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)\rangle$ if $b_i \leq d_j$ whenever $a_i \wedge c_j \neq 0$.

If we identify those elements $p.r \in S$ for which both $p \leq r$ and $r \leq p$ then the set of all corresponding equivalence classes is the required sum. Its elements will be written in square brackets, i.e. $[(a_1,b_1), (a_2,b_2),\ldots,(c_m,d_m)]$ is the equivalence class containing the element $\langle (a_1,b_1), (a_2,b_2),\ldots,(a_n,b_n)\rangle$. Note that it can be chosen so that $a_i \neq 0$ for each $i = 1,2,\ldots,n$.

The orthocomplementation in $\underline{L}$ is defined in the following way: If

$$p = [(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)],$$

then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$p' = [(a_1,b_1'),(a_2,b_2'),\dots,(a_n,b_n')].$$

It is not difficult to prove that if

$$p = [(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n)],$$

$$r = [(c_1,d_1),(c_2,d_2),\dots,(c_m,d_m)],$$

then

$$p \vee r = [(a_i \wedge c_j,\ b_i \vee d_j)]_{i=1\ j=1}^{n\ m},$$

$$p \wedge r = [(a_i \wedge c_j,\ b_i \wedge d_j)]_{i=1\ j=1}^{n\ m}.$$

The functions $\underline{f}$, $\underline{f}_1$ from the definition of sum are such that

$$f(a) = [(a,1),\ (a',0)],\ f_1(b) = [(1,b)].$$

The state $\tilde{\underline{s}}$ corresponding to states $\underline{s}$, $\underline{s}_1$ from the definition of the sum is given by the formula

$$\tilde{s}([(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n)]) = \sum_{i=1}^{n} s(a_i)s_1(b_i).$$

For further details see [2].

3. Algebraic properties

In this section we present necessary and sufficient conditions for sums to have some algebraic properties. Let us recall definitions of terms we focus on:

A logic $\underline{L}$ is called

(i) modular if for each $a,b,c \in L$ there is

$$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c),$$

(ii) complete if the greatest lower bound (g.l.b.) of an arbitrary nonvoid subset of $\underline{L}$ is contained in $\underline{L}$,

(iii) chain-finite if all chains in $\underline{L}$ are finite,

(iv) atomic if for each $a \in L$, $a \neq 0$ there is an atom $b \in L$ such that $b \leq a$ ($b \in L$ is an atom in $\underline{L}$ if $b \neq 0$ and $c \neq b$, $c \leq b$ imply $c = 0$).

Lattices with these properties are studied e.g. in [1].

Throughout this section the symbols $\underline{B}$, $\underline{L}_1$, $\underline{L}$ denote a Boolean algebra, a logic and their sum respectively.

It is easy prove that if (E) is an equation of logic then $\underline{L}$ satisfies (E) iff $\underline{L}_1$ satisfies (E). Using this fact for the equation of modularity we get:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 3.1. $\underline{L}$ is modular if and only if $\underline{L}_1$ is modular.

The following example shows that completeness of the both $\underline{B}$ and $\underline{L}_1$ does not guarantee completeness of $\underline{L}$:

Example 3.1. Let $\underline{B}$ be the system of all subsets of the interval $[0,\infty[$ ordered by the usual set inclusion. Let $L_1 = B$ and denote $a_m^n = [m,n[$. Let

$$M = \left\{[(a_0^1,a_0^1),(a_1^2,a_1^2),\ldots,(a_n^{n+1},a_n^{n+1}),(a_{n+1}^{\infty},a_{n+1}^{\infty})]\right\}_{n=0}^{\infty}.$$

It is easy to verify that $\underline{M}$ is a subset of $\underline{L}$ which has no g.l.b. in $\underline{L}$. Therefore $\underline{L}$ is not complete though the both $\underline{B}$ and $\underline{L}_1$ are complete.

Unlike completeness the rest two properties are inherited in the sum.

Theorem 3.2. $\underline{L}$ is chain-finite if and only if $\underline{B}$ and $\underline{L}_1$ are both chain-finite.

Proof. If $\underline{B}$ is chain-finite then it is atomic and finite, too. All the elements of $\underline{L}$ can then be written in the form

$$[(p_1,b_1),(p_2,b_2),\ldots,(p_n,b_n)],$$

where $\{p_1,p_2,\ldots,p_n\}$ is the set all the atoms of $\underline{B}$. If moreover $\underline{L}_1$ is chain-finite then it can be easily seen that $\underline{L}$ is chain-finite, too.

As the mappings $\underline{f}$ and $\underline{f}_1$ preserve order, the inverse statement is obvious.

Theorem 3.3. $\underline{L}$ is atomic if and only if both $\underline{B}$ and $\underline{L}_1$ are atomic.

Proof. Let $\underline{L}$ be atomic and let

$$p = [(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)],$$

be an atom in $\underline{L}$. Without loss of generality suppose $a_1 \neq 0$, $b_1 \neq 0$. Then $[(a_1,b_1),(a_1',0) \leq p$.

Thus each atom of $\underline{L}$ is of the form $[(a,b),(a',0)$. If there is an element $\bar{a} \in B$, $\bar{a} \neq 0$, $\bar{a} \leq a$, then

$$[(\bar{a},b),\ (\bar{a}',0)] \leq [(a,b),\ (a',0)]$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and these elements are different. Thus $\bar{a} = 0$ and $\underline{a}$ is an atom in $\underline{B}$. By the same reasoning it can be shown that $\underline{b}$ is an atom in $\underline{L}_1$. Now let $\underline{c},\underline{d}$ be non-zero elements in $\underline{B}$ and $\underline{L}_1$ respectively. Then there is an atom $[(c_0,d_0),(c_0',0)]$ in $\underline{L}$ such that

$$[(c_0,d_0),(c_0',0)] \leq [(c,d),(c',0)],$$

and so $c_0 \leq c$, $d_0 \leq d$ which proves that $\underline{B}$ and $\underline{L}_1$ are atomic.

If both $\underline{B}$ and $\underline{L}_1$ are atomic, $a = [(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]$ is a non-zero element in $\underline{L}$ such that $a_1 \neq 0$, $b_1 \neq 0$ and $\underline{a}_0$, b_0 are atoms in $\underline{B}$ and $\underline{L}_1$ such that $a_0 \leq a_1$, $b_0 \leq b_1$ then it is obvious that the element $[(a_0,b_0),(a_0',0)]$ is an atom in $\underline{L}$ contained in $\underline{a}$. Therefore $\underline{L}$ is atomic.

4. States on sums

A state $\underline{s}$ on a logic $\underline{L}$ is called faithful if for each $a \in L$, $a \neq 0$ there is $s(a)>0$.

If $\underline{M}$ is a system of states on a logic $\underline{L}$ then $\underline{M}$ is called

(i) separating if for each $a \in L$, $a \neq 1$ there is $s \in M$ such that $s(a) > 0$.

(ii) unital if for each $a \in L$, $a \neq 1$ there is $s \in M$ such that $s(a) = 0$,

If $\underline{M},\underline{M}_1$ are systems of states on $\underline{B}$, $\underline{L}_1$ respectively, if $\underline{L}$ is the sum of $\underline{B}$ and $\underline{L}_1$ then for each $s \in M$, $s_1 \in M_1$ it is possible to find a corresponding state $\tilde{\underline{s}}$ on $\underline{L}$ (see sect.2). Denote the set of all these states by $\tilde{\underline{M}}$.

Assuming these notations the following theorems are valid:

Theorem 4.1. The state $\tilde{\underline{s}}$ is faithful if and only if both $\underline{s}$ and $\underline{s}_1$ are faithful.

Theorem 4.2. $\tilde{\underline{M}}$ is separating if and only if both $\underline{M}$ and $\underline{M}_1$ are separating.

Both proofs are obvious.

Theorem 4.3. $\tilde{\underline{M}}$ is unital if and only if both $\underline{M}$ and $\underline{M}_1$ are unital.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let $\underline{M}$ and $\underline{M}_1$ be unital, let

$$p = [(a_1,b_1),(a_2,b_2),\dots,(a_n,b_n)] \in L,\ p \neq [(1,1)].$$

Then at least one $\underline{b}_i$ is different from 1. Without the loss of generality let $b_1 \neq 1$. Then

$$p \leq [(a_1,b_1),(a_2,b_2),\dots,(a_n,1)],$$

and for $\tilde{s} \in \tilde{M}$ there is

$$\tilde{s}(p) = s(a_1)s_1(b_1) + \sum_{i=2}^{n} s(a_i) = s(a_1)s_1(b_1) + s(\bigvee_{i=2}^{n} a_i).$$

If we choose $\underline{s}$, $\underline{s}_1$ so that $s(\bigvee_{i=2}^{n} a_i) = 0$, $s_1(b_1) = 0$ then we have $\tilde{s}(p) = 0$.

The inverse statement is obvious.

5. Topologies on sums

If $\underline{L}$ is a logic, $\underline{s}$ is a state on $\underline{L}$, then it is possible to define a topology τ_s on $\underline{L}$ which is in a certain sense connected to this state. Such a topology is studied in [3] and [4]. It is defined in the following way:

Let $\underline{L}$ be a logic with a state $\underline{s}$, let $(a_\alpha)_{\alpha\in A}$ be a net in $\underline{L}$, $a \in \underline{L}$. Then $a_\alpha \longrightarrow a$ in τ_s ($a_\alpha \xrightarrow{\tau_s} a$) if for each $x \in L$ there is $s(a_\alpha \vee x) \longrightarrow s(a \vee x)$ and $s(a_\alpha \wedge x) \longrightarrow s(a \wedge x)$.

It is proved in [3] that for any logic $\underline{L}$ and state $\underline{s}$ on $\underline{L}$ the topology τ_s exists. It is called a state induced topology.

Another widely used topology on partially ordered sets (therefore also on logics) is the order topology described e.g. in [5]. Recall the definition:

Let $\underline{X}$ be a partially ordered set. A net $(x_\alpha)_{\alpha\in A}$ in $\underline{X}$ is increasing (decreasing) to $x \in X$, if $x_\alpha \leq x_\beta$ ($x_\beta \leq x_\alpha$) for $\alpha \leq \beta$ and $x = \bigvee_{\alpha\in A} x_\alpha$ ($x = \bigwedge_{\alpha\in A} x_\alpha$). (We denote $x_\alpha \nearrow x$ or $x_\alpha \searrow x$.) A net $(x_\alpha)_{\alpha\in A}$ is said to be order-convergent to $x \in X$ ($x_\alpha \xrightarrow{O} x$) if there are nets $(y_\alpha)_{\alpha\in A}, (z_\alpha)_{\alpha\in A}$ in $\underline{X}$ such that $y_\alpha \leq x_\alpha \leq z_\alpha$ for each $\alpha \in A$, $y_\alpha \nearrow x$, $z_\alpha \searrow x$. The strongest topology in which all order-convergent nets are convergent is called the order topology and denoted by the symbol τ_0.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The aim of this section is to show if we use usual notation the embeddings $\underline{f}$ and $\underline{f}_1$ from the definition of the sum are homeomorphisms both in τ_0 and in $\tau_{\tilde{s}}$, $\tilde{s}$ being a state on $\underline{L}$. More precisely, if $\approx$ denotes a homeomorphism of topological spaces, then

$$(B,\tau_0^B) \approx (L \cap f(B), \tau_0^L/_{f(B)}),$$

$$(L_1,\tau_0^{L_1}) \approx (L \cap f_1(L_1), \tau_0^L/_{f_1(L_1)})$$

where τ_0^B, $\tau_0^{L_1}$, τ_0^L are order topologies on $\underline{B}$, $\underline{L}_1$, $\underline{L}$ respectively, and

$$(B,\tau_s) \approx (L \cap f(B), \tau_{\tilde{s}/f(B)}),$$

$$(L_1,\tau_{s_1}) \approx (L \cap f_1(L_1), \tau_{\tilde{s}/f_1(L_1)}),$$

where $\underline{s}$, $\underline{s}_1$ are states on $\underline{B}$ and $\underline{L}_1$ respectively, $\tilde{s}$ is the corresponding state on $\underline{L}$ and τ_s, τ_{s_1}, $\tau_{\tilde{s}}$ are state induced topologies.

Theorem 5.1. Let $\underline{s}$ be state on $\underline{B}$, $\underline{s}_1$ be a state on $\underline{L}_1$, let $\tilde{s}$ be the corresponding state on $\underline{L}$, let $\underline{f}$, $\underline{f}_1$ be the embeddings from the definition of the sum. Then

(i) $f: (B,\tau_s) \longrightarrow (L \cap f(B), \tau_{\tilde{s}/f(B)})$ is a homeomorphism,

(ii) $f_1: (L_1,\tau_{s_1}) \longrightarrow (L \cap f_1(L_1), \tau_{\tilde{s}/f_1(L_1)})$ is a homeomorphism.

Proof: As both $\underline{f}$ and $\underline{f}_1$ are isomorphisms, all we have to prove is that $\underline{f}$, $\underline{f}^{-1}$, $\underline{f}_1$, $\underline{f}_1^{-1}$ are continuous.

(i) Let $(a_\alpha)_{\alpha\in A}$ be a net in $\underline{B}$, $a \in B$, $a_\alpha \longrightarrow a$ in τ_s. Then for any there is $z = [(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)] \in L$

$$\tilde{s}(f(a_\alpha) \vee z) = \tilde{s}([(a_\alpha,1),(a'_\alpha,0)] \vee z) =$$
$$= s([(a_\alpha \wedge x_1,1),(a_\alpha \wedge x_2,1),\ldots,(a_\alpha \wedge x_n,1),$$
$$(a'_\alpha \wedge x_1,y_1),(a'_\alpha \wedge x_2,y_2),\ldots,(a'_\alpha \wedge x_n,y_n)]) =$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\sum_{i=1}^{n} s(a_\alpha \wedge x_i) + \sum_{i=1}^{n} s(a'_\alpha \wedge x_i) s_1(y_i) \longrightarrow$$

$$\longrightarrow \sum_{i=1}^{n} s(a_\alpha \wedge x_i) + \sum_{i=1}^{n} s(a'_\alpha \wedge x_i) s_1(y_i) = \tilde{s}(f(a) \vee z).$$

We have used the fact that if $a_\alpha \xrightarrow{\tau_s} a$ then the complements $\underline{a}'_\alpha$ also converge to $\underline{a}'$ in τ_s. Thus $\tilde{s}(f(a_\alpha) \vee z) \longrightarrow \tilde{s}(f(a) \vee z)$.

By the same method we could show that

$$\tilde{s}(f(a_\alpha) \wedge z) \longrightarrow \tilde{s}(f(a) \wedge z),$$

therefore $f(a_\alpha)_{\alpha \in A}$ be a net in $\underline{B}$, $a \in B$ is such that $f(a_\alpha) \xrightarrow{\tau_{\tilde{s}}} f(a)$. Suppose $(a_\alpha)_{\alpha \in A}$ does not converge to $\underline{a}$ in τ_s. Then

a) there exists $x_0 \in B$ such that $s(a_\alpha \wedge x_0) \not\longrightarrow s(a \wedge x_0)$, or

b) for each $x \in B$ there is $s(a_\alpha \wedge x) \longrightarrow s(a \wedge x)$ and there exists $x_0 \in B$ such that $s(a_\alpha \vee x_0) \not\longrightarrow s(a \vee x_0)$.

In the a) case put $z = [(x_0,1),(x'_0,0)]$. It is easy to prove that

$$\tilde{s}(f(a_\alpha) \wedge z) = s(a_\alpha \wedge x_0),\ \tilde{s}(f(a) \wedge z) = s(a \wedge x_0)$$

which is a contradiction.

In the b) case it is sufficient to realise that $s(a_\alpha \vee x_0) \not\longrightarrow s(a \vee x_0)$ implies $s(a'_\alpha \vee x'_0) \not\longrightarrow s(a' \vee x'_0)$ and putting $z = [(x'_0,1),(x_0,0))]$ we come to a contradiction again.

Thus $\underline{f}$ a homeomorphism.

The proof of (ii) is similar.

Theorem 5.2. Let τ_0^B, $\tau_0^{L_1}$, τ_0^L be order topologies on $\underline{B}$, $\underline{L}_1$, $\underline{L}$ respectively, let $\underline{f}$, $\underline{f}_1$ be the embeddings from the definition of the sum.

Then

(i) $f:(B,\tau_0^B) \longrightarrow (L \cap f(B), \tau_0^L/_{f(B)})$ is a homeomorphism.

(ii) $f_1:(L_1,\tau_0^{L_1}) \longrightarrow (L \cap f_1(L_1), \tau_{0/f_1(L_1)}^L)$ is a homeomorphism.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. It is sufficient to show that a net is order-convergent in $B(L_1)$ if the net of its images is order-convergent in $f(B)$ $(f_1(L_1))$.

(i) Let $(a_\alpha)_{\alpha\in A}$ be a net in $\underline{B}$, let $a \in B$ such that $a_\alpha \xrightarrow{0} a$. As $\underline{f}$ preserves order we obtain $f(a_\alpha) \xrightarrow{0} f(a)$.

Now let $(a_\alpha)_{\alpha\in A}$ be a net in $\underline{B}$, $a \in B$ such that $f(a_\alpha) \xrightarrow{0} f(a)$.

Then there are nets in $\underline{L}$ such that

$$[(b_{1\alpha},c_{1\alpha}),(b_{2\alpha},c_{2\alpha}),\dots,(b_{n_\alpha\alpha},c_{n_\alpha\alpha})] \nearrow [(a,1),(a',0)],$$

$$[(d_{1\alpha},e_{1\alpha}),(d_{2\alpha},e_{2\alpha}),\dots,(d_{m_\alpha\alpha},e_{m_\alpha\alpha})] \searrow [(a,1),(a',0)]$$

and

$$[(b_{1\alpha},c_{1\alpha}),(b_{2\alpha},c_{2\alpha}),\dots,(b_{n_\alpha\alpha},c_{n_\alpha\alpha})] \le [(a_\alpha,1),(a'_\alpha,0)] \le$$

$$\le [(d_{1\alpha},e_{1\alpha}),(d_{2\alpha},e_{2\alpha}),\dots,(d_{m_\alpha\alpha},e_{m_\alpha\alpha})]$$

for each $\alpha \in A$.

Let for $\alpha \in A$ the set $\underline{M}_\alpha$ contains those $\underline{i}$ for which $b_{i\alpha} \wedge a'_\alpha \neq 0$, let $\underline{N}_\alpha$ be the set of those $\underline{i}$ for which $d_{i\alpha} \wedge a_\alpha \neq 0$. Let

$$M'_\alpha = \left\{b_{ix}\right\}_{i=1}^{n_\alpha} \setminus M_\alpha,\quad N'_\alpha = \left\{d_{i\alpha}\right\}_{i=1}^{m_\alpha} \setminus N_\alpha.$$

Put $x_\alpha = \bigvee_{i\in M'_\alpha} b_{i\alpha}$, $y_\alpha = \bigwedge_{i\in N'_\alpha} d_{i\alpha}$. It is not difficult to prove that $x_\alpha \nearrow a$, $y_\alpha \searrow a$, $x_\alpha \le a_\alpha \le y_\alpha$ for each $\alpha \in A$. Therefore $a_\alpha \xrightarrow{0} a$ and $\underline{f}$ is a homeomorphism.

(ii) The proof for $\underline{f}_1$ is similar.

REFERENCES

[1] G. Kalmbach: Orthomodular Lattices. Academic Press, London 1983.

[2] P. Pták: Summing of Boolean algebras and logics, Demonstratio Math. 19 (1986), 349-357.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] Z. Riečanová: Topology in a quantum logic induced by a measure, Proceedings of the Conference Topology and Measure V, Wissenschaftliche Beiträge der EMA-Universität, Greifswald 1988.

[4] Z. Riečanová: Some properties of topology in quantum logics induced by measures, Proceedings of the First Winter school on measure theory, SAV, Bratislava 1988.

[5] T. A. Sarymsakov: Uporjadočennyje algebry. Izdatelstvo FAN, Taskent 1983.

DEPARTMENT OF MATHEMATICS, FACULTY OF ELECTRICAL ENGINEERING, SLOVAK TECHNICAL UNIVERSITY, 812 19 BRATISLAVA, CZECHOSLOVAKIA

Recived February 13, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

R. A. Rashwan

ON THE CONVERGENCE OF MANN ITERATES TO A COMMON FIXED POINT FOR A PAIR OF MAPPINGS

1. Introduction

Let T be a self-mapping on a Banach space X. In recent years several authors have obtained results on the convergence of Mann iterates of a single mapping T to a fixed point of T under various contractive conditions [2], [4], [5] and [6]. In this note we extend some results if these authors under other different contractive conditions in a normed space to the case of a pair of mappings.

Let T and S be self-mappings on a Banach space X. For any x_0 in X, we shall consider the Mann iterates process $\{x_n\}$ [3], [4] associated with S as:

$$x_{n+1} = (1 - c_n)\, x_n + c_n\, Sx_n \quad \text{for } n > 0 ,$$

where $\{c_n\}$ satisfies

(i) $c_0 = 1$,

(ii) $0 \le c_n \le 1$,

(iii) $\underset{n\to\infty}{\text{Lim}} \sup c_n > 0$.

AMS (1980) Subject Classification. Primary 47H10
Key words and phrases, normed spaces, common fixed point, Mann iterates, contractive conditions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Main theorem

We present our result in the form of a single theorem.

Theorem . Let C be a convex and closed subset of a normed space N. Further, suppose that S and T are two self-mappings of C satisfying one of the following conditions:

(I) $$\|Sx - Ty\| \le a\|x-y\| + b\,[\|x-Sx\| + \|y-Ty\|] + c\,[\|x-Ty\| + \|y-Sx\|],$$

for all x,y in C and a,b,c $\ge$ 0, 1-b-c > 0.

(II) $$\|Sx - Ty\| \le K \max\{\|x-y\|\ ,\ \|x-Sx\| + \|y-Ty\|,\ \|x-Ty\| + \|y-Sx\|\},$$

for all x,y in C and 0 < K < 1.

(III) $$\|Sx - Ty\| \le K \max\{\|x-y\|\ ,\ \|x-Sx\| + \|y-Ty\|,\ \|x-Ty\|\ ,\ \|y-Sx\|\},$$

for all x,y in C and 0 < K < 1.

If for some x_0 in C, the sequence $\{x_n\}$ of Mann iterates associated with either S or T is convergent, then its limit is a fixed point of each S and T. This is a common fixed point of S and T and is unique if (III) holds.

Proof. Suppose first that Su = u for a point $u \in C$. Then putting x = y = u into any of the inequalities (I) - (III) we easily see that Tu = u. Similarly Tu = u implies Su = u. Thus to prove our theorem it is enough to show that S has a unique fixed point. Let $\{x_n\}$ be a sequence of Mann iterates associated with S and convergent to u. From (iii) we infer that $p := \lim_{k\to\infty} c_{n_k} > 0$ for a subsequence $\{c_{n_k}\}$ of $\{c_n\}$. Thus we obtain

(•) $$Sx_{n_k} = (x_{n_k+1} - (1-c_{n_k})x_{n_k}) / c_{n_k} \longrightarrow pu/p = u.$$

Now suppose that one of the conditions (I) - (III) holds. Putting $x = x_{n_k}$, y = u get inequalities which lead to (when we let $k \longrightarrow \infty$ and take into account (•))

$$\|u-Tu\| \le (b+c)\,\|u-Tu\| \quad \text{or} \quad \|u-Tu\| \le K\,\|u-Tu\|,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

whence $Tu = u$ follows. By our observation at the beggining of the proof we get $Su = u$ as well. To prove uniqueness in the case (III), suppose that $Sv = v$ for a $v \in C$. Put $x = v$ and $y = u$ in III. We obtain $\|v-u\| \leq K\|v-u\|$ whence $v = u$ follows. An analogous argument is valid if a convergent sequence $\{x_n\}$ is associated with T. This ends the proof.

Remark. Note that if (I) holds and in addition $a + 2b + 2c < 1$ then by Theorem 1 from [1] each of S, T has a unique fixed point and these points coincide. Thus in this case the above Theorem adds fact that this common fixed point is the limit of any convergent sequence of Mann iterates associated with either S or T. On the other hand taking in (I) $a = 1$, $b = c = 0$ we see that the above Theorem holds in a situation not covered by Theorem 1 from [1] although uniqueness of a common fixed point does not hold anymore as the example of S = T = shows.

Acknowledgement. The author is thankful to the referee for his helpful suggestions for the improvement of this paper.

REFERENCES

[1] **Chi Song Wong** : Common fixed points of two mappings, Pacific J. Math., 48 (1973), 299 - 312.

[2] **K.M. Ghosh** : A note on a Theorem of Rhoades, Math. Seminar Notes, Koba Univ., 8 (1980), 505 - 707.

[3] **L.A. Khan** : On the convergence of Mann iterates to a common fixed point of two mappings, J. Pure Appl. Sci. 5, (1986), 57 - 58.

[4] **B.E. Rhoades** : Fixed point thorems using infinite matrices, Trans. Amer. Soc., 196 (1974).

[5] **B.E. Rhoades** : Extensions of some fixed point theorems of Ciric, Maiti, and Pal, Math. Seminar Notes, Kobe Univ., (1978), 41 - 46.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[6] **K.L. Singh :** Fixed point iterations using infinite matrices, Proc. of an International Conference on Applied Nonlinear Analysis. Academic Press, (1979), 689 - 703.

DEPARTAMENT OF MATHEMATICS, FACULTY OF SCIENCE,
ASSIUT UNIVERSITY, ASSIUT, EGYPT

Received March 14, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Bogdan Rzepecki

ON THE HYPERBOLIC EQUATION $z_{xy} = f(x,y,z)$ IN BANACH SPACES

Let a, b be positive real numbers or positive infinity, $Q = \overline{[0,a) \times [0,b)}$ (here the closure is usual), $(E, \|\cdot\|)$ a Banach space and f a function from $\Omega = Q \times E$ into E. We deal with the Darboux problem

$$\text{(PD)} \qquad \begin{cases} z_{xy} = f(x,y,z) & \text{for } (x,y) \in Q \\ z(x,0) = z(0,y) = 0 & \text{(the zero of E)} \end{cases}$$

in the classical sense, i.e. a solution z and its partial derivatives z_x, z_y, z_{xy} are E-valued continuous functions of x,y.

Let α denote the Kuratowski measure of noncompactness in E (see e.g. [1]). By the fixed point theorem of Sadovskii ([10], Theorem 3.4.3) we prove the existence of solutions of (PD) assuming for f some regularity conditions expressed in terms of α. Our method is closely patterned on Bielecki's norm changing technique in the theory of differential equations.

Assume below that $\|f(x,y,z)\| \le G(x,y,\|z\|)$ for $(x,y,x) \in \Omega$, where G is a nonnegative continuous nondecreasing in the last variable such that the scalar inequality

(s) $$\int_0^x \int_0^y G(s,t,g(s,t))\,dsdt \le g(x,y)$$

or

(s') $$G\Big(x,y,\int_0^x \int_0^y g(s,t)\,dsdt\Big) \le g(x,y)$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

has a locally bounded solution, say $g_0(x,y)$, on Q. Moreover, we shall mean by $\mathfrak{X}$ the set of all $z \in C(Q,E)$ (the standard Fréchet space of all continuous E-valued functions defined on Q) with

$$\|z(x,y)\| \leq g_0(x,y) \quad \text{for } (x,y) \in Q.$$

P. Negrini [5] studied the Darboux problem for the equation $z_{xy} = f(x,y,z)$ in Banach spaces applying the method of Euler polygons. The following theorem extend his result.

Theorem 1. Let f be continuous on Ω and (+) hold. If for each compact subset P of Q there exists a nonnegative constant k(P) (depending on P) such that

$$\alpha(f[P\times W]) \leq k(P)\alpha(W)$$

for any nonempty bounded subset W of E, then (PD) has at least one solution in $\mathfrak{X}$.

The proof of this theorem, although more difficult, resembles one given in [2] (cf. [7] - [9]) and therefore if will be omitted.

Theorem 2. Let f be uniformly continuous on bounded subsets of Ω and (++) hold. Suppose that

$$\alpha\,(f[x,y,W]) \leq L(x,y)\ \alpha(W)$$

for $(x,y) \in Q$ and a nonempty bounded subset W of E, where L is a nonnegative Lebesgue locally integrable function. Then there exists a solution of (PD) in the set $\mathfrak{X}$.

Proof. Without loss of generality we may assume that a, b are positive infinity. Put

$$(Fz)(x,y) = (x,y, \int_0^x \int_0^y z(s,t)\,dsdt \quad \text{for } z \in C(Q,E).$$

Let $P_n = [0,n] \times [0,n]$ $(n = 1,2,\ldots)$ and $\omega_n(\sigma)$, $\sigma \geq 0$, denote the supremum of the numbers $\|(Fz)(x',y') - (Fz)(x'',y'')\|$ with $z \in \mathfrak{X}$ and $|x' - x''| + |y' - y''| \leq \sigma$, whenever (x',y'), $(x'',y'') \in P_n$. Consider the set $\mathfrak{X}_0$ of all $z \in \mathfrak{X}$ such that

$$\|z(x',y') - z(x'',y'')\| \leq \omega_n(|x' - x''| + |y' - y'') \quad (n = 1,2,\ldots)$$

for (x',y'), $(x'',y'') \in P_n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Obviously, $\mathfrak{X}_0$ is a closed convex subset of $C(Q,E)$ and F is a continuous mapping of $\mathfrak{X}_0$ into itself. Since f is uniformly continuous on $Q \subset \Omega$, so $\lim_{\sigma \to 0+} \omega_n(\sigma) = 0$ $(n \geq 1)$ and $\mathfrak{X}_0$ is equicontinuous in the topology of $C(Q,E)$.

Define $\Phi(Z) = (\Phi_1(Z), \Phi_2(Z), \ldots)$ for a nonempty subset Z of $\mathfrak{X}_0$. Here

$$\Phi_n(Z) = \sup_{(x,y) \in P_n} \exp\left(-r_0 \int_0^x \int_0^y L(s,t)\,dsdt\right) \int_0^x \int_0^y \alpha\big(Z(s,t)\big)\,dstd$$

with a nonnegative constant r_0 and $Z(t,s)$ standing for the set of all $z(s,t)$ such that $z \in Z$.

Denote by S_∞ the set of all nonnegative real sequences with standard partial order. Evidently, $\Phi(Z) \in S_\infty$. By the properties of α and the Ascoli theorem, $\Phi(\overline{\text{conv}}\ Z) = \Phi(Z)$ ($\overline{\text{conv}}\ Z$ is the closed convex hull of Z), $\Phi(\{z\} \cup Z) = \Phi(Z)$ for $z \in \mathfrak{X}_0$, and if $\Phi(Z) = \mathbb{0}$ (the zero sequence) then Z is conditionally compact in $C(Q,E)$. Therefore the Sadovskii theorem states that F has fixed points provided that $\Phi(F(Z)) < \Phi(Z)$ for arbitrary nonempty subset Z of $\mathfrak{X}_0$ with $\Phi(Z) > \mathbb{0}$.

Let Z be a nonempty subset of $\mathfrak{X}_0$. Fix a positive integer n and (x,y) in P_n. By immediate adaptation of corresponding result of [3], we get

$$\alpha\left(\int_0^x \int_0^y Z(s,t)\,dsdt\right) = \alpha\left(\left\{\int_0^x \int_0^y z(s,t)\,dsdt : z \in Z\right\}\right) \leq$$

$$\leq \int_0^x \int_0^y \alpha\,\big(Z(s,t)\big)\,dsdt.$$

Therefore

$$\int_0^x \int_0^y \alpha\big(F[Z](s,t)\big)\,dsdt \leq$$

$$\leq \int_0^x \int_0^y L(s,t)\ \alpha\left(\int_0^x \int_0^t Z(u,v)\,dudv\right)dsdt \leq$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq \int_0^x \int_0^y L(s,t) \left(\int_0^s \int_0^t \alpha(Z(u,v))dudv \right) dsdt \leq$$

$$\leq \Phi_n(Z) \cdot \int_0^x \int_0^y L(s,t) \exp\left(r_0 \int_0^x \int_0^t L(u,v)dudv \right) dsdt \leq$$

$$\leq r_0^{-1} \cdot \Phi_n(Z) \cdot \exp\left(r_0 \int_0^x \int_0^y L(s,t)dsdt \right) .$$

If $r_0 > 1$, the last inequality implies $\Phi(F[Z]) < \Phi(Z)$, whenever $\Phi(Z) > 0$. Thus, all assumptions of the Sadovskii theorem being satisfied, F has a fixed point in X_0 and the proof is complete.

We conclude with some application of Theorem 1 to the theory of multivalued hyperbolic equation.

Denote by CC(E) the family of all nonempty closed convex subset of E. We put $\| W \| = \sup \{\| w \| : w \in W\}$ for $W \in 2^E$ (the family of all nonempty subsets of E), and write d(z,W) for the distance $\inf \{\| z - w \| : w \in W\}$ from the point $z \in E$ to the set $W \in 2^E$.

Let Γ be a mapping from Ω to CC(E) such that $\xi \longrightarrow d\big(z,\Gamma(\xi)\big)$ is a continuous function for each $z \in E$. Suppose that $\| \Gamma(x,y,z) \| \leq G(x,y,\|z\|)$ on Ω, where G is a continuous function nondecreasing in the last variable and the inequality (+) has a locally bounded solution, say $g_0(x,y)$, existing on Q.

Let $\mu > 1$ and ω is a E-valued continuous function defined on Ω. Applying Lemma 7.1 of Michael [4] and modifying the reasoning from the proof of Lemma [6], we infer that there is continuous selection h from the multivalued mapping Γ satisfying

$$\| h(x,y,z) - \omega(x,y,z) \| \leq \mu \cdot d\big(\omega(x,y,z), \Gamma(x,y,z)\big)$$

for every (x,y,z) in Ω.

Assume in addition that for each compact subset P of Q there exists a constant $k(P) \geq 0$ such that $\alpha(\Gamma[P \times W]) \leq k(P)\alpha(W)$ for any bounded $W \in 2^E$. Then the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

multivalued partial differential problem

$$\begin{cases} z_{xy} \in \Gamma(x,y,z) & \text{for } (x,y) \in Q \\ z(x,0) = z(0,y) = 0 \end{cases}$$

has a classical solution z with $\| z(x,y) \| \leq g_0(x,y)$ and

$$\| z(x,y) - \omega(x,y,z(x,y)) \| \leq$$

$$\leq \mu \cdot d\Big(\omega(x,y,z(x,y)),\ \Gamma(x,y,z(x,y))\Big)$$

for $(x,y) \in Q$.

Indeed, according to Theorem 1 applied for $f = h$, there exists a function $z \in \mathcal{X}$ which satisfies the equation $z_{xy} = h(x,y,z)$ with $z(x,0) = z(0,y) = 0$. Since

$$h(x,y,z(x,y)) \in \Gamma(x,y,z(x,y))$$

and

$$\| h(x,y,z(x,y)) - \omega(x,y,z(x,y)) \| \leq$$

$$\leq \mu \cdot d\Big(\omega(x,y,z(x,y)),\ \Gamma(x,y,z(x,y))\Big)$$

on Q, so we are done.

REFERENCES

[1] J. Banaś, K. Goebel : Measure of noncompactness in Banach spaces, Lect. Notes Pure Applied Math. 60, Marcel Dakker, New York 1980.

[2] M. Dawidowski, I. Kubiaczyk, B. Rzepecki : An existence theorem for the hyperbolic equation $z_{x,y} = f(x,y,z)$ in Banach spaces, Demonstratio Math. 20 (1987) 489 - 493.

[3] K. Goebel, W. Rzymowski : An existence theorem for the equation $x' = f(t,x)$ in Banach spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 18 (1970) 367 - 370.

[4] E. Michael : Continuous selections I, Ann. Math. 63 (1956) 361 - 382.

[5] P. Negrini : Sul problema Darboux negli spazi di Banach, Boll. Un. Mat. Ital. (5) 17-A (1980) 156 - 160.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[6] **B. Rzepecki** : A fixed point theorem of Krasnoselskii type for multivalued mappings, Demonstratio Math. 17 (1984) 767 - 776.

[7] **B. Rzepecki** : An existence theorem for bounded solutions of differential equations in Banach spaces, Rend. Sem. Mat. Univ. Padova 73 (1985) 89 - 94.

[8] **B. Rzepecki** : An application of a fixed point principle of Sadovskii to differential equations on the real line, Comm. Math. Univ. Carolinae 26 (1985) 713 - 717.

[9] **B. Rzepecki** : On hyperbolic partial differential equations in Banach spaces, Rend. Atti Acc. Naz. Lincei 80 (1986) 540 - 544.

[10] **B. N. Sadovskii** : Limit compact and condensing operators, Russian Math. Surveys 27 (1972) 86 - 114.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY,
62-246 ZIELONA GÓRA, POLAND

Received March 24, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

T. D. Narang , S. K. Garg

FARTHEST POINTS IN ULTRAMETRIC SPACES

The theory of farthest points in normed linear spaces or metric spaces is comparatively less developed than that of the theory of nearest points. Only a few mathematicians have persued this study in sporadic attempts. For a survey of the literature on farthest points we refer to Narang [2]. In this note we shall discuss farthest points in ultrametric spaces. To begin with, we recall a few definitions.

Definition 1. Let X be any non-empty set. A function $d: X \times X \longrightarrow \mathbb{R}$ is said to be an ultrametric on X if for all x,y,z in X,

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,y) \leq \max \{d(x,z), d(z,y)\}$.

The pair (x,d) is called an ultrametric space.

Clearly every non-archimedean normed linear space (see e.g.[3]) is an ultrametric space with $d(x,y) = \| x-y \|$. Also every ultrametric space is a metric space but not conversely.

Definition 2. Let K be subset of an ultrametric space (X,d). An element $k \in K$ is said to be farthest to $p \in X$ if for all $x \in K$,

(*) $$d(p,k) \geq d(p,x) \quad \text{i.e.,} \quad d(p,k) = \sup_{x \in K} d(p,x)$$

If such a k exists for each p in X, then K is said to be remotal and if it is also unique for each p in X, then K is

The first author is thankful to the U.G.C. for financial support.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

said to be uniquely remotal.

Certain results on tho existence of points in metric spaces were proved by Ahuja, Narang and Trechan [1] (e.g. it was shown in [1] that every compact set in a metric space is remotal). These results also hold in ultrametric spaces.

Next, we make a few observations from (*) using the fact that d is an ultrametric.

(1) If $k \in K$ is farthest to $p \in X$ then (*) implies

(**) $d(k,x) \leq \max \{d(k,p), d(p,x)\} = d(k,p)$ for all $x \in K$

(2) From (*) and (**), we obtain

$$d(p,k) \geq \max \{d(p,x), d(k,x)\}.$$

But $d(p,k) \leq \max \{d(p,x), d(k,x)\}$ is always true. Therefore $k \in K$ is farthest to p if for all $x \in K$ we have

$$d(p,k) = \max \{d(p,x), d(k,x)\}.$$

Geometrically, this equality means that for all $x \in K$, triangle x k p is either an equilateral triangle, or an isosceles triangle, of which k p is of the equal sides.

(3) From the inequality (**) we also have:

If $k \in K$ is farthest to p and $p \in K$, the p is farthest to k.

The following result shows that if $k \in K$ is farthest to $p \in X$, then k is also farthest to all those points of X whose distances from k are not less than of p from k.

Theorem 1. Let K be a subset of an ultrametric space (X,d) and $k \in K$ be farthest to $p \in X$. If $q \in X$ satisfies $d(q,k) \geq d(p,k)$, then k is also farthest to q.

Proof. Let $x \in K$, then

$$d(x,q) \leq \text{Max } \{d(x,k), d(k,q)\} \leq$$
$$\leq \text{Max } \{d(p,k), d(k,q)\}, \text{ as } d(p,k) \geq d(x,k) =$$
$$= d(k,q).$$

Therefore $k \in K$ is farthest to q.

In particular, if $p \in K$, then we have the following:

Theorem 2. If $k \in K$ is farthest to $p \in K$ and $q \in X$ be such that $d(k,q) < d(p,k)$, then p is farthest to q.

Proof. Firstly, we observe that

$$d(k,q) < d(p,k) \Longrightarrow d(p,q) = d(p,k).$$

Now for $x \in K$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$d(x,q) \leq \max\{d(k,x), d(k,q)\} \leq$
$\leq \max\{d(k,x), d(p,k)\}$, as $d(p,k) > d(k,q) =$
$= d(p,k)$, as k is farthest to p and so $d(p,k) \geq d(k,x) =$
$= d(p,q)$.

Remark. A consequence of Theorems 1 and 2 is that even if one element of K has point farthest to it in K, then every point of X has a farthest point in K i.e. is remotal.

It is worth noting that such a result is not true in metric spaces as shown by the following example.

Example 1. Let $\mathbb{R}$ be the set of real numbers with the usual metric. Take $K = [0,1]$. Then for all the points in $[\frac{1}{2},1]$, 0 is the farthest point. But no point interval $[0,\frac{1}{2}]$, has a farthest point in K.

Regarding the uniqueness of farthest points, we have the following:

Lemma. Let $k \in K$ be farthest to $p \in X$. If $t \in K$ is such that $d(t,k) < (d(p,k)$, then t is also farthest to p.

Proof. Since $d(t,k) < d(p,k)$ we have

$$d(p,t) = d(p,k) = \sup_{x \in K} d(p,x).$$

Remark. If $k \in K$ is farthest to $p \in X$ then an element $t \in K$ satisfying $d(t,k) > d(p,k)$ can not exist because this will mean that $d(t,p) = d(t,k) > d(p,k)$.

From this remark and the above lemma, we have the following theorem

Theorem 3. (i) If $k \in K$ is unique farthest point to $p \in X$, then there is no $t \in K$ such that $d(t,k) < d(p,k)$,
(ii) If there is no $t \in K$ such that $d(t,k) \leq d(p,k)$ then $k \in K$ is unique farthest point to $p \in X$.

The following example shows that in the second part of Theorem 3, the inequality cannot be replaced by strict inequality.

Example 2. Let X be the space of rational numbers with the usual 3-adic valuation (Let $0 < \alpha < 1$ be a fixed real number. Let $3^n \frac{p}{q}$, $3 \times p$, $3 \times q$, p and q integers, be a non-zero

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

rational number. Then 3-adic valuation of $3^n \frac{p}{q}$ is defined as $|3^n \frac{p}{q}|_3 = \alpha^n, |0|_3 = 0)$.

Let K = {3,6}, p = 0, t = 6, k = 3. Then

$$d(t,k) = |6-3|_3 = |3|_3 = \alpha,$$

$$d(p,k) = |6-3|_3 = |3|_3 = \alpha,$$

and

$$d(p,t) = |6-0|_3 = |6|_3 = \alpha.$$

Thus k and t are both farthest points to p.

REFERENCES

[1] **G.C. Ahuja, T.D. Narang:** Swaran Trehan: On existence of farthest points, Math. Student, 43(1975), 443-446.

[2] **T. D. Narang:** A study of parthest points, Nieuw Archief Voor Wiskunde, 25(1977), 54-79.

[3] **A. C. M. Van Rooij:** Non-archimedean functional analysis, Marcel Dekker Inc., New York (1978).

DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, ARMITSAR - 143005 (INDIA).

Received March 30, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Mariusz Strześniewski

AN EXAMPLE OF AMBIVALENT, APPROXIMATELY CONTINUOUS BUT NOT STRONGLY BAIRE 1 FUNCTION

R.J.O'Malley [1] investigated two classes of functions, important in relation to the r topology. These are: the class strongly Baire 1, approximately continuous functions and the class of ambivalent approximately continuous functions. It is also shown in [1], that every strongly Baire 1 function is ambivalent. Professor Z.Grande posed a question of the existence of a function which is ambivalent and approximately continuous but not strongly Baire 1 function. In this paper we construct such a function.

1. Preliminaries

All functions will be real-valued and defined on the interval $[0,1]$. The restriction of a function f to a set A is denoted by $f_{|A}$. The metric density of a set A at a point x is denoted by $d(x,A)$, analogously the lefthanded density is denoted by $d_-(x,A)$. N and R denote the sets to all natural and real numbers, respectively. A set is said to be ambivalent if it is both an F_δ - set and G_δ- set. A function f is said to be ambivalent if for every $a \in R$ the sets $\{x:f(x)< a\},\{x:f(x) >a\}$ are ambivalent. A function f is strongly Baire 1 if every nonempty closed set $F \subseteq [0,1]$ there is an open interval (a,b) with $F \cap (a,b) \neq 0$ such that the restriction of f to $F \cap (a,b)$ is continuous. Let C denote the Cantor set and let $\{I_n = (a_n,b_n)\}_{n=1}^{\infty}$ be a sequence of all contiguous to C (in $[0,1]$) open intervals.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 1. There exists a double-indexed sequence $\{I_{n,k}\}$ of contiguous to C open intervals such that:

(i) $d_-(a_n, \bigcup_{k=1}^{\infty} I_{n,k}) = 1$ for every $n \in N$,

(ii) a_n is the only accumulation point of the set $\bigcup_{k=1}^{\infty} \{a_{n,k}, b_{n,k}\}$ for every $n \in N$,

(iii) if $(n,k) \neq (m,l)$ then $I_{n,k} \cap I_{m,l} = \emptyset$ for all $(n,k),(m,l) \in N \times N$,

(iv) $\bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{n,k}$.

Lemma 2. For the double - indexed intervals $I_{n,k}$ of Lemma 1 there exist double - indexed sequences $\{c_{n,k}\}$, $\{d_{n,k}\}$, $\{e_{n,k}\} \subseteq R$ such that:

(i) For every $n, k \in N$, $a_{n,k} < d_{n,k} < c_{n,k} < e_{n,k} < b_{n,k}$,

(ii) For every $n \in N$, $d_-\left(a_n, \bigcup_{k=1}^{\infty} (c_{n,k}, e_{n,k})\right) = 1$,

(iii) For every $x \in C$, $d\left(x, \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (d_{n,k}, c_{n,k})\right) = 0$.

Proof. It is not hard be satisfy (ii). In order to get (iii) it is sufficient to assume that $c_{n,k} - d_{n,k} < (d_{n,k} - a_{n,k})/(nk)$ and $c_{n,k} - d_{n,k} < (b_{n,k} - c_{n,k})/(nk)$ for $n,k \in N$.

2. The main result

Theorem. There exists $f : [0,1] \longrightarrow R$ which is ambivalent and approximately continuous but not strongly Baire 1 function.

Proof. Let the double - indexed sequences of reals $\{a_{n,k}\}$, $\{b_{n,k}\}$, $\{c_{n,k}\}$, $\{d_{n,k}\}$, $\{e_{n,k}\}$ be chosen as in Section 1.

Let $g_{n,k}$ be a function on $[a_{n,k}, b_{n,k}]$ as follows,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$g_{n,k}(x) = \begin{cases} b_{n,k}-a_{n,k} & \text{if } x \in [a_{n,k}, d_{n,k}], \\ b_n-a_n & \text{if } x \in [c_{n,k}, e_{n,k}], \\ 1 & \text{if } x = (d_{n,k}+c_{n,k})/2, \\ 0 & \text{if } x = b_{n,k}, \\ g_{n,k} \text{ is linear on } & [d_{n,k}, (d_{n,k}+c_{n,k})/2], \\ [(d_{n,k}+c_{n,k})/2, c_{n,k}], & [e_{n,k}, b_{n,k}], \text{ respectively.} \end{cases}$$

Define,

$$g(x) = \begin{cases} g_{n,k}(x) & \text{if } x \in [a_{n,k}\ b_{n,k}], \\ 0 & \text{if } x \in C \setminus \bigcup_{n=1}^{\infty} \{a_n\}. \end{cases}$$

Obviously, g is an approximately continuous function. We put $f(x) = x + g(x)$ for every $x \in [0,1]$. The function f is approximately continuous as the sum of two approximately continuous functions. For the proof that f is ambivalent, let $a \in R$ and $f^a = \{x: f(x) < a\}$. If $a \leq 0$ then $f^a = \emptyset$. If $a > 0$ then f^a is a sum of open intervals and a set of the form $[0,a) \cap C$. Hence f^a is both an F_σ - set and a G_δ -set. Analogously $f_a = \{x: f(X) > a\}$ is empty or it is a sum of open intervals and a set of the form $(a,1] \cap C$, hence f_a is ambivalent. Because $f_{|C}$ is discontinuous at every point a_n, n = 1,2,..., and $\bigcup_{n=1}^{\infty} \{a_n\}$ is dense in C we have that f is not strongly Baire 1. This ends the proof.

Let us note that the function g given in the proof is not ambivalent, even though the functions $f.\ id_{|[01]}$ are. It means that the class of ambivalent approximately continuous functions is not closed under the addition of functions.

The author would like to express his hearty thanks to Professor J. S. Lipiński for suggesting this problem and helpful discussions.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] **Richard J. O'Malley:** Approximately differentiable functions: the r topology, Pacific J.Math.,72 (1977).

INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDAŃSK,
80-952 GDAŃSK, POLAND

Received April 7, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Danuta Ozdarska

ON THE RELATIONSHIP BETWEEN BOUNDED SOLUTIONS OF LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

Let $J = \langle 0, \infty)$ and let E be a Banach space with norm $\|\cdot\|$. In this paper we study some properties of bounded solutions of the quasilinear differential equation

(1) $$x' = A(t)x + f(t,x)$$

depending on properties of the equation

(2) $$x' = A(t)x .$$

More precisely, the third part of the work gives sufficient conditions for the existence of bounded solutions of (1). Moreover, in the fourth part it is shown that under similar assumptions for each bounded solution $v : J \longrightarrow E$ of (2) there exists a bounded solution u of (1) such that

$$\lim_{t\to\infty} \|v(t) - u(t)\| = 0$$

Our considerations are concerned with the well-known Massera-Schäffer book [8], and also with some works of Conti, Hartman, Lovellady and Szufla [2, 3, 7, 12]. Our methods of proving are quite different from those used in the papers mentioned above. They are based on the use of a recent Heinz theorem [4] on the measure of noncompactness of sets of integrable functions, and a certain modification of the Mönch fixed point theorem [9].

Let us recall that the Kuratowski measure of noncompactness of the bounded subset X of the space E is called the infimum $\alpha(X)$ of all $\varepsilon > 0$ such that there exists a finite covering of X by the sets of diameter $\leq \varepsilon$.

Let T be a measurable subset of R. For any set W of strongly measurable function $x : T \longrightarrow E$ let us put $W(t) = \{x(t) : x \in W\}$, $(t \in T)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1 (Heinz [4]). Suppose that W is a countable set of strongly measurable functions from T into E such that there exists $g : T \to R$ such that $\|x(t)\| \le g(t)$ for all $t \in T$ and $x \in W$. Then the function $t \to \alpha(W(t))$ is integrable on T and

$$\alpha(\{\int_T x(s)ds : x \in W\}) \le 2\int_T \alpha(W(s))ds .$$

Repeating Mönch's argumentation from the proof of Theorem 2.2 from [9] and using the Schauder-Tichonow theorem instead of the Schauder theorem, we can obtain the following fixed point theorem.

Theorem 2. Assume that:

- Y is a B_0 space;
- K is a closed, convex and bounded subset of Y;
- F is a continuous mapping from K into itself satisfying the following condition:

there exists $x_0 \in K$ such that for any countable subset X of K the following implication holds

$$\overline{X} = \overline{\text{conv}}(\{x_0\} \cup F(X)) \Longrightarrow X \text{ is relatively compact.}$$

Then F has a fixed point in K.

2. Preliminaries

Likewise in the book [8] we introduce now the following denotations:

- $\tilde{E}$ - the space of continuous linear mappings $E \to E$;
- μ - the Lebesgue measure in R
- $L^1(J,E)$ - the space of Bochner integrable functions $x : E \to E$, provided with the norm $\|x\|_1 = \int_J \|x(s)\|ds$;
- L(J,E) - the space of strongly measurable functions $x : E \to E$, Bochner integrable in every finite subinterval J' of J, with the topology of the convergence in the mean on every such J', i.e. convergence in $L^1(J',E)$ of the restriction to J' ;
- C = C(J,E) - the space of continuous functions $J \to E$ with the topology of uniform convergence on compact subintervals of J;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

- $B = B(J,E)$ - the Banach function space of all strongly measurable functions $x : J \to E$ such that $\|x\| \in B(J,R)$, provided with the norm $\|x\|_B = \|\|x\|\|_B$, where $B(J,R)$ is a Banach function space such that:

(i) $B(J,R) \subset L(J,R)$ and $B(J,R)$ is stronger than $L(J,R)$;

(ii) $B(J,R)$ is not stronger than $L^1(J,R)$;

(iii) $B(J,R)$ contains all essentially bounded functions with compact support;

(iv) if $u \in B(J,R)$ and v is a real-valued measurable function on J such that $|v| \leq |u|$, then $v \in B(J,R)$ and $|v|_B \leq \|u\|_B$;

(v) if $u \in B(J,R)$ and (v_n) is a sequence in $L(J,R)$ such that $|v_n| \leq u$ and $\lim_{n\to\infty} v_n(t) = 0$ almost everywhere on J, then $\lim_{n\to\infty} \|v_n\|_B = 0$;

(vi) if (u_n) is a sequence in $B(J,R)$ such that $\lim_{n\to\infty} u_n(t) = 0$ for almost every $t \in J$, and (u_n) has equi-absolutely continuous norms in $B(J,R)$, then $\lim_{n\to\infty} \|u_n \chi_{<0,a)}\|_B = 0$ for any $a > 0$;

- B' - the associate space to B ([8], p.50).

Suppose now that $A \in L(J,\tilde{E})$. Let E_0 denote the set of all points of E which are values for $t = 0$ of bounded solutions of (2). We assume that E_0 is closed and has a closed complement E_1. Let P be the projection of E onto E_0. Moreover, let $U : J \to \tilde{E}$ denote the solution of the differential equation $U' = A(t)U$ with the initial condition $U(0) = I$. By Theorem 33.B of [8] there exists a positive number M such that

(3) $$\|U(t)P\| \leq M \quad \text{for } t \in J.$$

Moreover, by Theorem 62.D of [8], we have the inequality

(4) $$\|U(t)P\| \leq N\|P\|e^{-\nu t} \quad \text{for } t \in J,$$

where N and ν are some positive numbers. Let us put

$$G(t,s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \leq s \leq t \\ -U(t)(I-P)U^{-1}(s) & \text{for } 0 \leq t \leq s. \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Throughout the whole work we assume that:

I. $G(t,\cdot) \in B'$ and $\|G(t,\cdot)\|_{B'} \leq k$ for every $t \in J$;

II. $p : J \to J$ is a nondecreasing continuous function such that $p(0) = 0$ and $p(r) < r$ for all $r > 0$;

III. $(t,x) \to f(t,x)$ is a function from $J \times E$ into E which satisfies the Caratheodory conditions;

IV. h is a nonnegative function belonging to $B(J,R)$ and such that $2k\|h\|_B \leq 1$, and

$$\alpha(f(t,X)) \leq h(t)p(\alpha(X)) \quad \text{for } t \in J \text{ and } X \subset E.$$

3. The existence of bounded solutions

Theorem 3. If the conditions I - IV hold and there exists a function $m \in B(J,R)$ such that

(5) $\quad k\|m\|_B \leq b$ and $\|f(t,x)\| \leq m(t)$ for $(t,x) \in J \times V$,

where $V = \{x \in E : \|x\| \leq b\}$, then for each $x_0 \in E_0$ such that $\|x_0\| \leq (b - k\|m\|_B)/M$ there exists at least one bounded solution of (1) such that $Px(0) = x_0$.

Proof. Let $\tilde{V}$ be the set of all continuous functions $x : J \to V$ such that

$$(6) \qquad \|x(t) - x(\tau)\| \leq b\int_\tau^t \|A(s)\|ds + \int_\tau^t m(s)ds \quad \text{for } 0 \leq \tau \leq t.$$

It is obvious that $\tilde{V}$ is a closed, convex and bounded subset of C.

Next, let us take any $x_0 \in E_0$ such that $\|x_0\| \leq (b - k\|m\|_B)/M$. Then the function $t \to w(t) = U(t)x_0$ is a solution of (2) and, by (3), $\|w(t)\| \leq M\|x_0\|$ for every $t \in J$.

Since the function f satisfies the Caratheodory conditions, for any $x \in \tilde{V}$ the function $s \to f(s,x(s))$ is strongly measurable on J. Additionally, by (5)

$$\|f(s,x(s))\| \leq m(s) \quad \text{for } s \in J.$$

To show the thesis of our theorem, we shall make use of Theorem 2 putting $Y = C$, $K = \tilde{V}$ and

$$F(x)(t) = w(t) + \int_0^{+\infty} G(t,s)f(s,x(s))ds \quad \text{for } x \in K \text{ and } t \in J.$$

At first, we shall show that F is continuous operator and its

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

values belong to K.

By (I), (5) and the Hölder inequality, we have

$$\|F(x)(t)\| \le \|w(t)\| + \int_0^{+\infty} \|G(t,s)\| m(s)ds \le M\|x_0\| + k\|m\|_B \le b \tag{7}$$

and

$$\|F(x)(t) - F(y)(t)\| \le \int_0^{+\infty} \|G(t,s)\| \|f(s,x(s)) - f(s,y(s))\| ds \le \tag{8}$$

$$\le k\|f(\cdot,x) - f(\cdot,y)\|_B$$

for each $x,y \in K$ and $t \in J$.

In addition, the function $z = F(x)$ is a solution of the differential equation $z' = A(t)z + f(t,x(t))$. Thus, by (7), (5) and the mean value theorem, we conclude that

$$\|F(x)(t) - F(x)(\tau)\| \le b\int_\tau^t \|A(s)\| ds + \int_\tau^t m(s)ds$$

for $x \in K$ and $0 \le \tau \le t$. From this and (7) we see that $F : K \to K$.

Suppose now that (x_n) is a sequence in K such that $\lim_{n\to\infty} x_n = x$ in C. Because $\lim_{n\to\infty} \|f(s,x_n(s)) - f(s,x(s))\| = 0$ and $\|f(s,x_n(s)) - f(s,x(s))\| \le 2m(s)$, by (v) we get

$$\lim_{n\to\infty} \|f(\cdot,x_n) - f(\cdot,x)\|_B = 0 .$$

Gurukul Kangri Vishwavidyalaya HARDWAR

From the above and (8) it follows that $\lim_{n\to\infty} \|F(x_n)(t) - F(x)(t)\| = 0$ uniformly in $t \in J$. This shows the continuity of F.

Assume now, that $X = \{x_n : n \in N\}$ is a subset of K such that

$$X \subset \overline{\text{conv}}(\{0\} \cup F(X)).$$

We shall prove that X is relatively compact in C.

Let us first notice that by (6) the set K is equicontinuous on every compact subinterval of J. Therefore, by the Ascoli theorem [5, th. 7.18], for any subset X of K, X is relatively compact in C if and only if X(t) is relatively compact in E for every $t \in J$. Fix $t \in J$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

To show that X(t) is relatively compact we shall make use of the properties of α and Theorem 1, putting

$$W = \{G(t,\cdot)f(\cdot,x_n) : n \in N\} \text{ and } T = \langle 0,\infty).$$

It is clear that the function $G(t,\cdot)f(\cdot,x_n) : J \to E$ is strongly measurable for every $n \in N$ and $t \in J$,

$$\|G(t,s)f(s,x_n(s))\| \le \|G(t,s)\|\|f(s,x_n(s))\| \le$$
$$\le \|G(t,s)\|m(s) \text{ for } s \in J$$

and

$$\int_0^{+\infty} \|G(t,s)\|m(s)ds \le k\|m\|_B < b.$$

From this we deduce that

$$\alpha(X(t)) \le \alpha(F(X)(t)) \le \alpha(\{\int_0^{+\infty} G(t,s)f(s,x_n(s))ds : n \in N\}) \le$$

$$\le 2\int_0^{+\infty} \alpha(G(t,s)f(s,X(s)))ds \le 2\int_0^{+\infty} \|G(t,s)\|\alpha(f(s,X(s)))ds \le$$

$$\le 2\int_0^{+\infty} \|G(t,s)\|h(s)p(\alpha(X(s)))ds \le 2k\|h\|_B p(\sup_{s\in J} \alpha(X(s))) \le$$

$$\le p(\sup_{s\in J} \alpha(X(s))).$$

Since the above inequalities are satisfied for any $t \in J$, we obtain

$$\sup_{s\in J} \alpha(X(s)) \le p(\sup_{s\in J} \alpha(X(s))).$$

As $p(r) < r$ for $r > 0$, from the above inequality we conclude that $\alpha(X(t)) = 0$ for every $t \in J$. On the basis of the earlier argumentation, the above equality proves that X is relatively compact in C. Consequently, by Theorem 2 we deduce that F has a fixed point in K, i.e. there exists $x : J \to E$ such that $F(x) = x$. It is obvious that x is a solution of (1), $\sup_{t\in J} \|x(t)\| \le b$ and $Px(0) = x_0$ because

$$x(0) = F(x)(0) = x_0 - (I - P)\int_0^{+\infty} U^{-1}(s)f(s,x(s))ds.$$

This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. The asymptotic relationship between solutions of linear and quasilinear differential equations

Now, we give some conditions which guarantee that for each bounded solution $v : J \to E$ of (2) there exist $a > 0$ and a solution $u : \langle a,\infty) \to E$ of (1) such that

$$\lim_{t\to\infty} \|v(t) - u(t)\| = 0. \tag{9}$$

Next, by appropriate assumptions we extend the above solution v of (1) on whole interval J such that the condition (9) will be true.

Theorem 4. We assume that the conditions I - IV hold and $(t,r) \to \omega(t,r)$ is a function from $J \times J$ into J such that

(10) ω is measurable in t and nondecreasing in r;

(11) for each $r > 0$ the function $\omega(\cdot,r) \in B(J,R)$ and

$$\lim_{\mu(z)\to 0} \|\omega(\cdot,r)\chi_Z\|_B = 0 \text{ and } \lim_{\alpha\to\infty} \|\omega(\cdot,r)\chi_{\langle d,\infty)}\|_B = 0;$$

(12) $\|f(t,x)\| \le \omega(t,\|x\|)$ for $(t,x) \in J \times E$.

Then for each bounded solution $v : J \to E$ of (2) there exist $a > 0$ and a solution $u : \langle a,\infty) \to E$ of (1) such that the condition (9) holds.

Proof. Let v be a bounded solution of (2) such that $\|v(t)\| \le b$ for $t \in J$.

By (11) there exists $a > 0$ such that

$$\|\omega(\cdot,2b)\chi_{\langle a,\infty)}\|_B \le b/k. \tag{13}$$

In this proof we make use of an application of Theorem 2 putting: Y is the space of all continuous functions $x : J_a = \langle a,\infty) \to E$, with the topology of uniform convergence on compact subintervals of J_a, and K is the set of all $x \in Y$ which satisfy the inequalities

$$\|x(t)\| \le 2b \quad \text{for } t \in J_a \tag{14}$$

and

$$\|x(t) - x(\tau)\| \le 2b\int_\tau^t \|A(s)\|ds + \int_\tau^t \omega(s,2b)ds \tag{15}$$

for $t \ge \tau \ge a$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Moreover, F is defined by

$$F(x)(t) = v(t) + \int_a^{+\infty} G(t,s)f(s,x(s))ds \quad \text{for } x \in K \text{ and } t \in J_a$$

It is clear that Y is B_0 space and K is closed, convex and bounded.

We shall prove that F is continuous and its values belong to K. By (I), (11), (12), (13) and the Hölder inequality, we have

$$(16) \qquad \|F(x)(t)\| \le \|v(t)\| + \int_a^{+\infty} \|G(t,s)\|\omega(s,2b)ds \le$$

$$\le b + \|G(t,\cdot)\|_{B'} \|\omega(\cdot,2b)\chi_{<a,\infty)}\|_B \le 2b$$

and

$$(17) \qquad \|F(x)(t) - F(y)(t)\| \le k\|f(\cdot,x) - f(\cdot,y)\|_B$$

for all $x,y \in K$ and $t \in J_a$. In addition, the function $z = F(x)$ is a solution of the differential equation $z' = A(t)z + f(t,x(t))$. Hence, by (12), (16) and the mean value theorem

$$\|F(x)(t) - F(x)(\tau)\| \le 2b \int_\tau^t \|A(s)\|ds + \int_\tau^t \omega(s,2b)ds$$

for $x \in K$ and $t \ge \tau \ge a$. From the above and (16) we infer that $F : K \longrightarrow K$.

Next, let us take any sequence (x_n) in K such that $\lim_{n\to\infty} x_n = x$ in Y. From (11) it follows that for a given $\varepsilon > 0$ we can choose $d \ge a$ such that $\|\omega(\cdot,2b)\chi_{<d,\infty)}\|_B \le \varepsilon/2k$. On the other hand $\lim_{n\to\infty} \|f(s,x_n(s)) - f(s,x(s))\| = 0$ and

$$\|f(s,x_n(s)) - f(s,x(s))\| \le 2\omega(s,2b) \quad \text{for } s \in J_a.$$

Hence, by (11) and (vi)

$$\lim_{n\to\infty} \|(f(\cdot,x_n) - f(\cdot,x))\chi_{<a,d)}\|_B = 0.$$

Therefore there exists n_0 such that

$$\|(f(\cdot,x_n) - f(\cdot,x))\chi_{<a,d)}\|_B \le \varepsilon/2k \quad \text{for } n \ge n_0.$$

Thus, for $n \ge n_0$

$$\|f(\cdot,x_n) - f(\cdot,x)\|_B \le$$

$$\le \|(f(\cdot,x_n) - f(\cdot,x))\chi_{<a,d)}\|_B + \|\omega(\cdot,2r)\chi_{<d,\infty)}\| \le \varepsilon/k.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Finally, by (17)

$$\|F(x_n)(t) - F(x)(t)\| \le \varepsilon \quad \text{for } n \ge n_0 \text{ and } t \in J_a,$$

which proves the continuity of F.

Let us consider the countable subset $X = \{x_n : n \in N\}$ of the set K such that

$$X \subset \overline{\text{conv}}\,(\{0\} \cup F(X)).$$

We shall show that X is relatively compact in Y applying a similar argumentation as in the proof of Theorem 3. We fix $t \in J_a$. Let

$$W = \{G(t,\cdot)f(\cdot,x_n) : n \in N\} \quad \text{and} \quad T = \langle a,\infty).$$

It is clear that the function $G(t,\cdot)f(\cdot,x_n) : J_a \longrightarrow E$ is strongly measurable for any $n \in N$ and $\|G(t,s)f(s,x_n(s))\| \le$ $\le |G(t,s)\|\omega(s,2b)$ for $s \in J_a$, and

$$\int_a^{+\infty} \|G(t,s)\|\omega(s,2b)\,ds \le k\|\omega(\cdot,2b)\chi_{\langle a,\infty)}\|_B \le b$$

Next, repeating the argument from the proof of Theorem 3, we have

$$\sup_{s\in J_a} \alpha(X(s)) \le p(\sup_{s\in J_a} \alpha(X(s))) .$$

From the above, by properties of p we conclude that X is relatively compact in Y.

Thus, by Theorem 2 we deduce that there exists $u : J_a \longrightarrow E$ such that $F(u) = u$. It is now obvious that u is a solution of (1). It still remains to show that the equality (9) holds.

By (11), for a given $\varepsilon>0$ there exists $d \ge a$ such that

$$\|\omega(\cdot,2b)\chi_{\langle d,\infty)}\|_B \le \varepsilon/2b .$$

On the other hand, from (4) it follows that there exists $t_0 \ge d$ such that

$$\|U(t)P\|\|\int_a^d U^{-1}(s)f(s,u(s))ds\| \le \varepsilon/2 \quad \text{for } t \ge t_0.$$

Since for $t \ge d$

$$u(t) - v(t) = U(t)P\int_a^d U^{-1}(s)f(s,u(s))ds + \int_d^{+\infty} G(t,s)f(s,u(s))ds,$$

by (I), (12) and the Hölder inequality we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|u(t) - v(t)\| \le \|U(t)P\| \left\| \int_a^d U^{-1}(s) f(s,u(s)) ds \right\| +$$

$$+ \|G(t,\cdot)\|_{B'} \|\omega(\cdot,2b)\chi_{<d,\infty)}\|_B \le \varepsilon \quad \text{for } t \ge 0.$$

This shows (9) and close the proof of the theorem.

Theorem 5. Suppose that the assumptions of the theorem 4 hold and there exists nonnegative functions $m,q \in L(J,R)$ such that

(18) $\|f(t,x)\| \le m(t) + q(t)\|x\|$ for all $(t,x) \in J\times E$.

Then for each bounded solution $w : J \longrightarrow E$ of (2) there exists a solution $v : J \longrightarrow E$ of (1) such that (9) holds.

We shall need the following theorem.

Theorem 6. (Harten - Mönch [10]). Assume that :

- the function $g : <t_0,t_0+a>\times B \longrightarrow E$ satisfies the Caratheodory conditions, where $B = \bar{k}(x_0,b) \subset E$, $b > 0$ and $t_0 \in R$;
- there exists a function $\varphi : <t_0,t_0+a>\times J \longrightarrow J$ such that $\alpha(g(t,X)) \le \varphi(t,\alpha(X))$ for all $t \in <t_0,t_0+a>$ and $X \subset B$, and $(t,z) \longrightarrow 2\varphi(t,z)$ is a Kamke function.

Then there exist $\delta > 0$ and the function $x : <t_0,t_0+\delta> \longrightarrow B$ such that x is a solution of the initial value problem

$$x' = g(t,x) , \quad x(t_0) = x_0 .$$

Now, we return to the proof of Theorem 5. Let $w : J \longrightarrow E$ be a bounded solution of (2) and let $u : <a,\infty) \longrightarrow E$ be a solution of (1) given by Theorem 4. Let us consider the Cauchy problem

(19) $x' = g(t,x) , \quad x(0) = U^{-1}(a)u(a) ,$

where

(20) $g(t,x) = -U^{-1}(a-t)f(a-t,U(a-t)x)$

for $t \in <0,a>$ and $x \in E$.

Denote $c = \max \{\|U^{-1}(t)\| : 0 \le t \le a\}$ and $d = \max \{\|U(t)\| : 0 \le t \le a\}$. By the definition of the function g and properties of α and f we obtain

$$\alpha(g(t,X)) = \alpha(U^{-1}(a-t)f(a-t,U(a-t)X) \le c\alpha(f(a-t,U(a-t)X)) \le$$

$$\le ch(a-t)\alpha(U(a-t)X) \le cdh(a-t)\alpha(X)$$

for all $t \in <0,a>$ and $X \subset E$. From the above we obtain

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(21) $\alpha(g(t,X)) \leq \varphi(t,\alpha(X))$ for all $t \in \langle 0,a\rangle$ and $X \subset E$,

where the function $\varphi : \langle 0,a\rangle \times J \longrightarrow J$ is defined by $\varphi(t,z) = cdh(a-t)z$.

Since the function $(t,x) \longrightarrow g(t,x)$ satisfies the Caratheodory conditions and the function $(t,z) \longrightarrow 2\varphi(t,z)$ is a Kamke function, therefore by (21) and the Harten - Mönch theorem we deduce that for any $\tau \in \langle 0,a)$ and $y \in E$ there exists a solution x of the Cauchy problem

$x' = g(t,x)$, $x(\tau) = y$ defined on some interval $\langle \tau, \tau+\delta\rangle$.

On the other hand by (18)

$$\|g(t,x)\| \leq cm(a-t) + cdq(a-t)\|x\| \quad (0 \leq t \leq a\ ;\ x \in E).$$

Thus, by (20), for any solution x defined on $\langle 0,\eta)$, $\eta < a$

$$\|x(t)\| \leq \|x(0)\| + \int_0^t (cm(a-s) + cdq(a-s)\|x(s)\|)ds.$$

So, by the Gronwall lemma, we have

$$\|x(t)\| \leq (\|x(0)\| + \int_0^\eta cm(a-s)ds)\exp(\int_0^t cdq(a-s)ds)$$

for $t \in \langle 0,\eta)$. From the above we deduce that there exists a solution x of (19) defined on $\langle 0,a\rangle$.

Let us notice that the function $t \longrightarrow y(t) = U(t)x(a-t)$ satisfies the integral equation

$$y(t) = U(t)U^{-1}(a)u(a) + \int_a^t U(t)U^{-1}(s)f(s,y(s))ds$$

for $t \in \langle 0,a\rangle$. Thus, the function v defined by

$$v(t) = \begin{cases} y(t)\ ; & 0 \leq t \leq a \\ u(t)\ ; & t \geq a \end{cases}$$

is a solution of (1). This completes the proof.

REFERENCES

[1] **A. Ambrosetti:** Un teorema di esistenza per le equazioni differenziali negli spazi di Banach, Rend. Semin. Mat. Univ. Padova 39 (1967), 349-360.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] **R. Conti:** On the boundedness of solutions of ordinary differential equations, Funkcial. Ekvac. 9 (1966), 23-26.

[3] **P. Hartman:** Ordinary differential equations, New York - London 1964.

[4] **H.P. Heinz:** On the behaviour of measures of noncompactness with respect to differentiation and integration of vector - valued functions, Nonlinear Analysis 7 (1983), 1351-1371.

[5] **I.L. Kelly:** General topology, Toronto - New York - London 1957

[6] **K. Kuratowski:** Topologie, Warszawa 1958.

[7] **D.L. Lovellady:** Nonlinear Stepanoff-bounded perturbation problems, J. Math. Anal. Appl. 50 (1975), 350-360.

[8] **J.L. Massera, J.J. Schäffer:** Linear differential equations and function spaces, New York - London 1966.

[9] **H. Mönch:** Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, Nonlinear Analysis 4 (1980), 10-12.

[10] **H. Mönch, G. Harten:** On the Cauchy problem for ordinary differential equations in Banach spaces, Arch. Math. 39 (1982), 153-160.

[11] **B.N. Sadovskii:** On the fixed point principle, Funkcjon. Analiz. 1 (1967), 74-76.

[12] **S. Szufla:** On the asymptotic relationship between solutions of linear and quasilinear differential equations in Banach spaces, Comm. Math. 26 (1986), 155-162.

INSTITUTE OF MATHEMATICS, TECHNICAL - AGRICULTURAL ACADEMY
85-736 BYDGOSZCZ, POLAND

Received April 10, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

D.Przeworska-Rolewicz and S.Rolewicz

ON THE SPECTRUM OF RIGHT INVERSES

A linear operator A acting in a linear space over the field $\mathcal{F}$ of scalars is said to be a Volterra operator, if $I-\lambda A$ is invertible for all $\lambda \in \mathcal{F}$. The set of all Volterra operators acting in X will be denoted by $V(X)$.

For right inverses of the operator d/dt we have obtained the following results.

Theorem 1 ([4], [5]).

(i) The only continuous Volterra right inverse for the operator d/dt in the spaces

(a) $C([0,1],\mathbb{C})$ of all complex-valued functions, defined and continuous on $[0,1]$,

(b) $C^{\infty}([0,1],\mathbb{C})$ of all complex-valued functions,defined and infinitely differentiable on $[0,1]$,

(c) a locally convex space X of functions such that the space $C^{\infty}([0,1],\mathbb{C})$ of all complex-valued and infinitely differentiable functions defined on $[0,1]$ is continuously embedded in X and dense in X,

are of the form $R = \int_a^t$, where $a \in [0,1]$.

(ii) Let R be a continuous right inverse of the operator d/dt in the space $C^k([0,1],\mathbb{C})$ $(k = 0,1,2,\ldots)$ of all functions k-times continuously differentiable on $[0,1]$. If R has at least $k+2$ eigenvalues, then it has an infinite number of eigenvalues.

(iii) In the space $C([0,1],\mathbb{R})$ there exist continuous right inverses to d/dt which are not of the form $\int_a^t$ for $a \in [0,1]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) If F is an initial operator[1] for d/dt corresponding to a continuous right inverse R which is not of the form $\int_a^t$ for an $a \in [0,1]$, then the initial value problem

(1) $\quad x' - \lambda x = 0, \quad Fx = x_0, \quad (\lambda \in \mathbb{C})$

has infinitely many eigenvalues.

It should be pointed out that the assumption on the continuity of right inverses is essential and that in proofs the fact that $\dim \ker \frac{d}{dt} = 1$ has been strongly used. However, point (i) of Theorem 1 can be generalized for right invertible operators in the following way.

Theorem 2 (cf. Example 8.2.5 in [2]). Let X be a complete linear metric locally convex space over the field $\mathbb{C}$. Suppose that $D \in R(X)$ (i.e. D is a right invertible operator with the domain and range in X), $\dim \ker D = 1$ and F_0 is a continuous initial operator for D corresponding to a continuous right inverse R_0. Write

(2) $\quad P(R_0) = \operatorname{lin}\{R_0^k z : z \in \ker D, k = 0,1,2,\ldots\}$,

(3) $\quad E(R_0) = \operatorname{lin}\{(I-\lambda R_0)^{-1} z : z \in \ker D, \lambda \in \mathbb{C}\}$, if $R_0 \in V(X)$.

(Note that both sets, $P(R_0)$ and $E(R_0)$ do not depend on the choice of R_0). Let $A(\mathbb{R}) = \mathbb{R}_+$ (or $\mathbb{R}$). Suppose that either

(i) $\overline{P(R_0)} = X$ and $\{S_h\}_{h \in A(\mathbb{R})}$ is a strongly continuous semigroup (group) of R_0-shifts

or

(ii) $R_0 \in V(X)$, $\overline{E(R_0)} = X$ and $\{S_h\}_{h \in A(\mathbb{R})}$ is a strongly continuous semigroup (group) of D-shifts.

(cf. Definitions 5.2.1 and 5.2.2 in [2]). Then the only continuous Volterra right inverses of D are the operators

(4) $\quad R_h = R_0 - F_h R_0$, where $R_h = F_0 S_h$, $h \in A(\mathbb{R})$.

Note. If X is a Banach space and R is quasinilpotent,

---

[1] For this notion and other notions and properties connected with right invertible operators we refer to [2].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

then R-shifts and D-shifts coincide (cf. Theorem 8.1.7 in [2]).

Observe that topological assumptions admitted in Theorem 2, as well as the property dim ker D = 1, are necessary in order to prove that the right inverses R_h defined by formula (4) are the only continuous Volterra right inverses. Without these assumptions we can prove the following

Proprosition 1. Suppose that X is a linear space over $\mathbb{C}$, $D \in R(X)$, $\ker D \neq \{0\}$, $R \in V(X)$ is a right inverse and $\{S_h\}_{h \in A(\mathbb{R})}$ is a family of D-shifts. Then the right inverses $R_h = R_0 - F_h R_0$, where $F_h = F_0 S_h$, have not eigenvalues for all $h \in A(\mathbb{R})$.

Proof. By our assumptions, we have $e_\lambda = (I-\lambda R_0)^{-1}$ and

$$F_h e_\lambda(z) = F_0 S_h e_\lambda(z) = F_0 e^{\lambda h} e_\lambda(z) = e^{\lambda h} F_0 e_\lambda(z) = e^{\lambda h} z \neq 0$$

for all $\lambda \in \mathbb{C}$, $h \in A(\mathbb{R})$, $0 \neq z \in \ker D$. This, and Theorem 2.3.4 in [2] together imply that R_h have not eigenvalues.

Now we shall generalize (ii) of Theorem 1 (without strong topological assumptions used in Theorem 2). In order to do this we shall prove the following

Theorem 3. Suppose that X is a linear space over $\mathcal{F}$ (where either $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$), $D \in R(X)$, $\dim \ker D \neq 0$ and F_0 is an initial operator for D corresponding to a Volterra right inverse R_0. Let $k = 0,1,2,\ldots,\infty$ be arbitrarily fixed and let be given a polynomial

$$(5) \quad a(t) = \sum_{m=0}^{n} a_m t^m, \quad a_m \in \mathcal{F} \quad (0 \leq n \leq k; \text{ if } k = \infty, \text{ then } n < \infty).$$

Then for every $0 \leq n \leq k$ the operator

$$(6) \quad R = R_0 + F_0 a(D) = R_0 + \sum_{m=0}^{n} a_m F_0 D^m$$

is a right inverse to D in the space D_k, where $D_0 = X$, $D_k = \operatorname{dom} D^k$ for $k \in \mathbb{N}$ and $D_\infty = \bigcap_{k \geq 0} D_k$. Moreover

$$(7) \quad (I - \lambda R) e_\lambda(z) = [1 - \lambda a(\lambda)] z \quad \text{for all } \lambda \in \mathcal{F},\ z \in \ker D,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where

(8) $e_\lambda = (I - \lambda R_0)^{-1}$ for $\lambda \in \mathcal{F}$.

Proof. By our assumptions, $DR_0 = I$, $F_0R_0 = 0$ and $DF_0 = 0$. The operator R maps D_k into itself. Hence R is a right inverse of D in D_k. If the operator R has an eigenvector, it must be of the form $e_\lambda(z)$, where $z \in \ker D$ (cf. Theorem 2.3.2 in [2]). We also have

(9) $a(D)e_\lambda(z) = a(\lambda)e_\lambda(z)$ for all $z \in \ker D$, $\lambda \in \mathcal{F}$.

This implies that for all $z \in \ker D$, $\lambda \in \mathcal{F}$ we have

$$(I - \lambda R)e_\lambda(z) = (I - \lambda R_0)e_\lambda(z) - \lambda F_0 a(D)e_\lambda(z) =$$
$$= z - \lambda F_0 a(\lambda)e_\lambda(z) = \lambda a(\lambda)F_0 e_\lambda(z) = z - \lambda a(\lambda)z = [1 - \lambda a(\lambda)]z.$$

Corollary 1. Suppose that all assumptions of Theorem 3 are satisfied. If $\mathcal{F} = \mathbb{C}$, then the right inverse R has exactly $n+1$ eigenvalues (hence $R \notin V(X)$) in the space D_k (we do not assume that all eigenvalues are different, each of them is counted as many times as its multiplicity).

The proof is an immediate consequence of formula (7). Indeed, the polynomial $1 - \lambda a(\lambda) = 1 - \sum_{m=0}^{n} a_m \lambda^m$ has in the complex domain exactly $n+1$ roots.

Similarly, we obtain

Corollary 2. Suppose that all assumptions of Theorem 3 are satisfied and $\mathcal{F} = \mathbb{R}$. Then the right inverse R has at most $n+1$ eigenvalues in the space D_k. However, if there exist $a_0, \ldots, a_n \in \mathbb{R}$ such that $1 - \lambda a(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$, then R is a Volterra operator.

It is so, for instance, if $n = 1$, $a_0 = 0$, $a_1 = 1$, since in this case we have $1 - \lambda a(\lambda) = 1 + \lambda^2 \neq 0$.

We will show that there are right inverses which are not of the form (6) and have infinitely many eigenvalues.

Theorem 4. Suppose that all assumptions of Theorem 3 are satisfied. Let $\{S_h\}_{h \in A(\mathbb{R})}$ be a family of D-shifts induced by R_0, i.e. we have (by definition) $S_0 = I$ and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(10) $S_h e_\lambda(z) = e^{\lambda h} e_\lambda(z)$ for all $\lambda \in \mathcal{F}$, $h \in A(\mathbb{R})$, $z \in \ker D$.

Let initial operators be defined as in formula (4), i.e.

(11) $F_h = F_0 S_h$ for $h \in A(\mathbb{R})$.

Let $k = 0,1,2,\ldots,\infty$ be arbitrarily fixed, let $0 \le n \le k$ and numbers $h_1,\ldots,h_n \in A(\mathbb{R})$, different than zero, be given. Consider a quasipolynomial:

(12) $$q(t) = \sum_{m=0}^{n} a_m(t) e^{h_m t},$$

(we admit here: $h_0 = 0$), where

(12') $$a_m(t) = \sum_{j=0}^{n} a_{mj} t^j, \quad a_{mj} \in \mathcal{F},$$

and not all $a_1(t),\ldots,a_n(t)$ vanish (identically) simultaneously. Then the operator

(13) $R = R_0 + \sum_{m=0}^{n} F_{h_m} a_m(D)$ is a right inverse of D in D_k

and

(14) $(I - \lambda R)e_\lambda(z) = [1 - \lambda q(\lambda)]z$ for all $\lambda \in \mathcal{F}$, $z \in \ker D$.

Proof. Similarly, as in proof of Theorem 3, we prove that R is a right inverse to D in D_k. Indeed, $DF_{h_m} = DF_0 S_{h_m} = 0$ for $m = 0,1,\ldots,n$. Hence, formulae (10), (12), (13) together imply that for all $\lambda \in \mathcal{F}$, $z \in \ker D$ we have

$$(I - \lambda R)e_\lambda(z) = (I - \lambda R_0)e_\lambda(z) -$$

$$- \lambda \sum_{m=0}^{n} F_{h_m} a_m(D) e_\lambda(z) = z - \lambda \sum_{m=0}^{n} F_{h_m} a_m(\lambda) e_\lambda(z) =$$

$$= z - \lambda \sum_{m=0}^{n} a_m(\lambda) F_0 S_{h_m} e_\lambda(z) = z - \lambda \sum_{m=0}^{n} a_m(\lambda) e^{\lambda h_m} F_0 e_\lambda(z) =$$

$$= z - \lambda \sum_{m=0}^{n} a_m(\lambda) e^{\lambda h_m} z = [1 - q(\lambda)]z.$$

Corollary 3. Suppose that all assumptions of Theorem 4 are satisfied and $\mathcal{F} = \mathbb{C}$. Then the right inverse R defined by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(11) has infinitely many eigenvalues.

Indeed, by our assumptions, the quasipolynomial $1 - \lambda q(\lambda)$ is not of the form $p(\lambda)(\lambda)e^{g(\lambda)}$, where $p(\lambda)$ is a polynomial and $g(\lambda)$ is an entire function. Thus, by the Picard theorem, this function has infinitely many zeros.

Till now all initial operators under consideration were of "local" type. Indeed, in the classical case, when $D = \frac{d}{dt}$, we have $(F_h x)(t) = x(h)$. Now we shall consider some initial operators of "non-local" type. In the classical case, an example of a non-local initial operator was given by Tasche (1974): $(Fx)(t) = \frac{1}{b-a}\int_a^b x(s)$ (cf. formula (2.2.42′) in [2]).

Theorem 5. Suppose that all assumptions of Theorem 4 are satisfied. Consider a quasipolynomial

$$r(t) = \sum_{m=1}^{n} h_m^{-1} a_m(t)(e^{h_m t} - 1), \tag{15}$$

where $a_m(t)$ are defined by (12′) and not all vanish (identically) simultaneously. Then the operator

$$R = R_0 + a_0 F_0 + \sum_{m=1}^{n} h_m^{-1} F_{h_m} R_0 a_m(D) \tag{16}$$

is a right inverse of D in D_k and

$$(I - \lambda R)e_\lambda(z) = [1 - a_0\lambda - r(\lambda)]z \quad \text{for all} \quad z \in \ker D, \tag{17}$$

$\lambda \in \mathcal{F}$.

Proof. Let $0 \neq h \in A(\mathbb{R})$ be arbitrarily fixed. Then the operator $h^{-1}F_h R_0 = h^{-1}F_0 S_h R_0$ is a projection onto $\ker D$, hence an initial operator for D. Similarly, as in the proof of Theorem 3, we prove that R defined by (16) is a right inverse of D in D_k. By definition, we have $(I - \lambda R_0)e_\lambda(z) = z$, which implies $\lambda R_0 e_\lambda(z) = e_\lambda(z) - z$ for all $z \in \ker D$, $\lambda \in \mathcal{F}$. $h \in$ D-shifts preserve constants: $S_h z = z$ for all $z \in \ker D$, $h \in A(\mathbb{R})$. Thus, by arguments similar to those used in Theorem 4, for all $z \in \ker D$, $\lambda \in \mathcal{F}$ we find

$$(I - \lambda R)e_\lambda(z) = (I - \lambda R_0)e_\lambda(z) - \lambda a_0 F_0 e_\lambda(z) -$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$- \lambda \sum_{m=1}^{n} h_m^{-1} F_{h_m} R_0 a_m(D) e_\lambda(z) =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} F_0 S_{h_m} \lambda R_0 a_m(\lambda) e_\lambda(z) =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} a_m(\lambda) F_0 S_{h_m} \lambda R_0 e_\lambda(z) =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} a_m(\lambda) F_0 S_{h_m} [e_\lambda(z) - z] =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} a_m(\lambda) F_0 [e^{h_m \lambda} e_\lambda(z) - z] =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} a_m(\lambda) (e^{h_m \lambda} z - z) =$$

$$= z - a_0 \lambda z - \sum_{m=1}^{n} h_m^{-1} a_m(\lambda) (e^{h_m \lambda} - 1) z =$$

$$= [1 - a_0 \lambda - r(\lambda)] z.$$

Corollary 4. Suppose that all assumptions of Theorem 4 are satisfied and $\mathcal{F} = \mathbb{C}$. Then the right inverse R defined by (16) has infinitely many eigenvalues.

The proof is going along the same lines as the proof of Corollary 3.

Recall that any linear combination of initial operators, whose sum of coefficients is equal to one, is again an initial operator. We therefore can easily prolongate the list of similar results.

Theorem 5. Suppose that all assumptions of Theorem 4 are satisfied, $\mathcal{F} = \mathbb{C}$ and F is an initial operator for D corresponding to a right inverse R. Consider an initial value problem

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(18) $(D - \lambda I)x = 0, \quad Fx = 0, \quad \lambda \in \mathbb{C}.$

(i) If $F = F_0$, i.e. $R = R_0$, then the problem (18) has not eigenvalues (this problem has only a trivial solution, hence it is well-posed, cf. Section 4.1 in [2]).

(ii) If F corresponds to the right inverse R defined by (6) then the problem has $n+1$ eigenvalues in D_k $(0 \le n \le k)$ (we do not assume that all are different, each eigenvalue is counted so many times as its multiplicity).

(iii) If F corresponds to the right inverse R defined either by (13) or by (16) then the problem (18) has infinitely many eigenvalues in D_k $(0 \le n \le k)$.

Proof. Since $D - \lambda I = D(I - \lambda R)$ and $Fx = 0$, the problem (18) can be written in an equivalent form

(19) $(I - \lambda R)x = 0 \qquad (\lambda \in \mathbb{C}).$

If $R = R_0$ then, by our assumption, the operator $I - \lambda R$ is invertible for all $\lambda \in \mathbb{C}$, which implies $x = 0$. If R is defined by (6), then Corollary 1 implies point (ii). If R is defined by (13), then, by Corollary 3, it has infinitely many eigenvalues in D_k. If R is defined by (16), then, by Corollary 4, it has infinitely many eigenvalues in D_k.

Theorem 6. Suppose that X is a linear space over $\mathcal{F}$ (where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$), $D \in R(X)$, $\dim \ker D \neq 0$ and F_0 is an initial operator corresponding to a Volterra right inverse R_0. Consider the boundary value problem

(20) $(D^2 - \lambda I)x = 0 \qquad (\lambda \in \mathbb{R}),$

(21) $F_0 x = 0, \qquad Fx = 0,$

where $F \neq F_0$ is an initial operator corresponding to a right inverse R satisfying the condition

(22) $(I - tR)e_t(z) = [1 - w(t)]z$ for all $t \in \mathcal{F}$, $z \in \ker D$,

and $w(t) \neq 0$ is a quasipolynomial (in particular, a polynomial), $e_\lambda = (I - \lambda R_0)^{-1}$. Then the only eigenvalues of the problem (20)-(21) are zeros of the function

(23) $w^0(t) = 1 - w_+(t)$, where $w_+(t) = \frac{1}{2}[w(t) + w(-t)]$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Since the equation (20) can be written as $D^2(I - \lambda R^2)x = 0$, by the Taylor formula we find $(I - \lambda R^2)x = RFDx + Fx = RFDx$, for the condition $Fx = 0$ should be satisfied. Observe that

(24) $x \in \ker (I - \lambda R^2)$ implies $x \in \ker (I - \lambda R^2 - RFD)$.

Indeed, if $x \in \ker (I - \lambda R^2)$ then $x = \lambda R^2 x$. Hence $FDx = \lambda FDR^2 x = \lambda FRx = 0$, for $FR = 0$. Thus $RFDx = 0$, which implies $x \in \ker (I - \lambda R^2 - RFD)$. Hence, in order to find non-trivial solutions of the equation $(I - \lambda R^2)x = RFDx$, it is enough to find solutions of the equation

(25) $(I - \lambda R^2)x = 0$.

On the other hand, it is not difficult to verify that every solution of the equation (20) is of the form:

$$e_\mu(z) + e_{-\mu}(z_1) \quad \text{if} \quad \lambda \geq 0, \quad \text{where} \quad \mu = \sqrt{\lambda},$$

$$e_{\mu i}(z) + e_{-\mu i}(z_1) \quad \text{if} \quad \lambda < 0, \quad \text{where} \quad \mu = \sqrt{-\lambda},$$

where $z, z_1 \in \ker D$ are arbitrary. If we apply the condition $F_0 x = 0$, then in both cases we obtain $z + z_1 = 0$. Hence $z_1 = -z$ and solutions, we are looking for, should be of the form:

(26) $$x_\mu = e_\mu(z) - e_{-\mu}(z) \quad \text{if} \quad \lambda \geq 0, \quad \text{where} \quad \mu = \sqrt{\lambda},$$

$$x_{\mu i} = e_{\mu i}(z) - e_{-\mu i}(z) \quad \text{if} \quad \lambda < 0, \quad \text{where} \quad \mu = \sqrt{-\lambda},$$

and $z \in \ker D$ is to be determined from (25). In particular, if $\lambda = 0$ then $\mu = 0$ and $x = z - z = 0$. Hence, $\lambda = 0$ is not an eigenvalue of the problem (20)-(21).

Write now

(27) $A_t = I - t^2R^2$, $x_t = e_t(z) - e_{-t}(z)$ for $t \in \mathbb{R}$, $z \in \ker D$.

Since, by definition, $tRe_t(z) = e_t(z) - z$ for all $t \in \mathbb{R}$, $z \in \ker D$, we find from (22)

$$t[1-w(t)]Rz = tR[1-w(t)]z = tR(1-tR)e_t(z) =$$

$$= tR[e_t(z) - tRe_t(z)] = tRe_t(z) - t^2R^2e_t(z) =$$

$$= e_t(z) - z - t^2R^2e_t(z) = (I-t^2R^2)e_t(z) - z = A_t e_t(z) - z.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It means that

(28) $A_t e_t(z) = t[1 - w(t)]Rz + z$ for $t \in \mathbb{R}$, $z \in \ker D$.

Since $A_{-t} = A_t$, formulae (27) and (28) together imply

$$(I-t^2R^2)x_t = A_t x_t = A_t[e_t(z) - e_{-t}(z)] =$$

$$= A_t e_t(z) - A_t e_{-t}(z) = A_t e_t(z) - A_{-t} e_{-t}(z) =$$

$$= t[1 - w(t)]Rz + z - (-t)[1 - w(-t)]Rz - z =$$

$$= t[2 - w(t) - w(-t)]Rz = 2t[1 - w_+(t)]Rz,$$

where $w_+(t) = \frac{1}{2}[w(t) + w(-t)]$ is the even part of $w(t)$ for all $t \in \mathbb{R}$, $z \in \ker D$. Finally, we get

(29) $(I - t^2R^2)x_t = 2t[1 - w_+(t)]Rz$ for $t \in \mathbb{R}$, $z \in \ker D$.

Since $\lambda = 0$ is not an eigenvalue of the problem under consideration, the only eigenvalues are determined by zeros of the function $1 - w_+(t)$. If for some $t \neq 0$ we find $1 - w_+(t) = 0$, then $\lambda = t^2$ is an eigenvalue of the problem (20)-(21).

Remark 1. If $w(t) \equiv 0$, then, by (22), we have

$$(I-tR)e_t(z) = z \quad \text{for all} \quad t \in \mathbb{R}, \quad z \in \ker D.$$

This implies that $R = R_0$. Hence $F = F_0$, which contradicts our assumption.

Corollary 5. Suppose that all assumptions of Theorem 3 are satisfied and $a(t) \not\equiv 0$. Then the problem (20)-(21), where F is an operator corresponding to the right inverse R defined by (6), has at most $n+1$ eigenvalues in D_k $(k \geq 2)$ which are squares of zeros of the function $1 - ta_-(t)$, where $a_-(t) = \frac{1}{2}[a(t) - a(-t)]$.

Indeed, in this case $w(t) = ta(t)$. Hence

$$1 - w_+(t) = 1 - \frac{1}{2}[ta(t) - ta(-t)] = 1 - ta_-(t).$$

Corollary 6. Suppose that all assumptions of Theorem 4 are satisfied and $\mathcal{F} = \mathbb{C}$. Then the problem (20)-(21), where F is an initial operator corresponding to the right inverse R defined either by (13) or (16), has infinitely many eigenvalues in D_k $(k \geq 2)$. These eigenvalues are squares of zeros of the function

$1-tq_-(t) = 1 - \frac{1}{2}t[q(t) - q(-t)]$ in the first case, where

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$q(t)$ is determined by formula (12), and the function $1-r_+(t) = 1 - \frac{1}{2}[r(t) + r(-t)]$ in the second case, where $r(t)$ is determined by formula (15).

Indeed, in the first case we have $w(t) = tq(t)$, in the second case $w(t) = a_0 t + r(t)$. This and Corollary 3 (resp. Corollary 4) together imply that the problem under consideration has infinitely many eigenvalues in D_k $(k \geq 2)$.

Remark 2. Suppose that all assumptions of Theorem 6 are satisfied. Then we find

(30) $Fe_\lambda(z) = [1-w(\lambda)]z$ for all $\lambda \in \mathcal{F}$, $z \in \ker D$.

Indeed, $Fe_\lambda(z) = F(I-\lambda R)e_\lambda(z) = F[1-w(\lambda)]z = [1-w(\lambda)]Fz = [1-w(\lambda)]z$. On the other hand, we have proved (cf. Theorem 2.3.4 in [2]) that $1/\lambda$ is an eigenvalue of R if and only if there exists $z \in \ker D$, $z \neq 0$, such that $Fe_\lambda(z) = 0$. The corresponding eigenvector is $e_\lambda(z)$, i.e. $e_\lambda(z) \in \ker (I-\lambda R)$.

It was shown in [3] that to every right inverse of weighted difference operators there corresponds a polynomial with scalar coefficients (which is uniquely determined). Thus, these operators have <u>at most</u> a finite number of eigenvalues. It should be also mentioned that recently N. Bozhinov studied the spectrum of certain non-local boundary value problems for the Sturm - Liouville operator (cf. [1], in particular, Appendix to [1]).

In Theorem 3 (and also in our further considerations) we have assumed that the right inverse R_0 is a Volterra operator. Without this assumption we have a different situation as it is shown by

Theorem 7. Suppose that all assumptions of Theorem 3 are satisfied, however, R_0 is not necessarily a Volterra right inverse. Write

(31) $sp(A) = \{\lambda \in \mathcal{F} : \ker (I-\lambda A) \neq \{0\}\}$ for $A \in L_0(X)$.

If R is defined by formula (6), then

(32) $sp(R) = sp(R_0) \cup \{\lambda \in \mathcal{F} : 1-\lambda a(\lambda) = 0\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Firstly, we will show that

(33) $\quad sp\ R \supset sp\ R_0$.

If $sp\ R_0 = \emptyset$, then this inclusion is trivial. Suppose that $\lambda \in sp\ R_0$. Then there exists an $x_\lambda \in \ker (I-\lambda R_0)$, $x_\lambda \neq 0$. Thus, we have $\lambda R_0 x_\lambda = x_\lambda$ and $F_0 x_\lambda = 0$. By an easy induction we prove that $D^n x_\lambda = \lambda^n x_\lambda$ for all $n \in \mathbb{N}$. Since

$$\lambda R x_\lambda = \lambda R_0 x_\lambda + \lambda F_0 a(D) x_\lambda = x_\lambda + \lambda a(\lambda) F_0 x_\lambda = x_\lambda,$$

we conclude that $x_\lambda \in \ker (I-\lambda R)$, i.e. $\lambda \in sp(R)$.

An initial operator F corresponding to the right inverse R is by definition, $F = I - RD = I - [R_0 + F_0 a(D)]D = I - R_0 D - F_0 Da(D) = F_0[I - Da(D)]$ on dom D. Suppose that $\lambda \in sp(R)$. Then $Fx_\lambda = \lambda FRx_\lambda = 0$ and $0 = Fx_\lambda = F_0[I-Da(D)]x_\lambda = F_0[1-\lambda a(\lambda)]x_\lambda = [1-\lambda a(\lambda)]F_0 x_\lambda$. This implies that either $1 - \lambda a(\lambda) = 0$ or $F_0 x_\lambda = 0$. The second equality means that $\lambda \in sp(R_0)$. This proves formula (32).

REFERENCES

[1] **N. Bozhinow:** Convolutional representations of commutators and multipliers. Publish. House of the Bulg. Acad. Sci., Sofia, 1988 (with Appendix: Uniqueness and completeness theorems concerning root function expansions of the nonlocal Sturm-Liouville operator, Institute of Mathematics with Computing Centre, Bulg. Acad. Sci., Sofia, 1988, to appear in Differencial'nye Uravnienija).

[2] **D. Przeworska-Rolewicz:** Algebraic Analysis. PWN - Polish Scientific Publishers and D. Reidel, Warszawa - - Dordrecht, 1988.

[3] **D. Przeworska-Rolewicz:** Advantages of one dimensional kernels. Math. Nachrichten 148 (1990), 143-158.

[4] **D. Przeworska-Rolewicz, S. Rolewicz:** The only continuous Volterra right inverses in $C_c[0,1]$ for the operator d/dt are $\int_a^t$, Colloq. Math. 57(1983) 279-283.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] D. Przeworska-Rolewicz, S. Rolewicz: On a representation of continuous Volterra right inverses to the derivative in the space of infinitely differentiable functions. J. Australian Math. Soc. 41(1986) 138-142.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES,
WARSZAWA

Received April 12, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 3 1990

Eberhard Wagner

ÜBER EIN ABZÄHLBARES SYSTEM GEWÖHNLICHER LINEAR DIFFERENTIALGLEICHUNGEN VON BANDSTRUKTUR

1. Einleitung

In einer Population mit abzählbar vielen Merkmalen kann ein Geburts-Todes-Prozeß, bei welchem Individuen mit dem Merkmal n aus Individuen mit den Merkmalen n-1, n+1 und n+2 entstehen, durch ein abzählbares System gewöhnlicher Differentialgleichungen der Form

$$(S)\quad x_n' = \delta_{n-1}x_{n-1} - \alpha_n x_n + \beta_{n+1}x_{n+1} + \gamma_{n+2}x_{n+2} + f_n$$
$$(n = 0,1,\ldots;\ x_{-1} = 0)$$

beschrieben werden. Dabei bedeutet $x_n = x_n(t)$ die Anzahl der Individuen der Merkmalsklasse n oder den relativen Anteil dieser Merkmalsklasse an der Gesamtpopulation zum Zeitpunkt t. Die Koeffizienten α_n, β_n, γ_n, δ_n werden als nichtnegativ vorausgesetzt. Als ein Beispiel sei das in [1] aufgestellte Modell für die Radikalpopulation bei gewissen Polymerisationsprozessen genannt, wo x_n den relativen Anteil von Polymerteilchen mit n freien Radikalen bezeichnet, die Koeffizienten die von t unabhängigen Werte $\alpha_n = \beta_n + \gamma_n + \delta_n$,$\beta_n = n\beta$, $\gamma_n = n(n-1)\gamma$, $\delta_n = \delta$ $(\beta,\gamma,\delta \geq 0)$ besitzen und sämtliche Quellfunktionen f_n verschwinden; die zugehörigen Anfangssbedingungen sind $x_0(0) = 1$, $x_1(0) = x_2(0) = \ldots = 0$.

Hinsichtlich der Existenz und Berechnung einer Lösung werden in [1] keine bzw. nur vage und teilweise unzutreffende Bemerkungen gemacht. Erwähnenswert ist bei diesem Beispiel, daß die Koeffizienten, bis auf den rekursiv lösbaren Ausnahmefall $\beta = \gamma = 0$, unbeschränkt sind.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In [6] wird für das System (S) mit von t unabhängigen Koeffizienten, homogenen Anfangsbedingungen und nur endlich vielen nicht identisch verschwindenden Quellfunktionen $f_{n'}$ unter Voraussetzungen, die bei dem angeführten Beispiel erfüllt sind, die Existenz einer nichtnegativen Lösung nachgewiesen. Diese Lösung kann man durch koordinatenweisen Grenzübergang aus den Lösungen "abgeschnittener" endlicher Systeme für die Funktionen x_0 bis x_N erhalten. Die Konvergenzgüte wird durch Fehlerabschätzungen quantifiziert, aus denen man ersehen kann, daß die Konvergenz um so besser ist, je stärker die Koeffizienten α_n gegen Unendlich streben. Wir bemerken noch, daß bei beliebig stark anwachsenden Koeffizienten das System (S) nicht als Differentialgleichung $x' = Ax + f$ in einem l_p-Raum mit einem beschränkten Operator A behandelt werden kann.

In der vorliegenden Arbeit werden die Untersuchungen des Systems (S) mit nichtnegativen, konstanten Koeffizienten auf den Fall inhomogener Anfangsbedingungen

(A) $\quad x_n(0) = C_n \geq 0 \qquad (n = 0,1,\ldots)$

und beliebig vieler nicht identisch verschwindender Quellfunktionen $f_n \geq 0$ verallgemeinert. Darüber hinaus werden

- ein Eindeutigkeitssatz bewiesen (Abschnitt 2.),
- die konstruierte nichtnegative Lösung von (S), (A) durch eine Extremaleigenschaft charakterisiert (Abschnitt 6.),
- die Fehlerabschätzungen im Fall homogener Anfangsbedingungen und endlich vieler Quellfunktionen vereinfacht (Abschnitt 7.),
- je ein Beispiel für Existenz und Nichtexistenz der Matrix exp(At) (A = Koeffizientenmatrix von (S)) unter den Voraussetzungen der vorliegenden Arbeit angegeben (Abschnitt 9.),
- ein Existensatz für den Fall nichtkonstanter Koeffizienten bewiesen (Abschnitt 10.).

2. Ein Eindeutigkeitssatz

Im folgenden werden die in (S) nicht auftretenden Koeffizienten

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.1) $\beta_0 = \gamma_0 = \gamma_1 = 0$

gesetzt. Ohne Einschränkung der Allgemeinheit kann angenommen werden, daß (S) keine Differentialgleichungen mit $\delta_{n-1} = \beta_{n+1} = \gamma_{n+2} = 0$ enthält. Die Differentialgleichungen in (S) denken wir uns mit (0),(1),... durchnumeriert. (S_n) bezeichne das aus den Gleichungen (0) bis (n) bestehende endliche Teilsystem; mit $(i)_h$, $(S)_h$ und $(S_n)_h$ werden die zugehörigen homogenen Gleichungen bzw. Systeme bezeichnet.

Satz 1: Sind α_n, β_n, γ_n, δ_n von t unabhängig und nicht-negativ mit

(2.2) $\delta_{n-1} + \beta_{n+1} + \gamma_{n+2} > 0 \quad (n = 0,1,...),$

so besitzt das Anfangswertproblem (S), (A) genau dann eine eindeutige Lösung, wenn eine echt monoton wachsende Folge $\{n_i\}$ natürlicher Zahlen mit

(2.3) $\beta_{n_i} = \gamma_{n_i} = \gamma_{n_i+1} = 0$

existiert. Andernfalls besitzt $(S)_h$ unendlich viele Lösungen mit verschwindenden Anfangswerten.

Beweis : Existiert eine Folge $\{n_i\}$ mit (2.3), so besitzt jedes endliche Teilsystem (S_{n_i-1}) bei beliebigen Anfangsbedingungen eine eindeutige Lösung, deren Komponenten sich beim Übergang zum System $(S_{n_{i+1}-1})$ nicht ändern, do daß (S), (A) eindeutig lösbar ist.

Im alternativen Fall gibt es wegen (2.1) einen Index $n_0 \geq 0$ mit $\beta_{n_0} = \gamma_{n_0} = \gamma_{n_0+1} = 0$ und $\beta_n + \gamma_n + \gamma_{n+1} > 0$ für $n > n_0$. Ist $n_0 > 0$, so existiert eine eindeutige Lösung des Teilsystems (S_{n_0-1}). Das aus den übrigen Differentialgleichungen bestehende System laßt sich wieder auf den Typ (S) bringen, so daß nachfolgend ohne Einschränkung der Allgemeinheit $n_0 = 0$, d.h.

(2.4) $\beta_n + \gamma_n + \gamma_{n+1} > 0 \quad (n = 1,2,...)$

angenommen werden kann. Es bleibt zu zeigen, daß dann $(S)_h$ unendlich viele Lösungen mit verschwindenden Anfangswerten besitz. Da jede Lösungskomponente des homogenen Anfangswert-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

problems für $t \geq 0$ beliebig oft differenzierbar sein und mit allen ihren Ableitungen in $t = 0$ verschwinden muß; sollen im folgenden alle als frei wählbar deklarierten Funktionen diese Bedingungen erfüllen, ohne identisch zu verschwinden.

Nehmen wir zuerst an, daß es einen Index $\bar{n} \geq 2$ mit $\gamma_n = 0$ für $n \geq \bar{n}$ gibt. Dann besitzt $(S_{\bar{n}-3})_h$ nach freier Wahl von beispielsweise $x_{\bar{n}-2}$ und $x_{\bar{n}-1}$ eine nichttriviale Lösung mit verschwindenden Anfangswerten, und das restliche homogene System kann sukzessiv nach $x_{\bar{n}}, x_{\bar{n}+1}, \ldots$ aufgelöst werden. Gibt es andererseits einen Index $\bar{n} \geq 1$ mit $\gamma_n = 0$ für $n \geq \bar{n}$, so ist nach (2.4) $\beta_n > 0$ für $n \geq \bar{n}$. In diesem Fall existiert nach freier Wahl von beispielsweise $x_{\bar{n}-1}$ wiederum eine nichttriviale Lösung von $(S_{\bar{n}-2})_h$ mit verschwindenden Anfangswerten, während $x_{\bar{n}}, x_{\bar{n}+1}, \ldots$ aus dem unendlichen homogenen Restsystem algebraisch berechenbar sind. Gibt es sowohl unendlich viele positive als auch unendlich viele verschwindende Koeffizienten γ_n, so sei $\{n_i\}$ die (echt monoton wachsende) Folge aller natürlicher Zahlen mit

$$\gamma_{n_i} = 0 \quad \gamma_{n_i+1} > 0. \tag{2.5}$$

Ist n_i ein beliebiges, festes Element dieser Folge, so besitzt $(S_{n_i-2})_h$ nach freier Wahl von x_{n_i-1} eine eindeutig bestimmte Lösung $x_j = g(x_{n_i-1})$ mit $x_j(0) = 0$ $(j = 0,1,\ldots,n_i-2)$. Das sich daran anschließende Teilsystem $(n_i-1)_h$ bis $(n_{i+1}-2)_h$ kann als inhomogenes (anormales, da die erste Gleichung algebraisch ist) Differentialgleichungssystem für $x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}$ aufgefaßt werden, das mittels Laplace-Transformation in ein lineares algebraisches Gleichungssystem mit der Koeffizientendeterminante

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(2.6)\quad B_{n_i}^{n_{i+1}-1} = \begin{vmatrix} \beta_{n_i} & \gamma_{n_i+1} & & & & 0 \\ -\lambda_{n_i} & \ddots & \ddots & & & \\ \delta_{n_i} & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \gamma_{n_{i+1}-1} \\ 0 & & \delta_{n_{i+1}-3} & -\lambda_{n_{i+1}-2} & \beta_{n_{i+1}-1} & \end{vmatrix}$$

übergeht, wobei zur Abkürzung $\lambda_j = s + \alpha_j$ gesetzt wurde. Da, wie nachfolgend gezeigt wird, diese Determinanten nicht identisch verschwinden, besitzt das System der Gleichungen $(n_i-1)_h$ bis $(n_{i+1}-2)_h$ eine eindeutig bestimmte Lösung $x_j = g_j(x_{n_{i-1}})$ mit verschwindenden Anfangswerten $(j=n_i,\ldots,n_{i+1}-1)$. (Die bei anormalen Systemen auftretenden Kompatibilitätsbedingungen sind offenbar erfüllt, da alle auftretenden Funktionen im Nullpunkt mit allen ihren Ableitungen verschwinden). Auf dieselbe Weise zeigt man sukzessive die Existenz von Lösungen mit verschwindenden Anfangswerten der nachfolgenden Teilsysteme $(n_{i+1}-1)_h$ bis $(n_{i+2}-2)_h$ usw.

Wir weisen nun noch $B_{n_i}^{n_{i+1}-1}(s) > 0$ für $s > 0$ nach. Nach (6) können alle Determinanten B_p^q rekursiv aus

$$(2.7)\quad B_p^q = \beta_q B_p^{q-1} + \lambda_{q-1}\gamma_q B_p^{q-2} + \delta_{q-2}\gamma_{q-1}\gamma_q B_p^{q-3} \quad (q=p+1,p+2,\ldots)$$

mit den Anfangswerten

$$(2.8)\quad B_p^{p-2} = 0,\ B_p^{p-1} = 1,\ B_p^p = \beta_p$$

berechnet werden, so daß die B_p^q Polynome in s mit nichtnegativen Koeffizienten sind. Wegen $\gamma_{n_i+1} > 0$, $\gamma_{n_{i+1}} = 0$ gibt es eine eindeutig bestimmte natürliche Zahl k $(n_i+1 \le k \le n_{i+1}-1)$ mit

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(2.9)\qquad \gamma_j \begin{cases} > 0 & \text{für } n_i+1 \le j \le k \\ = 0 & \text{für } k+1 \le j \le n_{i+1} \end{cases}.$$

Ist $k < n_{i+1}-1$, so sind wegen (2.3) die Koeffizienten $\beta_{k+1}, \ldots, \beta_{n_{i+1}-1}$ positiv, und man erhält aus (2.7)

$$B_{n_i}^{n_{i+1}-1} = B_{n_i}^{k} \prod_{j=k+1}^{n_{i+1}-1} \beta_j ,$$

so daß es genügt $B_{n_i}^{k}(s) > 0$ für $s > 0$ zu beweisen. Für $k = n_i + 2j + 1$ $(j = 0,1,\ldots)$ folgt dies aus $\gamma_{n_i+1} > 0, \ldots, \gamma_{n_i+2j+1} > 0$ und den Ungleichungen

$$(2.10)\qquad B_{n_i}^{n_i+2j+1} \ge \gamma_{n_i+2j+1}\lambda_{n_i+2j}\gamma_{n_i+2j-1}\lambda_{n_i+2j-2} \cdots$$

$$\cdots \gamma_{n_i+1}\lambda_{n_i} B_{n_i}^{n_i-1} \ge \gamma_{n_i+2j+1}\gamma_{n_i+2j-1} \cdots \gamma_{n_i+1} s^{j+1},$$

die man durch sukzessive Anwendung der Rekursionsformel (2.7) unter Beachtung der Anfangswerte (2.8) erhält. Für $k = n_i+2j+2$ $(j = 0,1,2,\ldots)$ folgt aus (2.7)

$$(2.11)\qquad B_{n_i}^{n_i+2j+2} \ge \beta_{n_i+2j+2} B_{n_i}^{n_i+2j+1} +$$

$$+ \delta_{n_i+2j}\gamma_{n_i+2j+1}\gamma_{n_i+2j+2} B_{n_i}^{n_i+2j-1}.$$

Wegen $\gamma_{n_i+2j+3} = 0$ ist nach (2.2) $\delta_{n_i+2j} + \beta_{n_i+2j+2} > 0$, so daß mindestens ein Summand auf der rechten Seite von (2.11) für $s > 0$ positiv ist und damit $B_{n_i}^{k} > 0$ für $s > 0$ und $k = n_i + 2j + 2$.

Bemerkung : Notwendige und hinreichende Bedingungen für die eindeutige Lösbarkeit eines allgemeinen unendlichen Systems $x' = Ax$ mit einer zeilenendlichen Matrix A findet man in der Arbeit [4].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Präzisierung der Voraussetzungen

Im folgenden betrachten wir das Anfangswertproblem (S),(A) mit von t unabhängingen Koeffizienten unter den Voraussetzungen

$$(3.1)\quad 0 = \delta_{-1} \le \delta_n \le \delta = \sup_{n\ge 0} \delta_n < \infty \qquad (n = 0,1,\dots),$$

$$(3.2)\quad 0 = \beta_0 \le \beta_n \qquad (n = 1,2,\dots),$$

$$(3.3)\quad 0 = \gamma_0 = \gamma_1 \le \gamma_n \qquad (n = 2,3,\dots),$$

$$(3.4)\quad \inf_{n\ge 0} (\alpha_n - \beta_n - \gamma_n) = \alpha > -\infty ,$$

$$(3.5)\quad \sum_{\substack{n=0 \\ \alpha_n>\alpha}}^{\infty} 1/(\alpha_n - \alpha) = \sigma < \infty ,$$

$$(3.6)\quad \sum_{\substack{n=0 \\ \alpha_{n+1}>\alpha_n=\alpha}}^{\infty} \delta_n = \rho < \infty ,$$

$$(3.7)\quad \sum_{n=0}^{\infty} C_n z_0^n < \infty \qquad (0 < z_0 = z_0^3 - 1 \approx 1{,}325\dots),$$

$$(3.8)\quad f_n \in C[0,\infty),\ f_n(t) \ge 0 \qquad (t \ge 0;\ n = 0,1,\dots),$$

$$\sum_{n=0}^{\infty} f_n(t) z_0^n < \infty \qquad (t \ge 0).$$

Bemerkungen : Beschreibt (S),(A) einen Geburts - Todes - Prozeß, so sind (3.1) bis (3.3) erfüllt und in der Regel die Spaltensummen der Koeffizientenmatrix A gleich Null, also $\alpha_n - \beta_n - \gamma_n = \delta_n \ge 0$, so daß auch (3.4) mit $\alpha \ge 0$ erfüllt ist. Die Voraussetzungen (3.5) bis (3.8) erweisen sich als hinreichend für die Konvergenz von Näherungslösungen gegen eine nichtnegative Lösung des unendlichen Cauchy-Problems (s. Abschnitt 6.). Die Bedingung (3.5) erfordert, daß die Teilfolge der Koeffizienten α_n mit $\alpha_n > \alpha$ hinreichend stark gegen Unendlich strebt. Je stärker diese Teilfolge wächst, um

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

so besser ist die Konvergenz der Näherungslösungen (s. Abschnitt 7.). (Das einfache System $x'_n = -ax_n + bx_{n+1}$ genügt demnach im Fall $b > 0$ nicht den genannten Bedingungen). Andererseits ist (3.5) auch dann erfüllt, wenn mit Ausnahme von höchstens endlich vielen Indizes $\alpha_n = \alpha = \inf\alpha_n$, also $\beta_n = \gamma_n = 0$ ist. Dieser Fall ist nach Abschnitt 2. uninteressant. Die Voraussetzung (3.6) ist stets erfüllt, wenn für nur endlich viele Indizes die Relationen $\alpha_{n+1} > \alpha_n = \alpha$ gelten, insbesondere also dann, wenn die α_n von einem gewissen Index an echt monoton wachsen. Das trifft in dem in der Einleitung angeführten Beispiel zu. Ob sich die Konvergenzbedingungen in (3.7), (3.8) zu $\{C_n\} \in l_1$, $\{f_n\} \in l_1$ abschwächen lassen, ist eine offene Frage, jedoch kann man davon ausgehen, daß bei vielen praktischen Anwendungen in der Regel nur endlich viele Anfangswerte und Quellfunktionen positiv sind.

4. Lösungen der abgeschnittenen Systeme

Betrachtet man das endliche Teilsystem (S_N) und setzt darin $x_{N+1} = x_{N+2} = 0$, so erhält man ein System (S^N) mit einer quadratischen Koeffizientenmatrix A^N, das unter den zugehörigen Anfangsbedingungen aus (A) die eindeutige Lösung

$$(4.1)\quad x_n^N(t) = \sum_{m=0}^{N} \left[C_m q_{nm}^N(t) + \int_0^t q_{nm}^N(t-\tau) f_m(\tau)\, d\tau \right] \quad (n=0,1,\ldots,N)$$

besitzt, wobei die q_{nm}^N für jedes feste $m \in \{0,1,\ldots,N\}$ die Lösung des homogenen Systems $(S^N)_h$ unter den Anfangsbedingungen

$$(4.2)\quad q_{nm}^N(0) = \begin{cases} 1 & \text{für } n=m \\ 0 & \text{für } n\neq m \end{cases} \qquad (n=0,1,\ldots,N)$$

bilden [3].

Wie aus [6] hervorgeht, aber auch unmittelbar mit dem bekannten Abschätzungssatz von Gerschgorin für Eigenwerte von Matrizen [5] gezeigt werden kann, sind die Realteile aller Eigenwerte der Matrizen A^N $(N=0,1,\ldots)$ nicht großer als $\delta - \alpha$, so daß für $\mathrm{Re}\, s > \delta - \alpha$ die Laplace-Transformierten

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Darstellungen

$$(4.3)\quad Q_{nm}^N = \begin{cases} \Delta_0^{m-1}(\Delta_{n+1}^N/\Delta_0^N)\prod_{\nu=m}^{n-1}\delta_\nu & (0\le m\le n\le N) \\ (B_{n+1}^m\Delta_0^{n-1} + \delta_{n-1}\gamma_{n+1}B_{n+2}^m\Delta_0^{n-2})(\Delta_{m+1}^N/\Delta_0^N) + & \\ + \delta_m\gamma_{m+1}(B_{n+1}^{m-1}\Delta_0^{m-1} + \delta_{n-1}\gamma_{n+1}B_{n+2}^{m-1}\Delta_0^{n-2})(\Delta_{m+2}^N/\Delta_0^N) & \\ & (0\le n<m\le N). \end{cases}$$

Dabei bezeichnen Δ_i^j und B_i^j durch gewisse Unterdeterminanten (vgl. (2.6)) der Matrix $sE-A^N$ (E Einheitsmatrix der Ordnung N+1) definierte Polynome in s, die sich nach [6] rekursiv aus

$$(4.4)\quad \Delta_i^j = \lambda_j\Delta_i^{j-1} - \delta_{j-1}\beta_j\Delta_i^{j-2} - \delta_{j-2}\delta_{j-1}\gamma_j\Delta_i^{j-3} \qquad (0\le i\le j)$$

mit den Anfangswerten $\Delta_i^{i-3} = \Delta_i^{i-2} = 0$, $\Delta_i^{i-1} = 1$ bzw. aus (2.7) und (2.8) berechnen lassen. Insbesondere ist $\Delta_0^N = \det(sE-A^N)$.

Gibt es einen Index $\bar{n}\ge 1$ mit $\beta_{\bar{n}} = \gamma_{\bar{n}} = \gamma_{\bar{n}+1} = 0$, so sind die Funktionen $x_n = x_n^{\bar{n}-1}$ $(n=0,1,\dots,\bar{n}-1)$ Lösung des Teilsystems $(S_{\bar{n}-1})$, da dieses mit $(S^{\bar{n}-1})$ zusammenfällt (vgl. Beweis von Satz 1). Da das restliche System wieder vom Typ (S) ist, werden wir diesem Fall im folgenden keine weitere Beachtung schenken.

5. Abschätzungen der Δ_i^j und B_i^j

Aus [6] erhält man die für $s>0$ gültigen Abschätzungen

$$(5.1)\quad 0 < d(s)\prod_{\nu=i}^{j}\lambda_\nu \le \Delta_i^j(s) \le \prod_{\nu=i}^{j}\lambda_\nu \qquad (0\le i\le j)$$

mit

$$(5.2)\quad d(s) = \begin{cases} (s+\alpha)^{-1}(s+\alpha-\delta)^1 e^{-\sigma(s+\alpha)} \\ \text{oder} \\ \exp\{-\dfrac{\rho}{s+\alpha-\delta} - \sigma(s+\alpha)\}\ , \end{cases}$$

je nachdem, ob die Relationen $\alpha_\nu > \alpha_{\nu-1} = \alpha$ für genau 1 natürliche Zahlen ν oder für unendlich viele ν gelten.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Außerdem entnehmen wir aus [6] die Abschätzungen

$$\Delta_i^{N+p}/\Delta_0^{N+p} - \Delta_i^N/\Delta_0^N \le \frac{\varepsilon_\beta^N+\varepsilon_\gamma^N}{d}\left(1 + \frac{\varepsilon_\beta^{N-2}+\varepsilon_\gamma^{N-2}}{d}\right)^{\prod_{\nu=0}^{i-1}\lambda_\nu} \tag{5.3}$$

mit den für $N \longrightarrow \infty$ und jedes $s > \delta - \alpha$ wegen (3.5), (3.6) gegen Null strebenden Reihenresten

$$\varepsilon_\beta^N = \varepsilon_\beta^N(s) = \sum_{\substack{\nu=N \\ \beta_{\nu+1}>0}}^{\infty} \delta_\nu/\tilde{\alpha}_\nu \,, \tag{5.4}$$

$$\varepsilon_\gamma^N = \varepsilon_\gamma^N(s) = \sum_{\substack{\nu=N \\ \gamma_{\nu+1}>0}}^{\infty} (\delta_{\nu-1}\delta_\nu)/(\tilde{\alpha}_{\nu-1}\tilde{\alpha}_\nu) \,,$$

wobei die $\tilde{\alpha}_\nu$ durch

$$\tilde{\alpha}_\nu = \begin{cases} \alpha_\nu - \alpha & \text{für } \alpha_\nu > \alpha \\ s + \alpha & \text{für } \alpha_\nu = \alpha \end{cases} \tag{5.5}$$

definiert sind. Abschätzungen der B_i^j liefert der folgende

Hilfssatz 1: Die durch (2.7), (2.8) definierten Polynome $B_i^j(s)$ erfüllen für alle $s > \delta - \alpha$ die Ungleichungen

$$0 \le B_i^j(s) \le p_{j-i-1}\prod_{\nu=i}^{j}\lambda_\nu \qquad (1 \le i \le j+2), \tag{5.6}$$

wobei die von s unabhängigen Koeffizienten p_k durch

$$p_{-3} = 0,\ p_{-2} = p_{-1} = 1,\ p_k = p_{k-2} + p_{k-3} \qquad (k=0,1,\ldots) \tag{5.7}$$

definiert sind.

Beweis: Aus (3.1) bis (3.4) folgt zunächst für $s > \delta - \alpha$

$$0 \le \beta_\nu + \gamma_\nu + \delta_\nu < \lambda_\nu \,. \tag{5.8}$$

Somit gilt (5.6) für $i-2 \le j \le i$. Unter der Voraussetzung, daß (5.6) für $i-2 \le j \le J$, $J \ge i$ gilt, erhält man aus (4.5) wegen (5.8) und unter Berücksichtigung des monotonen Wachstums der Folge $\{p_k\}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$0 \le B_i^{J+1} \le \prod_{\nu=i}^{J+1} \lambda_\nu \{(p_{J-i-3} + p_{J-i-4})(\beta_{J+1}/\lambda_{J+1}) +$$

$$+ p_{J-i-2}(\gamma_{J+1}/\lambda_{J+1}) + p_{J-i-3}(\delta_{J-1}\gamma_J\gamma_{J+1})/(\lambda_{J-1}\lambda_J\lambda_{J+1})\} \le$$

$$\le \prod_{\nu=i}^{J+1} \lambda_\nu (p_{J-i-3} + p_{J-i-2}) = p_{J-i} \prod_{\nu=i}^{J+1} \lambda_\nu .$$

Über das Wachstum der Folge $\{p_k\}$ gibt der folgende Hilfssatz 2 Auskunft, in dem z_0 die positive Lösung der Gleichung $z^3 - z - 1 = 0$ bezeichnet (vgl. (3.7)).

Hilfssatz 2: Für die in (5.7) definierte Folge $\{p_k\}$ gilt

(5.9) $$p_k = Cz_0^k + o(1) \qquad (k \to \infty;\ C > 0 \text{ konstant})$$

und

(5.10) $$0 \le p_k \le 2z_0^{k-1} \qquad (k \ge 0).$$

Beweis: Die allgemeine Lösung der Differenzengleichung (5.7) hat die Darstellung $p_k = Cz_0^k + C_1 z_1^k + C_2 z_2^k$, wobei z_1 und z_2 die konjugiert komplexen Wurzeln der charakteristischen Gleichung $z^3 - z - 1 = 0$ bezeichnen, so daß (5.9) aus $z_0 > 1$ und $|z_1| = |z_2| < 1$ folgt.
Die Ungleichungen (5.10) beweist man durch vollständige Induktion.

6. Existenz einer nichtnegativen Lösung von (S), (A)

Satz 2: Unter den Voraussetzungen (3.1) bis (3.8) gilt für alle $t \ge 0$

(6.1) $$0 \le x_n^N(t) \le x_n^{N+1}(t) \qquad (0 \le n \le N),$$

es existieren die Grenzfunktionen

(6.2) $$x_n(t) = \lim_{N \to \infty} x_n^N(t) \qquad (t \ge 0),$$

und die x_n bilden eine nichtnegative Lösung des Anfangswertproblems (S), (A) mit der Eigenschaft

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

 Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(6.3)\qquad \sum_{n=0}^{\infty} x_n(t) z_0^n < \infty \qquad (t \geq 0) .$$

Beweis: Aus [6] übernehmen wir

$$(6.4)\qquad q_{nm}^N(t) \geq 0 \qquad (0 \leq m,\ n \leq N,\ t \geq 0)$$

und die für jede natürliche Zahl p und jedes $s > \delta - \alpha$ geltenden Abschätzungen

$$(6.5)\qquad 0 \leq q_{nm}^{N+p}(t) - q_{nm}^N(t) \leq \lambda_n e^{st}\left[Q_{nm}^{N+p}(s) - Q_{nm}^N(s)\right] \quad (0 \leq m,\ n \leq N,\ t \geq 0).$$

Entsprechend beweist man

$$(6.6)\qquad q_{nm}^{N+p}(t) \leq \lambda_n e^{st} Q_{nm}^{N+p}(s) \qquad (0 \leq n \leq N < m \leq N+p,\ t \geq 0,\ s > \delta - \alpha).$$

Damit folgen aus (4.1) $x_n^N(t) \geq 0$ $(0 \leq n \leq N,\ t \geq 0)$ und für alle $s > \delta - \alpha$, $t \geq 0$ die Abschätzungen

$$(6.7)\qquad 0 \leq x_n^{N+p}(t) - x_n^N(t) \leq$$

$$\leq \lambda_n e^{st}\left\{ \sum_{m=0}^{N}\left[Q_{nm}^{N+p}(s) - Q_{nm}^N(s)\right]\tilde{f}_m(s,t) + \sum_{m=N+1}^{N+p} Q_{nm}^{N+p}(s)\tilde{f}_m(s,t) \right\}$$

mit den abkürzenden Bezeichnungen

$$(6.8)\qquad \tilde{f}_m(s,t) = c_m + \int_0^t e^{-s\tau} f_m(\tau)\, d\tau .$$

Setzt man in (6.7) die Darstellungen (4.3) ein und verwendet die Abschätzungen (5.1), (5.3) und (5.6) sowie (5.7), so ergeben sich nach einiger Rechnung die für alle $s > \delta - \alpha$, $t \geq 0$, $0 \leq n \leq N$ gültigen Ungleichungen

$$(6.9)\qquad 0 \leq x_n^{N+p}(t) - x_n^N(t) \leq$$

$$\leq \frac{e^{st}}{d}\left\{ (\varepsilon_\beta^N + \varepsilon_\gamma^N)\left(1 + \frac{\varepsilon_\beta^{N-2} + \varepsilon_\gamma^{N-2}}{d}\right)\left[\sum_{m=0}^{n} \tilde{f}_m \prod_{\nu=m}^{n-1} (\delta_\nu/\lambda_\nu) + \sum_{m=n+1}^{N} p_{m-n+2}\tilde{f}_m \right] + \right.$$

$$\left. + \sum_{m=N+1}^{N+p} p_{m-n+2}\tilde{f}_m \right\} .$$

Wegen (3.7), (3.8) und (5.10) sind die Reihen $\sum_{m=n+1}^{\infty} p_{m-n+2}\tilde{f}_m$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

konvergent, so daß aus (6.9) die Existenz der Grenzfunktionen (6.2) folgt. Wie man leicht sieht, ist die Konvergenz $x_n^N \longrightarrow x_n$ ($N \to \infty$, $n = 0,1,2,\ldots$) bezüglich t in jedem endlichen Intervall $[0,T]$ gleichmäßig. Daraus schließt man wie in [6] mittels der Differentialgleichungen des Systems (S) auf die gleichmäßige Konvergenz $x_n^{N'} \longrightarrow x_n'$ und schließlich durch Grenzübergang $N \to \infty$ in den Differentialgleichungen für die x_n^N, daß die x_n das System (S) lösen. Klar ist, daß die x_n auch die Anfangsbedingungen (A) erfüllen. Multipliziert man noch die Differentialgleichungen von (S^N) nach ihrer Reihenfolge mit z_0^n ($n = 0,1,2,\ldots,N$) und addiert die erhaltenen Gleichungen, so bekommt man wegen $\delta_N \geq 0$ und $z_0 \geq 1$ sowie (3.4)

$$\frac{d}{dt}\sum_{n=0}^{N} x_n^N(t) z_0^n \leq (\delta z_0 - \alpha)\sum_{n=0}^{N} x_n^N(t) z_0^n + \sum_{n=0}^{N} f_n(t) z_0^n ,$$

woraus mit einem bekannten Vergleichssatz (s. [7], Ch.II, 12., Th. IX) zunächst

$$(6.10)\quad \sum_{n=0}^{N} x_n^N(t) z_0^n \leq e^{(z_0\delta-\alpha)t}\left\{ \sum_{n=0}^{N} c_n z_0^n + \right.$$
$$\left. + \int_0^t e^{-(z_0\delta-\alpha)\tau}\left[\sum_{n=0}^{N} f_n(\tau) z_0^n \right] d\tau \right\}$$

für alle $t \geq 0$, $N \geq 0$ und schließlich mit $N \longrightarrow \infty$ wegen (3.7), (3.8) die Behauptung (6.3) folgt.

Bemerkung 1: Wegen (5.9) sind die Konvergenzbedingungen (3.7), (3.8) notwending und hinreichend für die Konvergenz der Reihen $\sum_{m=n+1}^{\infty} p_{m-n+2}\tilde{f}_m$.

Bemerkung 2: Im Fall $\alpha = d = 0$ und $f_n = 0$ $(n = 0,1,\ldots)$ folgt aus (6.10)

$$\sum_{n=0}^{\infty} x_n(t) z_0^n \leq \sum_{n=0}^{\infty} x_n(0) z_0^n .$$

Bemerkung 3: Wenn es gelingt, statt (5.6) eine schärfere

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Abschätzung $B_i^j(s) \le M \prod_{\nu=i}^{j} \lambda_\nu$ $(1 \le i \le j+2$, M konstant, $s > \delta-\alpha)$ zu beweisen, können die Konvergenzbedingungen (3.7), (3.8) zu $\{C_n\} \in l_1$, $\{f_n\} \in l_1$ abgeschwächt werden. In diesem Fall gilt dann $\{x_n\} \in l_1$ und $\sum_{n=0}^{\infty} x_n(t) \le \sum_{n=0}^{\infty} C_n$.

Die durch (6.2) definierte Lösung von (S), (A) läßt sich durch die folgende Extremaleigenschaft charakterisieren.

Satz 3: Ist $\tilde{x} = \{\tilde{x}_n\}_{n\ge 0}$ irgendeine nichtnegative Lösung von (S), (A), so gilt $x(t) \le \tilde{x}(t)$ für alle $t \ge 0$.

Beweis: Setzt man $g_n(x_0,\ldots,x_N) = \delta_{n-1}x_{n-1} - \alpha_n x_n + \beta_{n+1}x_{n+1} + \gamma_{n+2}x_{n+2} + f_n$ $(n=0,1,\ldots,N)$ mit $\beta_{N+1} = \gamma_{N+1} = \gamma_{N+2} = 0$, so gilt wegen $\tilde{x}_n(t) \ge 0$ für $n = 0,1,\ldots,N$

$\tilde{x}_n'(t) \ge g_n(\tilde{x}_0,\ldots,\tilde{x}_N)$, $(x_n^N)' = g_n(x_0^N,\ldots,x_N^N)$, $x_n^N(0)=\tilde{x}_n(0)=C_n$.

Die g_n sind quasimonoton wachsende Funktionen, so daß nach dem schon im vorigen Beweis zitierten Vergleichssatz $x_n^N(t) \le \tilde{x}_n(t)$ $(t \ge 0,\ n = 0,1,\ldots N)$ und mit $N \longrightarrow \infty$ die Behauptung von Satz 3 folgt.

Bemerkung: Offen ist nach Kenntnis des Verfassers die Frage nach der Existenz mehrerer nichtnegativer Lösungen von (S), (A).

7. Fehlerabschätzungen

Setzt man

$$(7.1)\qquad \varepsilon_C^n = \sum_{\nu=n}^{\infty} C_\nu z_0^\nu,\quad \varepsilon_f^n = \sum_{\nu=n}^{\infty} f_\nu(t) z_0^\nu,\quad \delta_{nm} = \prod_{\nu=m}^{n-1} (\delta_\nu/\lambda_\nu)$$

$$(0 \le m \le n),$$

so erhält man aus (6.9) mit $p \longrightarrow \infty$ und (5.10)

$$(7.2)\qquad 0 \le x_n(t) - x_n^N(t) \le$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq \inf_{s>\delta-\alpha}\left\{ \frac{e^{st}}{d}\left(\left(\varepsilon_\beta^N + \varepsilon_\gamma^N\right)\left(1 + \frac{\varepsilon_\beta^{N-2}+\varepsilon_\gamma^{N-2}}{\alpha}\right)\left[\sum_{m=0}^{n}\delta_{nm}C_m + \right.\right.\right.$$

$$+ \frac{2\varepsilon_C^{n+1}}{z_0^{n-1}} + \int_0^t e^{-s\tau}\left(\sum_{m=0}^{n}\delta_{nm}f_m(\tau) + \frac{2\varepsilon_f^{n+1}}{z_0^{n-1}}\right)d\tau\Big] +$$

$$\left.\left.\left. + \frac{2}{z_0^{n-1}}\left[\varepsilon_C^{N+1} + \int_0^t e^{-s\tau}\varepsilon_f^{N+1}d\tau\right]\right)\right\}.$$

Zur Vereinfachung können noch die δ_{nm} durch 1 ersetzt werden.

Einen nicht kleineren Näherungswert für das Infimum erhält man, wenn für s die von t abhängige Funktion

$$(7.3)\qquad s_0 = s_0(t) = \begin{cases} \delta-\alpha+1/(t+\sigma) & \text{für } 0\leq l<\infty\,, \\ \delta-\alpha+\sqrt{\rho/(t+\sigma)} & \text{für } l=\infty \end{cases}$$

eingesetzt wird [6]. Dann ist

$$(7.4)\qquad \exp(s_0t)/d(s_0) = M(t)\,\exp[(\delta-\alpha)t + \delta\sigma]$$

mit

$$(7.5)\qquad M(t) = \begin{cases} 1 & \text{für } l=0 \\ [e+e\delta(t+\sigma)/l]^l & \text{für } 0<l<\infty \\ \exp\{2\sqrt{\rho/(t+\sigma)}\} & \text{für } l=\infty\,, \end{cases}$$

so daß für $\delta < \alpha$, also im Falle der Stabilität, der Faktor $\exp(s_0t)/d(s_0)$ mit $t \longrightarrow \infty$ exponentiell gegen Null strebt.

Gibt es einen Index n_0 mit $\delta_n\beta_{n+1} = \delta_{n-1}\delta_n\gamma_{n+1} = 0$ für $n \geq n_0$, so sind nach (5.4) $\varepsilon_\beta^N = \varepsilon_\gamma^N = 0$ für $N \geq n_0$, und (7.2) reduziert sich mit $s = s_0$ auf

$$(7.6)\qquad 0 \leq x_n(t) - x_n^N(t) \leq \frac{2M(t)}{z_0^{n-1}}e^{(\delta-\alpha)t+\delta\sigma}\left(\varepsilon_C^{N+1} + \right.$$

$$\left. + \int_0^t e^{-(\delta-\alpha)\tau}\varepsilon_f^{N+1}(\tau)d\tau\right) \qquad (0 \leq n \leq N,\ N \geq n_0).$$

Das gilt speziell für $\delta = 0$, also wenn A eine obere Dreiecksmatrix vom hier betrachteten Typ ist. Sind außerdem noch $c_n = f_n = 0$ für $n \geq n_0 + 1$, so stimmen die x_n^N für $0 \leq n \leq N$, $N \geq n_0$ mit den exakten Lösungskomponenten x_n überein.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

8. Ein Anwendungsbeispiel

Im einleitend dargestellten Beispiel der Radikalpopulation bei Polymerisationsprozessen sind $\alpha_n = \beta_n + \gamma_n + \delta_n$, $\beta_n = n\beta$, $\gamma_n = n(n-1)\gamma$, $\delta_n = \delta$, $f_n = 0$, $C_0 = 1$, $C_1 = C_2 = \ldots = 0$. Zur Vereinfachung beschränken wir uns noch auf den praktisch relevanten Fall $\beta = 0$, $\gamma > 0$, $\delta > 0$. Dann erhält man aus (3.4) $\alpha = \delta$, aus (3.5) $\sigma = 1/\gamma$ und aus (3.6) $\rho = \delta$, so daß nach Satz 2 eine nichtnegative Lösung $\{x_n\}_{n\geq 0}$ existiert, die durch die eindeutigen Lösungen der abgeschnittenen Systeme (S^N) mit beliebiger Genauigkeit approximiert werden kann. Eine einfache Schranke für die Abweichung der Näherungslösungen erhält man nach dem vorigen Abschnitt in folgender Weise. Nach (5.4) sind $\varepsilon_\beta^N = 0$ und, mit der abkürzenden Bezeichnung $q = \delta/\gamma$,

$$\varepsilon_\gamma^N = q^2 \sum_{\nu=N}^{\infty} \frac{1}{\nu(\nu-1)^2(\nu-2)} \leq \frac{q^2}{3(N-3)(N-2)^2} \qquad (N \geq 4),$$

woraus für $N \geq 8$ die etwas einfachere Abschätzung $\varepsilon_\gamma^N \leq q^2/N^3$ folgt. Nach (7.1) sind $\varepsilon_C^0 = 1$, $\varepsilon_C^n = 0$ $(n=1,2,\ldots)$, $\varepsilon_f^n = 0$ $(n=0,1,\ldots)$,

$$\delta_{n0} = \delta^n / \prod_{\nu=0}^{n-1} [s+\nu(\nu-1)\gamma+\delta] \qquad (n=0,1,\ldots).$$

Somit ergibt sich aus (7.2) für $N \geq 6$ bzw. $N \geq 8$

$$0 \leq x_n(t) - x_n^N(t) \leq$$

$$\leq \frac{q^2}{3(N-3)(N-2)^2} \inf_{s>0}\left\{\frac{e^{st}}{d(s)}\left[1 + \frac{q^2}{3(N-5)(N-4)^2 d(s)}\right]\delta_{n0}\right\} \leq$$

$$\leq \frac{q^2}{N^3}\inf\left\{\frac{e^{st}}{d(s)}\left[1 + \frac{q^2}{(N-2)^3 d(s)}\right]\delta_{n0}\right\}$$

mit der durch (5.2) definierten Funktion $d(s) = s\exp\{-(s+\delta)/\gamma\}/(s+\delta)$ (die Relationen $\alpha_\nu > \alpha_{\nu-1} = \alpha$ gelten nur für $\nu=2$, so daß $l=1$ ist). Die δ_{n0} lassen sich noch ohne

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

wesentliche Einbußen an Genauigkeit durch die von s unabhängigen oberen Schranken

$$\bar{\delta}_n = \begin{cases} 1 & \text{für } n=0,1,2 \\ q^{n-2}/[(n-2)!(n-1)!] & \text{für } n=3,4,\ldots \end{cases}$$

ersetzen. Das Infimum läßt sich approximieren, indem man für s die durch (7.3) definierte Funktion $s_0 = \gamma/(1+\gamma t)$ einsetzt. (Im Punkt s_0 hat die Funktion $e^{st}/[(s+\delta)d(s)]$ bezüglich s ein relatives Minimum). Dann ist

$$1/d(s_0) = [1+q(1+\gamma t)]\exp[q+1/(1+\gamma t)],$$

und man erhält schließlich die numerisch unmittelbar auswertbare Fehlerabschätzung

$$0 \le x_n(t) - x_n^N(t) \le \frac{q^2\bar{\delta}_n}{N^3}\left[1 + q(\gamma t+1)\right] e^{q+1}\left[1 + \frac{q^2}{(N-2)^3 d(s_0)}\right]$$

$$(N \ge 8, \quad t \ge 0, \quad n \ge 0),$$

woraus ersichtlich ist, daß die Konvergenz für beschränkte Werte t mindestens von der Größenordnung $1/N^3$ und für $t \longrightarrow \infty$ mindestens von der Größenordnung t/N^3 ist.

9. Existenz von exp(At)

Für ein unendliches System linearer gewöhnlicher Differentialgleichungen $x' = Ax + f$, $x(0) = C$ kann man unter gewissen Voraussetzungen, zu denen die Existenz der unendlichen Matrix $\exp(At) = \sum_{n=0}^{\infty} A^n t^n/n!$ gehört, in geeigneten Folgenräumen die Konvergenz der durch sukzessive Approximation erhältlichen Näherungen gegen eine Lösung mit der Darstellung

$$x(t) = \exp(At)C + \int_0^t \exp\{A(t-\tau)\}f(\tau)d\tau \quad (t\ge 0) \tag{9.1}$$

beweisen. (Nach Theorem 6.3 aus [2] konvergieren die oben mit x_n^N bezeichneten Funktionen für $N \longrightarrow \infty$ und jedes $n = 0,1,\ldots$ gleichmäßig in jedem endlichen Intervall $[0,T]$ gegen die Lösung (9.1), falls $\exp(|A|t)C$ und $\exp(|A|t)f$ existieren, wobei $|A|$ die aus den Beträgen der Elemente von A gebildete Matrix bedeutet. Der Beweis ist allerdings in der angegeben

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Form nicht richtig. Ein ähnlicher Satz aus [4] gilt nur für eindeutig lösbare Systeme).

Die beiden folgenden Beispiele belegen, daß zwischen Existenzsätzen dieser Art und dem oben bewiesenen Existenzsatz kein unmittelbarer Zusammenhang bestehen kann. Sie zeigen, daß unter den Bedingungen (3.1) BIS (3.6) für die Koeffizientenmatrix A von (S) keine allgemeingültige Aussage über Existenz oder Nichtexistenz von exp(At) gemacht werden kann.

Beispiel 1: Ist $\delta_n=0$ $(n=0,1,\ldots)$, so ist die Koeffizientenmatrix A von (S) eine obere Dreiecksmatrix mit höchstens drei nichtverschwindenden Elementen in jeder Zeile. Man kann sich leicht überlegen, daß in diesem Fall, unabhängig vom Erfülltsein der Voraussetzungen (3.1) bis (3.6), exp(At) und exp(|A|t) für alle $t \geq 0$ existieren.

Beispiel 2: Wählt man in (S) für $n=0,1,\ldots$

(9.2) $\delta_n = 1$, $\alpha_n = (n+1)(2n+1)(2n+2)$, $\beta_{n+1} = \alpha_{n+1}$, $\gamma_{n+2} = 0$,

so ist leicht zu sehen, daß die Voraussetzungen (3.1) bis (3.6) erfüllt sind. Werden die Elemente der jeweils ersten Spalte der Matrixpotenzen A^n mit $a_i^{(n)}$ bezeichnet (A Koeffizientenmatrix von (S)), so können leicht die folgenden Relationen bewiesen werden:

$$a_{2i}^{(2n-1)} \begin{cases} \leq -\prod_{\nu=2i}^{n+i-1} \alpha_\nu & \text{für } i \leq n-1, \\ = 0 & \text{für } i > n, \end{cases}$$

$$a_{2i+1}^{(2n-1)} \begin{cases} \geq \prod_{\nu=2i+2}^{n+i} \alpha_\nu & \text{für } i < n-1, \\ = 1 & \text{für } i = n-1, \\ = 0 & \text{für } i \geq n, \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$a_{2i}^{(2n)} \begin{cases} \geq \prod_{\nu=2i+1}^{n+i} \alpha_\nu & \text{für } i<n, \\ = 1 & \text{für } i=n, \\ = 0 & \text{für } i>n, \end{cases} \qquad a_{2i+1}^{(2n)} \begin{cases} \leq -\prod_{\nu=2i+1}^{n+i} \alpha_\nu & \text{für } i<n, \\ = 0 & \text{für } i\geq n. \end{cases}$$

Insbesondere gilt demnach $|a_0^{(2n-1)}| \geq \prod_{\nu=0}^{n-1} \alpha_\nu = n!(2n)!$, so daß die Reihe $\sum_{n=0}^{\infty} a_0^{(n)} t^n/n!$ für alle $t \neq 0$ divergiert.

10. Ein Existenzatz für ein System mit variablen Koeffizienten

In [2] wird ein allgemeiner Existenzsatz für zeilenfinite Systeme abzählbar vieler (nicht notwendig linearer) gewöhnlicher Differentialgleichungen bewiesen (s. Theorem 6.3. in [2]) zu dessen Voraussetzungen die Existenz einer nichtnegativen Lösung eines Vergleichssystems gehört. Er gestattet in Verbindung mit Satz 2 den Beweis des folgenden Existenzsatzes für das Anfangswertproblem (S), (A) bei nicht konstanten Koeffizienten. Wir schreiben statt (S)

(10.1) $$y_i' = d_{i-1}y_{i-1} - (\alpha_i+a_i)y_i + b_{i+1}y_{i+1} + c_{i+2}y_{i+2} + f_i$$
$$(i = 0,1,\ldots;\ d_{-1} = 0)$$

und setzen zwecks einfacherer Schreibweise

(10.2) $$\rho_i(t) = \exp \int_0^t a_i(\tau)\, d\tau, \quad \sigma_i(t) = \rho_i(t)\exp(\alpha_i t).$$

Satz 4: Das unendliche System (10.1) besitzt in $[0,T]$ eine Lösung mit $y_i(0) = c_i \in \mathbb{R}$, falls folgende Bedingungen erfüllt sind:

(i) $d_{i-1}, a_i, b_{i+1}, c_{i+2}, f_i \in C[0,T] \quad (i = 0,1,\ldots)$;

(ii) $|d_{i-1}| \leq \delta_{i-1}\rho_{i-1}/\rho_i, \quad |b_{i+1}| \leq \beta_{i+1}\rho_{i+1}/\rho_i,$

(iii) $|c_{i+2}| \leq \gamma_{i+2}\rho_{i+2}/\rho_i \quad (0 \leq t \leq T,\ i = 0,1,\ldots)$;
$\alpha_i, \beta_i, \gamma_i, \delta_i$ genügen (3.1) bis (3.6);

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) $\sum_{i=0}^{\infty} |c_i| z_0^i < \infty$ und $\sum_{i=0}^{\infty} \rho_i(t) |f_i(t)| z_0^i < \infty$ $(0 \le t \le T)$.

Beweis: Das System (10.1) ist äquivalent dem System
$\tilde{y}_i' = \varphi_i(t, \tilde{y}_{i-1}, \tilde{y}_{i+1}, \tilde{y}_{i+2}) = \tilde{d}_{i-1}\tilde{y}_{i-1} + \tilde{b}_{i+1}\tilde{y}_{i+1} + \tilde{c}_{i+2}\tilde{y}_{i+2} + \tilde{f}_i \quad (i = 0,1,\ldots)$
mit $\tilde{y}_i = \sigma_i y_i$, $\tilde{d}_{i-1} = d_{i-1}\sigma_i/\sigma_{i-1}$, $\tilde{b}_{i+1} = b_{i+1}\sigma_i/\sigma_{i+1}$, $\tilde{c}_{i+2} = c_{i+2}\sigma_i/\sigma_{i+2}$, $\tilde{f}_i = \sigma_i f_i$. Wegen (ii) gilt für $0 \le t \le T$

$$|\varphi_i(t)| \le \psi_i(t, |\tilde{y}_{i-1}|, |\tilde{y}_{i+1}|, |\tilde{y}_{i+2}|)$$

mit

$$\psi_i(t, z_{i-1}, z_{i+1}, z_{i+2}) = e^{\alpha_i t}\left[\delta_{i-1} e^{-\alpha_{i-1} t} z_{i-1} + \beta_{i+1} e^{-\alpha_{i+1} t} z_{i+1} + \gamma_{i+2} e^{-\alpha_{i+2} t} z_{i+2} + \rho_i |f_i|\right].$$

Da das unendliche System

$$z_i' = \psi_i(t, z_{i-1}, z_{i+1}, z_{i+2}), \quad z_i(0) = |c_i|$$

durch die Substitution $z_i = \exp(\alpha_i t) x_i$ in das äquivalente System (S) mit $\rho_i |f_i|$ statt f_i und $x_i(0) = |c_i|$ übergeht, besitzt es nach Satz 2 eine nichtnegative Lösung. Aus deren Existenz und der Monotonie der ψ_i bezüglich der Variablen z_i folgt nach Theorem 6.3 aus [2] die Behauptung des Satzes.

Bemerkung 1: Bezüglich der Bedingungen (iv) sei auf die Bemerkung in Abschnitt 6. verwiesen.

Bemerkung 2: Für $a_i(t) = a(t)$ $(i = 0,1,\ldots)$ sind, die ρ_i unabhängig von i und die Bedingungen (ii) vereinfachen sich zu $|d_{i-1}| \le \delta_{i-1}$, $|b_{i+1}| \le \beta_{i+1}$, $|c_{i+2}| \le \gamma_{i+2}$.

LITERATUR

[1] **D. C. Blackley:** Theory of Kinetics of Compartmentalized Free-Radical Polymerization Reactions. In: I. Pirma (ed.): Emulsion Polymerization. New York, London, San Francisco: Academic Press 1982, 145-190.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] K. Deimling: Ordinary differential equations in Banach spaces, Lect. Notes in Mathematics Vol. 596. Berlin, New York: Springer-Verlag 1977.

[3] G. Doetsch: Handbuch der Laplace-Transformation II. Basel -Stuttgart: Birkhäuser-Verlag 1955.

[4] R. Lemmert, Ä. Weckbach: Charakterisierungen zeilenendlicher Matrizen mit abzählbarem Spektrum, Math. 188 (1984), 119-124.

[5] J. Stoer, R. Bulirsch: Einführung in die numerische Mathematik Bd. 2. Berlin-Heidelberg-New-York: Springer-Verlag 1973.

[6] E. Wagner: Lösungsapproximation und Fehlerabschätzungen für ein unendliches System linearer, gewöhnlicher Differentialgleichungen mit konstanter Bandmatrix, Z. Anal. Anw. 8 (1989) 445-462.

[7] W. Walter: Differential and Integral Inequalities. Berlin -Heidelberg-New-York: Springer-Verlag 1970.

SEKTION MATHEMATIK DER MARTIN-LUTHER-UNIVERSITÄT
GERMANY-4020 HALLE/S.

Received April 18, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Andrzej Borzymowski, Zawam A.Dalla

A BOUNDARY VALUE PROBLEM FOR A SYSTEM OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbb{R}^m$

1. Boundary value problems for hyperbolic partial differential equations of order higher than two, and systems of such equations, have been dealt with in papers [1] - [12] (see also the references therein). In many of these papers the boundary conditions of the problems were given on two curves (surfaces) or on the characteristics of the equation considered or on the characteristics and one non-characteristic curve (surface). The harder case of a greater number of non characteristic curves or surfaces was examined in papers [1] - [4], [8], [9] and [12]. The aim of this paper is to extend the results of paper [12], devoted to a 2 dimensional problem, to the case of the $\mathbb{R}^m$-space with arbitrary m. For m=2 our assumptions, as well as results, are partly different from those in [12] (see Remark 2 in the sequel). We reduce the problem to a system of nonlinear integral - functional equations and then prove the existence of a solution (global or local, depending on the assumptions) by using the Schauder fixed point theorem. We also prove the uniqueness of the (local) solution by applying the Banach fixed point theorem. Our reasoning is based on a number of lemmas given in Section 3 of the paper. Let us note that the present results generalize those in paper [5] (Chapter I) in which the case n=3 was examined.

AMS (MOS,1980) classification: 35L35

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Let $N \ni m \geq 2$, where N is the set of all positive integers. We consider the parallelepiped [1]

$$P = \{x=(x_i)\in\mathbb{R}^m : 0\leq x_i \leq A_i < \infty;\ i=1,2,\ldots,m\}$$

and denote by P_ν (where $\nu=1,2,\ldots,m$) the set of points $x^{(\nu)}=(x_i)\in\mathbb{R}^{m-1}$ such that $x_i\in\langle 0,A_i\rangle$; $i=1,2,\ldots,m$; $i\neq\nu$.

Given numbers $r_1,r_2,\ldots r_m\in N$, we introduce a system of surfaces $S_{\nu,k}$ $(k=1,2,\ldots,r_\nu;\ \nu=1,2,\ldots,m)$ of equations $x_\nu = f_{\nu,k}(x^{(\nu)})$ $(x^{(\nu)}\in P_\nu)$, respectively. We admit the notation $\hat{x}^{(\nu,k)} = (x_i)\in\mathbb{R}^m$, where x_ν is replaced by $f_{\nu,k}(x^{(\nu)})$.

Let $\mathfrak{k}$ be the class of all functions (vectors) $u=(u^l):P\longrightarrow\mathbb{R}^n$ ($n\in N$ being fixed) such that the derivatives $D^\beta u$, where $D^\beta = \prod_{\mu=1}^{m} D_\mu^{\beta_\mu}$ with $D_\mu^{\beta_\mu} = \dfrac{\partial^{\beta_\mu}}{\partial x_\mu^{\beta_\mu}}$ exist for $\beta_\mu=0,1,\ldots,r_\mu$; $\mu=1,2,\ldots,m$, are continuous and do not depend on the order in which the differentiation is performed.

For $u\in\mathfrak{k}$ we use the following functions (vectors)

(1) $$V=(v_{s,k})$$

$(k=1,2,\ldots,r_s;\ s=1,2,\ldots,m)$, where

(2) $v_{s,k} = D^\beta u$ with $\beta_s=k-1$; $\beta_\mu=r_\mu$ for $\mu\neq s$ $(\mu=1,2,\ldots,m)$;

(3) $$\Omega^\alpha = (\omega^\alpha_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}) \quad \text{when } m\geq 3$$

$(\alpha=1,2,\ldots,m-2;\ \vec{s}_{(\alpha)}=(s_i)$ with $i=1,2,\ldots,\alpha;\ N\ni s_i\leq m;\ s_i\neq s_j$ for $i\neq j$, $\vec{s}_{(1)}=s_1$; $\vec{k}_{(m-\alpha)}=(k_i)$ with $i=1,2,\ldots,m;\ i\neq s_1,\ldots,s_\alpha$; $k_i=1,2,\ldots,r_i)$, where

(4) $$\omega^\alpha_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}} = D^\beta u \text{ with } \beta_\mu = \begin{cases} r_\mu & \text{for } \mu=s_1,\ldots,s_\alpha \\ k_\mu-1 & \text{for } \mu\neq s_1,\ldots,s_\alpha \end{cases}$$

$(k_\mu=1,2,\ldots,r_\mu;\ \mu=1,2,\ldots,m)$;

---

1) Here and in the sequel (.) denotes a finite sequence

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(5) $Z = (z_{\vec{k}_{(m)}})$

$(\vec{k}_{(m)} = (k_i)$ with $i=1,2,\ldots,m;\ k_i=1,2,\ldots,r_i)$, where

(6) $z_{\vec{k}_{(m)}} = D^{\beta}u$ with $\beta_\mu = k_\mu - 1$

$(\mu=1,2,\ldots,m)$.

We examine the following problem (S):

Find a solution u of the system of partial differential equations[2]

(7) $$D^{\beta}u = F\left[x, Z(x), (\Omega^{\alpha}(x)), V(x)\right]$$

$(\beta_\mu = r_\mu$ for $\mu=1,2,\ldots,m)$ in P, that is a function (vector) $u \in \mathfrak{X}$ satisfying system (7) at each point $x \in P$, subject to the following conditions

(8) $$v_{\nu,k}(x) = e_{\nu,k}\left[x^{(\nu)}, Z(\hat{x}^{(\nu,k)}), (\Omega^{\alpha}(\hat{x}^{(\nu,k)})), V^{\nu,*}(\hat{x}^{(\nu,k)})\right]$$

$(x^{(\nu)} \in P_\nu;\ k=1,2,\ldots,r_\nu;\ \nu=1,2,\ldots,m$ and $V^{\nu,*} = (v_{s,k_s})$ with $s=1,2,\ldots,m;\ s\neq\nu)$ when $x_\nu = f_{\nu,k}(x^{(\nu)})$;

(9) $$\omega^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}(x) = a^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}(x_{s_1},\ldots,x_{s_\alpha})$$

$(x \in P)$ when $x_i = x_i^*$ for $i=1,2,\ldots,m;\ i \neq s_1,\ldots,s_\alpha$,

and

(10) $$z_{\vec{k}_{(m)}}(\overset{0}{x}^{\vec{k}_{(m)}}) = \overset{0}{u}^{\vec{k}_{(m)}}.$$

Above, F, $e_{\nu,k}$ and $a^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}$ are given functions (vectors), $\overset{\beta}{u}(a) = (\overset{0}{u}^{1,\vec{k}_{(m)}})$; $1=1,2,\ldots,n$ with $\overset{0}{u}^{1,\vec{k}_{(m)}}$ being given

[2] If $m=2$, then the argument $(\Omega^{\alpha}(x))$ in (7) does not appear and, consequently, condition (9) and assumption IV in the sequel are superfluous.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

number vectors, the points $\overset{*}{x}_i$ are arbitrarily fixed in $\langle o,A_i\rangle$ $(i=1,2,\ldots,m)$, respectively [3], and the points $\overset{\vec{k}}{q}(m)=(\overset{\vec{k}}{q}_1(m),\ldots,\overset{\vec{k}}{q}_m(m))$ arbitrarily fixed in P.

We assume the following

I. The functions $f_{\nu,k}\colon P_\nu\longrightarrow\langle o,A_\nu\rangle$ $(k=1,2,\ldots,r_\nu;$ $\nu=1,2,\ldots,m)$ are continuous.

II. The function $F\colon P\times\mathbb{R}^{n\sigma}\longrightarrow\mathbb{R}^n$ (where $\sigma=r+\sum_{\alpha=1}^{m-2}\sum_{s_1,\ldots,s_\alpha=1}^{m}\prod_{\mu=1,\,\mu\neq s_1,\ldots,s_\alpha}^{m} r_\mu+\prod_{\mu=1}^{m} r_\mu$ with $r=\sum_{\mu=1}^{m} r_\mu$) is continuous and satisfies the condition [4]

$$(11)\qquad \left|F\left[x,\xi,(\eta^\alpha),\zeta\right]\right| \leq K_1+K_2\Xi_o+K_2'\Xi_o'$$

$\left(\xi=(\xi_{\vec{k}(m)}),(\eta^\alpha)=(\eta^\alpha_{\vec{s}(\alpha),\vec{k}(m-\alpha)}),\ \alpha=1,2,\ldots,m-2,\text{ and }\zeta=(\zeta_{s,k})\right)$,

where K_1, K_2 and K_2' are positive constants, while Ξ_o and E_o' are given by

$$(12)\qquad \Xi_o=\sum_{k_1=1}^{r_1}\ldots\sum_{k_m=1}^{r_m}\left|\xi_{\vec{k}(m)}\right|^\kappa+\sum_{\alpha=1}^{m-2}\sum_{1\leq k_i\leq r_i}\left|\eta^\alpha_{\vec{s}(\alpha),\vec{k}(m-\alpha)}\right|^\kappa$$

$(i=1,2,\ldots,m;\ i\neq s_1,\ldots,s_\alpha)$ and

$$(13)\qquad \Xi_o'=\sum_{s=1}^{m}\sum_{k=1}^{r_s}\left|\zeta_{s,k}\right|^\kappa$$

respectively, with $\kappa\in(0,1\rangle$.

---

[3] Without loss of generality we assume in the sequel that $x_i^*=0$ for $i=1,2,\ldots,m$.

[4] Here and in the sequel $|\cdot|$ denotes the modulus of an n-dimensional vector

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

III. The functions $e_{\nu,k} : P_\nu \times \mathbb{R}^{n(\sigma-r_\nu)} \longrightarrow \mathbb{R}^n$ $(k=1,2,\ldots,r_\nu;$ $\nu=1,2,\ldots,m)$ are continuous and satisfy the inequality

$$\left|e_{\nu,k}\left[x^{(\nu)},\xi,(\eta^\alpha)\zeta^{\nu,*}\right]\right| \leq K_3+K_4\Xi_0+K_4'\Xi_1 \tag{14}$$

$(\zeta^{\nu,*}=(\zeta_{s,k})$; $k=1,2,\ldots,r_s$; $s=1,2,\ldots,m$; $s\neq\nu)$, where K_3, K_4 and K_4' are positive constants, Ξ_0 is given by (12) and Ξ_1 denotes the expression Ξ_0' (see (13)) with the term corresponding to $s=\nu$ being omitted.

IV. The functions $a^\alpha_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}} : \overset{\alpha}{\underset{i=1}{X}} <0,A_{s_i}> \longrightarrow \mathbb{R}^n$ $(\alpha=1,2,\ldots,m-2)$ are continuous.

3. In this section we give several lemmas, using the following notation:

$$\begin{cases} x^{(p_1,\ldots,p_\mu,0)} = (x_i)\in\mathbb{R}^m \text{ with } x_{p_1}=\ldots=x_{p_\mu}=0 \quad (N\ni\mu,p_\mu\leq m); \\ x^{(p_1,\ldots,p_\mu,0,t)} = (x_i)\in\mathbb{R}^m \text{ with } x_{p_1},\ldots,x_{p_\mu}=0,\ x_{p_{\mu+1}}=t \end{cases} \tag{15}$$

$(N\ni\mu\leq m-1;\ N\ni p_\mu\leq m)$;

$$x^{(\nu,t)} = (x_i)\in\mathbb{R}^m \text{ with } x_\nu=t \tag{16}$$

$(N\ni\nu\leq m)$.

Lemma 1. Let $u\in\mathfrak{K}$; $N\ni p_1,p_2,\ldots,p_m\leq m\leq 3$; $p_i\neq p_j$ for $i\neq j$ $(i,j=1,2,\ldots,m)$. The following equality is valid

$$z_{\vec{k}_{(m)}}(x) = \int_0^{x_{p_1}} \frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\,\omega^1_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt + \tag{17}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\sum_{\tau=1}^{m-1}\sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}}\cdots\sum_{\gamma_{p_\tau}=0}^{r_{p_\tau}-k_{p_\tau}}\int_0^{x_{p_{\tau+1}}}\frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!}\cdot$$

$$\cdot\,\omega^{1}_{p_{\tau+1},\vec{k}_{(m-1)}+\vec{\gamma}^{\tau}_{(m-1)}}(x^{(p_1,\ldots,p_\tau,0,t)})\,dt\cdot\prod_{i=1}^{\tau}\frac{x_{p_i}^{\gamma_{p_i}}}{\gamma_{p_i}!}+$$

$$+\sum_{\gamma_1=0}^{r_1-k_1}\cdots\sum_{\gamma_m=0}^{r_m-k_m}C^{\vec{k}_{(m)}}_{\vec{\gamma}_{(m)}}\cdot\prod_{i=1}^{m}\frac{x^{\gamma_i}}{\gamma_i!}$$

($x\in P$; $\vec{\gamma}_{(m)}=(\gamma_i)$; $\vec{k}_{(m)}=(k_i)$; $i=1,2,\ldots,m$), where $\vec{k}_{(m-1)}=(k_i)$; $i=1,2,\ldots,m$, with $i\neq p_1$ in the first term and $i\neq p_{\tau+1}$ in the second one, $\vec{\gamma}^{\tau}_{(m-1)}=(\gamma_{p_i})$; $i=1,2,\ldots,m$; $i\neq\tau+1$; $\gamma_{p_i}=0$ for $i=\tau+2,\ldots,m$, and $C^{\vec{k}_{(m)}}_{\vec{\gamma}_{(m)}}=z_{\vec{k}_{(m)}+\vec{\gamma}_{(m)}}(0,\ldots,0)$ (with $\vec{\gamma}_{(m)}=(\gamma_i)$; $i=1,2,\ldots,m$) are constans, depending on V, defined by the recurrent formula

$$(18)\quad C^{\vec{k}_{(m)}}_{\vec{\gamma}_{(m)}}=Q^{\vec{k}_{(m)}+\vec{\gamma}_{(m)}}-H^{\vec{k}_{(m)}+\vec{\gamma}_{(m)}}(Q^{\vec{k}_{(m)}+\vec{\gamma}_{(m)}})+$$

$$-\sum^{*}_{\vec{\mu}_{(m)}}C^{\vec{k}_{(m)}}_{\vec{\mu}_{(m)}}\prod_{i=1}^{m}\frac{\left(Q_i^{k_i+\gamma_i}\right)^{\mu_i-\gamma_i}}{(\mu_i-\gamma_i)!}$$

with

$$(19)\quad H^{\vec{k}_{(m)}+\vec{\gamma}_{(m)}}(x)=\prod_{\mu=1}^{m}D_\mu^{\gamma_\mu}\left\{\int_0^{x_{p_1}}\frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\cdot\right.$$

$$\left.\cdot\,\omega^{1}_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt+\right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\sum_{\tau=1}^{m-1}\sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}}\cdots\sum_{\gamma_{p_\tau}=0}^{r_{p_\tau}-k_{p_\tau}}\int_0^{x_{p_{\tau+1}}}\frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!}\cdot$$

$$\cdot\,\omega^{1}_{p_{\tau+1},\vec{k}_{(m-1)}+\vec{\gamma}^{\tau}_{(m-1)}}(x^{(p_1,\ldots,p_\tau,0,t)})\,dt\,\prod_{i=1}^{\tau}\frac{x_{p_i}^{\gamma_{p_i}}}{\gamma_{p_i}!}\Bigg\},$$

the summation $\sum^{*}_{\vec{\mu}_{(m)}}$ in (18) being taken over all systems $\vec{\mu}_{(m)}=(\mu_1,\ldots,\mu_m)$ such that $\gamma_i\le\mu_i\le r_i-k_i$, $i=1,2,\ldots,m$, and $\mu_1+\ldots+\mu_m>\gamma_1+\ldots+\gamma_m$.

Proof. Using definitions (3) ($\alpha=1$; $s_1=p_1$) and (6), and applying Taylor's formula with the integral form of the remainder, we have

$$z_{\vec{k}_{(m)}}(x)=\int_0^{x_{p_1}}\frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\,\omega^{1}_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt+$$

$$+\sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}}D^{\gamma_{p_1}}_{p_1}z_{\vec{k}_{(m)}}(x^{(p_1,0)})\,\frac{x_{p_1}^{\gamma_{p_1}}}{\gamma_{p_1}!}$$

($\vec{k}_{(m-1)}=(k_i)$; $i=1,2,\ldots,m$; $i\ne p_1$), whence and by (6) we get

$$z_{\vec{k}_{(m)}}(x)=\int_0^{x_{p_1}}\frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\,\omega^{1}_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt+ \tag{20}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}} z_{\hat{\vec{k}}_{(m)}}(x^{(p_1,0)})\,\frac{x_{p_1}^{\gamma_{p_1}}}{\gamma_{p_1}!}$$

($\hat{\vec{k}}_{(m)}$ is understood as $\vec{k}_{(m)}$ with k_{p_1} replaced by $k_{p_1}+\gamma_{p_1}$).

Dealing in a similar way with $z_{\hat{\vec{k}}_{(m)}}$ and combining the obtained result with (20), we have

$$(21)\quad z_{\vec{k}_{(m)}}(x) = \int_0^{x_{p_1}} \frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\,\omega^1_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt +$$

$$+\sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}} \int_0^{x_{p_2}} \frac{(x_{p_2}-t)^{r_{p_2}-k_{p_2}}}{(r_{p_2}-k_{p_2})!}\,\omega^1_{p_2,\vec{k}_{(m-1)}}(x^{(p_1,0,t)})\,dt\,\cdot$$

$$\cdot\,\frac{x_{p_1}^{\gamma_{p_1}}}{\gamma_{p_1}!} + \sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}}\ \sum_{\gamma_{p_2}=0}^{r_{p_2}-k_{p_2}} z_{\vec{k}_{(m)}+\vec{\gamma}^2_{(m)}}(x^{(p_1,p_2,0)})\prod_{i=1}^{2}\frac{x_{p_i}^{\gamma_{p_i}}}{\gamma_{p_i}!},$$

($\vec{\gamma}^2=(\gamma_{p_i})$; $i=1,2,\ldots,m$; $\gamma_{p_i}=0$ for $i=3,\ldots,m$).

By basing on (21) and using mathematical induction, one can prove that

$$(22)\quad z_{\vec{k}_{(m)}}(x) = \int_0^{x_{p_1}} \frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!}\,\omega^1_{p_1,\vec{k}_{(m-1)}}(x^{(p_1,t)})\,dt +$$

$$+\sum_{\tau=1}^{\alpha}\ \sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}}\cdots\sum_{\gamma_{p_\tau}=0}^{r_{p_\tau}-k_{p_\tau}}\int_0^{x_{p_{\tau+1}}} \frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!}\,.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

boundary value problem

$$\cdot \omega^{1}_{p_{\tau+1},\vec{k}_{(m-1)}+\gamma^{\tau}_{(m-1)}}(x^{(p_1,\ldots,p_\tau,0,t)})\,dt \cdot \prod_{i=1}^{\tau} \frac{x_{p_i}^{\gamma_{p_i}}}{\gamma_{p_i}!} +$$

$$+ \sum_{\gamma_{p_1}=0}^{r_{p_1}-k_{p_1}} \cdots \sum_{\gamma_{p_{\alpha+1}}}^{r_{p_{\alpha+1}}-k_{p_{\alpha+1}}} z_{\vec{k}_{(m)}+\vec{\gamma}^{\alpha+1}_{(m)}}(x^{(p_1,\ldots,p_{\alpha+1},0)}) \prod_{i=1}^{\alpha+1} \frac{x_{p_i}^{\gamma_{p_i}}}{\gamma_{p_i}!},$$

where $N \ni \alpha \le m-1$.

Setting $\alpha=m-1$, we get from (22) the required relation (17). Relation (18) results from (17) by taking the derivative $\prod_{i=1}^{n} D_i^{\gamma_i}$ and using definition (6) and condition (10).

Lemma 2. If $u \in \mathfrak{K}$, $m \ge 3$ and $\vec{s}_{(m-2)}=(s_1,\ldots,s_{m-2})$ with $s_1,\ldots,s_{m-2}$ being as in (3), then (see (4))

$$(23)\quad \omega^{m-2}_{\vec{s}_{(m-2)},\vec{k}_{(2)}}(x) = \int_0^{x_p} \frac{(x_p-t)^{r_p-k_p}}{(r_p-k_p)!}\, \upsilon_{q,k_q}(x^{(p,t)})\,dt +$$

$$+ \sum_{\gamma_p=0}^{r_p-k_p} \int_0^{x_q} \frac{(x_q-t)^{r_q-k_q}}{(r_q-k_q)!}\, \upsilon_{p,k_p+\gamma_p}(x^{(p,0,t)})\,dt\, \frac{x_p^{\gamma_p}}{\gamma_p!} +$$

$$+ \sum_{\gamma_p=0}^{r_p-k_p} \sum_{\gamma_q=0}^{r_q-k_q} a_{\vec{s}_{(m-2)},\vec{k}_{(2)}+\tilde{\vec{\gamma}}_{(2)}}(x_{s_1},\ldots,x_{s_{m-2}})\, \frac{x_p^{\gamma_p}}{\gamma_p!}\, \frac{x_q^{\gamma_q}}{\gamma_q!}$$

($x \in P$, $N \ni p,q \le m$; $p \ne q$; $p,q \ne s_1,\ldots,s_{m-2}$; $\vec{k}_{(2)}=(k_p,k_q)$; $\tilde{\vec{\gamma}}_{(2)}=(\gamma_p,\gamma_q)$).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 3. Let $u\in\mathfrak{X}$, $m\geq 4$, $\alpha\in N$; $2\leq\alpha\leq m-2$ and $\vec{s}_{(\alpha-1)}=(s_1,\ldots,s_{\alpha-1})$ with $s_1,\ldots,s_{\alpha-1}$ being as in (3). Let, moreover, $p_\alpha, p_{\alpha+1},\ldots,p_m\in N$; $p_i\leq m$; $p_i\neq s_1,\ldots,s_{\alpha-1}$; $p_i\neq p_j$ for $i\neq j$ $(i,j=\alpha,\alpha+1,\ldots,m)$. The following equality is valid

$$(24)\quad \omega^{\alpha-1}_{\vec{s}_{(\alpha-1)},\vec{k}_{(m-\alpha+1)}}(x)=\int_0^{x_{p_\alpha}}\frac{(x_{p_\alpha}-t)^{r_{p_\alpha}-k_{p_\alpha}}}{(r_{p_\alpha}-k_{p_\alpha})!}\cdot$$

$$\cdot\,\omega^{\alpha}_{\vec{s}_{(\alpha)},\hat{\vec{k}}_{(m-\alpha)}}(x^{(p_\alpha,t)})\,dt\;+$$

$$+\sum_{\tau=\alpha}^{m-1}\sum_{\gamma_{p_\alpha}=0}^{r_{p_\alpha}-k_{p_\alpha}}\cdots\sum_{\gamma_{p_\tau}=0}^{r_{p_\tau}-k_{p_\tau}}\int_0^{x_{p_{\tau+1}}}\frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!}\cdot$$

$$\cdot\,\omega^{\alpha}_{\vec{s}_{(\alpha)},\hat{\vec{k}}_{(m-\alpha)}+\hat{\vec{\gamma}}^{\tau}_{(m-\alpha)}}(x^{(p_\alpha,\ldots,p_\tau,0,t)})\,dt\cdot\prod_{\nu=\alpha}^{\tau}\frac{x_{p_\nu}^{\gamma_{p_\nu}}}{\gamma_{p_\nu}!}\;+$$

$$+\sum_{\gamma_{p_\alpha}=0}^{r_{p_\alpha}-k_{p_\alpha}}\cdots\sum_{\gamma_{p_m}}^{r_{p_m}-k_{p_m}}a^{\alpha-1}_{\vec{s}_{(\alpha-1)},\vec{k}_{(m-\alpha+1)}+\hat{\vec{\gamma}}^{m-\alpha+1}_{(m-\alpha+1)}}(x_{s_1},\ldots,x_{s_{\alpha-1}})\cdot$$

$$\cdot\prod_{\nu=\alpha}^{m}\frac{x_{p_\nu}^{\gamma_{p_\nu}}}{\gamma_{p_\nu}!}$$

$(x\in P)$, where $\vec{s}_{(\alpha)}$ is understood as $(s_1,\ldots,s_{\alpha-1},p_\alpha)$ in the first term and as $(s_1,\ldots,s_{\alpha-1},p_{\tau+1})$ in the second one, $\hat{\vec{k}}_{(m-\alpha)}=(k_{p_i})$; $i=\alpha,\alpha+1,\ldots,m$ with $i\neq\tau+1$, $\hat{\vec{k}}_{(m-\alpha+1)}=(k_{p_i})$; $i=\alpha,\alpha+1,\ldots,m$, $\hat{\vec{\gamma}}^{\tau}_{(m-\alpha)}=(\gamma_{p_i})$; $i=\alpha,\alpha+1,\ldots,m$ with $i\neq\tau+1$, and $\hat{\vec{\gamma}}^{m-\alpha+1}_{(m-\alpha+1)}=(\gamma_{p_i})$ with $i=\alpha,\alpha+1,\ldots,m$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proofs of Lemmas 2 and 3 are similar to that of Lemma 1.

Remark 1. It follows from Lemmas 1-3 that the functions (vectors) Z and Ω^{α} $(\alpha=1,2,\ldots,m-2)$ can be expressed in terms of V. Doing so, we shall write

(25) $Z=L_0[V(.),x]; \qquad \Omega^{\alpha}=L_{\alpha}[V(.),x] \qquad (\alpha=1,2,\ldots,m-2).$

Now, let us consider the following system of integral-functional equations

(26) $$v_{\nu,k}(x) = e_{\nu,k}\{x^{(\nu)}(L_{\mu}[V(.),\hat{x}^{(\nu,k)}]), V^{\nu,*}(\hat{x}^{(\nu,k)})\}+$$
$$+\int_{f_{\nu,k}(x^{(\nu)})}^{x_{\nu}} v_{\nu,k+1}(x^{(\nu,t)})dt, \quad (k=1,2,\ldots,r_{\nu}-1);$$

(27) $$v_{\nu,r_{\nu}}(x) = e_{\nu,r_{\nu}}\{x^{(\nu)},(L_{\mu}[V(.),\hat{x}^{(\nu,r_{\nu})}]), V^{\nu,*}(\hat{x}^{(\nu,r_{\nu})})\}+$$
$$+\int_{f_{\nu,r_{\nu}}(x^{(\nu)})}^{x_{\nu}} F\{x^{(\nu,t)},(L_{\mu}[V(.),x^{(\nu,t)}]),V(x^{(\nu,t)})\}dt,$$

where $x\in P$, $x^{(\nu)}\in P_{\nu}$, $x^{(\nu,t)}$ is given by (16), $\nu=1,2,\ldots,m$ and $\mu=0,1,\ldots,m-2$.

Lemma 4. If u is a solution of problem (S), then V is a continuous solution of system (26), (27).

The validity of Lemma 4 follows from formulas (1)-(6) and Lemmas 1-3.

Lemma 5. If relations (23), (24), (26) and (27) are satisfied, $m\geq 3$ and $\alpha\in N$ with $2\leq\alpha\leq m-1$, then

(28) $$\prod_{\mu=\alpha}^{m} D_{p_{\mu}}^{k_{p_{\mu}}-1}\omega^{\alpha-1}_{\vec{s}_{(\alpha-1)},\vec{k}_{(m-\alpha+1)}}(x) = \omega^{\alpha-1}_{\vec{s}_{(\alpha-1)},\vec{k}_{(m-\alpha+1)}}(x)$$

(29) $$\prod_{\mu=\alpha}^{m-1} D_{p_{\mu}}^{r_{p_{\mu}}-k_{p_{\mu}}+1}\omega^{\alpha-1}_{\vec{s}_{(\alpha-1)},\vec{k}_{(m-\alpha+1)}}(x) = v_{p_m,k}(x)$$

$(x\in P;\ k=1,2,\ldots,r_{p_m},\ \vec{s}_{(\alpha-1)}$ and $\vec{k}_{(m-\alpha+1)}$ are understood as in

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 3, and $\overset{\circ}{\vec{k}}_{(m-\alpha+1)}$ denotes $\vec{k}_{(m-\alpha+1)}$ with $k_{p_i}=\overset{\circ}{k}_{p_i}:=1$ for $i=\alpha,\alpha+1,\ldots,m)$.

We omit an inductive, rather lengthy, proof of Lemma 5.

Lemma 6. If V is a continuous solution of system (26), (27) and $z_{\vec{k}_{(m)}}$ is given by relation (17) with $\omega^1_{p_\mu,\vec{k}_{(m-1)}}$ $(\mu=1,2,\ldots,m)$ expressed in terms of V (see (23),(24)), then $z_{\overset{\circ}{\vec{k}}_{(m)}} \in \mathfrak{X}$ and

$$z_{\vec{k}_{(m)}}(x) = \prod_{\mu=1}^{m} D_\mu^{k_\mu-1} z_{\overset{\circ}{\vec{k}}_{(m)}}(x) \tag{30}$$

$(x\in P$, and $\overset{\circ}{\vec{k}}_{(m)}$ is understood similarly as $\overset{\circ}{\vec{k}}_{(m-\alpha+1)}$ in (28)).

Proof. The first part of the thesis $(z_{\overset{\circ}{\vec{k}}_{(m)}} \in \mathfrak{X})$ follows directly from (17),(23) and (24). In order to prove relation (28) we differentiate $z_{\overset{\circ}{\vec{k}}_{(m)}}$ (see (17) with $\vec{k}_{(m)}=\overset{\circ}{\vec{k}}_{(m)}$), getting

$$\prod_{\mu=1}^{m} D_\mu^{k_\mu-1} z_{\overset{\circ}{\vec{k}}_{(m)}}(x) = \int_0^{x_{p_1}} \frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!} \cdot \tag{31}$$

$$\cdot \prod_{\mu=2}^{m} D_{p_\mu}^{k_{p_\mu}-1} \omega^1_{p_1,\overset{\circ}{\vec{k}}_{(m-1)}}(x^{(p_1,t)})\,dt + \sum_{\tau=1}^{m-1} \sum_{\gamma_{p_1}=k_{p_1}-1}^{r_{p_1}-1} \cdots$$

$$\cdots \sum_{\gamma_{p_\tau}=k_{p_\tau}-1}^{r_{p_\tau}-1} \int_0^{x_{p_{\tau+1}}} \frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!} \cdot$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\cdot \prod_{\mu=\tau+2}^{m} D_{p_\mu}^{k_{p_\mu}-1} \omega_{p_{\tau+1},\vec{k}_{(m-1)}^{\circ}}^{1} (x^{(p_1,\ldots,p_\tau,0,t)}) \, dt \cdot$$

$$\cdot \prod_{i=1}^{\tau} \frac{x_{p_i}^{\gamma_{p_i}-k_{p_i}+1}}{\gamma_{p_i}-k_{p_i}+1} + \sum_{\gamma_1=k_1-1}^{r_1-1} \cdots \sum_{\gamma_m=k_m-1}^{r_m-1} C_{\vec{\gamma}_{(m)}}^{\vec{k}_{(m)}^{\circ}} \cdot$$

$$\cdot \prod_{i=1}^{m} \frac{x_i^{\gamma_i-k_i+1}}{(\gamma_i-k_i+1)!} \cdot$$

Basing on Lemma 5, we obtain from (31) the following equality

$$(32) \quad \prod_{\mu=1}^{m} D_\mu^{k_\mu-1} z_{\vec{k}_{(m)}^{\circ}}(x) = \int_0^{x_{p_1}} \frac{(x_{p_1}-t)^{r_{p_1}-k_{p_1}}}{(r_{p_1}-k_{p_1})!} \cdot$$

$$\cdot \omega_{p_1,\vec{k}_{(m-1)}}^{1} (x^{(p_1,t)}) \, dt +$$

$$+ \sum_{\tau=1}^{m-1} \sum_{\beta_{p_1}=0}^{r_{p_1}-k_{p_1}} \cdots \sum_{\beta_{p_\tau}=0}^{r_{p_\tau}-k_{p_\tau}} \int_0^{x_{p_{\tau+1}}} \frac{(x_{p_{\tau+1}}-t)^{r_{p_{\tau+1}}-k_{p_{\tau+1}}}}{(r_{p_{\tau+1}}-k_{p_{\tau+1}})!} \cdot$$

$$\cdot \omega_{p_{\tau+1},\vec{k}_{(m-1)}+\vec{\beta}_{(m-1)}^{\tau}}^{1} (x^{(p_1,\ldots,p_\tau,0,t)}) dt \prod_{i=1}^{\tau} \frac{x_{p_i}^{\beta_{p_i}}}{\beta_{p_i}!} +$$

$$+ \sum_{\beta_1=0}^{r_1-k_1} \cdots \sum_{\beta_m=0}^{r_m-k_m} C_{\tilde{\vec{\beta}}_{(m)}+\vec{k}_{(m)}-\vec{k}_{(m)}}^{\vec{k}_{(m)}^{\circ}} \cdot \prod_{i=1}^{m} \frac{x_i^{\beta_i}}{\beta_i!} \cdot$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

As (see (18))

$$C^{\overset{\circ}{\vec{k}}_{(m)}}_{\vec{\beta}_{(m)}+\vec{k}_{(m)}-\overset{\circ}{\vec{k}}_{(m)}} = C^{\vec{k}_{(m)}}_{\vec{\beta}_{(m)}},$$

we easily conclude from (32) and (17) that relation (30) holds good. Q.E.D.

Lemma 7. Under the assumptions of Lemma 6, the following equalities are valid

$$\omega^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}(x) = \prod_{\mu=1}^{m} D_{\mu}^{\beta_{\mu}} z_{\overset{\circ}{\vec{k}}_{(m)}}(x) \tag{33}$$

($x\in P$; $\alpha=1,2,\ldots,m-2$; $m\geq 3$), where $\vec{s}_{(\alpha)}$, $\vec{k}_{(m-\alpha)}$ and β_{μ} ($\mu=1,2,\ldots,m$) are understood as in (3) - (5);

$$v_{s,k}(x) = D^{\beta} z_{\overset{\circ}{\vec{k}}_{(m)}}(x) \tag{34}$$

($x\in P$; $k=1,2,\ldots,r_s$; $s=1,2,\ldots,m$), where D^{β} is as in (2).

Proof is similar to that of Lemma 6.

Lemma 8. Under the assumptions of Lemma 6, the function $z_{\overset{\circ}{\vec{k}}_{(m)}}$ is a solution of the problem (S).

Proof. By Lemma 6, $z_{\overset{\circ}{\vec{k}}_{(m)}} \in \mathfrak{K}$. We can write the following sequence of equalities (cp. (17), (26), (27) and (29))

$$\prod_{\mu=1}^{m} D_{\mu}^{r_{\mu}} z_{\overset{\circ}{\vec{k}}_{(m)}}(x) = \prod_{\mu=2}^{m} D_{p_{\mu}}^{r_{p_{\mu}}} \omega^{1}_{p_1,\overset{\circ}{\vec{k}}_{(m-1)}}(x) =$$

$$= D_{p_m}^{r_{p_m}} \Big(\prod_{\mu=2}^{m-1} D_{p_{\mu}}^{r_{p_{\mu}}} \omega^{1}_{p_1,\overset{\circ}{\vec{k}}_{(m-1)}} \Big)(x) = D_{p_m}^{r_{p_m}} v_{p_m,\varrho_{p_m}}(x) = D_{p_m}^{r_{p_m}} v_{p_m,r_{p_m}}(x)$$

$$= F\left[x, Z(x), (\Omega^{\alpha}(x)), V(x)\right]$$

($x\in P$), which shows that $z_{\overset{\circ}{\vec{k}}_{(m)}}$ satisfies system (7).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Finally, it follows from relations (26),(27) and Lemma 7 that conditions (8)-(10) are also satisfied. Thus, the proof of Lemma 8 is completed.

4. In this section we prove the existence of a solution of system (26),(27) and hence (see Lemma 8) of problem (S). To this end we apply the Schauder fixed point theorem.

Let us introduce the Banach space Λ of all continuous functions (vectors) $V = (v_{s,k})$: $P \to \mathbb{R}^{nr}$ with the norm

$$\| V \| = \max_{k,s} \sup_{P} |v_{s,k}(x)| \tag{35}$$

$(1 \le k \le r_s;\ 1 \le s \le m)$.

We consider the set $\Im$ of points $V \in \Lambda$ possessing equicontinuous components $v_{s,k}$ $(k=1,2,\ldots,r_s;\ s=1,2,\ldots,m)$ and satisfying the condition

$$\| V \| \le \rho, \tag{36}$$

where $\rho > 1$ is a parameter which will be suitably chosen in the sequel (see (47), (47′)).

Evidently, $\Im$ is a convex compact set.

In view of system (26), (27), we map the set $\Im$ by the following transformation

$$\tilde{V} = T\,V \qquad (V \in \Im;\ \tilde{V} = (\tilde{v}_{s,k})), \tag{37}$$

where

$$\tilde{v}_{\nu,k}(x) = e_{\nu,k}\{x^{(\nu)}, (L_\mu[V(\cdot), \hat{x}^{(\nu,k)}]), V^{\nu,*}(\hat{x}^{(\nu,k)})\} + \int_{f_{\nu,k}(x^{(\nu)})}^{x_\nu} \tilde{v}_{\nu,k+1}(x^{(\nu,t)})\,dt \tag{38}$$

$(k \in p;\ x^{(\nu)} \in p_\nu;\ k=1,2,\ldots r_\nu - 1;\ \nu=1,2,\ldots,m)$;

$$\tilde{v}_{\nu,r_\nu}(x) = e_{\nu,r_\nu}\{x^{(\nu)}, (L_\mu[V(\cdot), \hat{x}^{(\nu,r_\nu)}]), V^{\nu,*}(\hat{x}^{(\nu,r_\nu)})\} + \tag{39}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\int_{f_{\nu,r_\nu}(x^{(\nu)})}^{x_\nu} F\{x^{(\nu,t)},(L_\mu[V(.),x^{(\nu,t)}]),V(x^{(\nu,t)})\}dt\,,$$

$(x\in P;\ \nu=1,2,\ldots,m)$.

We are going to find sufficient conditions for the inclusion $T(3)\subset 3$.

It follows from (38), (39), Assumptions I-IV and the definition of the set 3 that the functions $\tilde{v}_{s,k}$ $(k=1,2,\ldots,r_s;\ s=1,2,\ldots,m)$ are equicontinuous.

Let us observe that, by formulas (17), (23) and (24), Assumptions I-IV and inequality (36), we have the following estimate

$$(40)\quad \max\Big\{\max_{1\le\alpha\le m-2}\max_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}\sup_P|\omega^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}(x)|,\ \max_{\vec{k}_{(m)}}\sup_P|z_{\vec{k}_{(m)}}(x)|\Big\}\le \hat{C}_1A\rho+\hat{C}_2M_*\,,$$

where $A=\max_{1\le\nu\le n}A_\nu$;

$$(41)\quad M_*=\max\Big(1,\max_{1\le\alpha\le m-2}\max_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}\sup_{x_{s_\mu}\in<0,A_{s_\mu}>}|a^{\alpha}_{\vec{s}_{(\alpha)}\vec{k}_{(m-\alpha)}}(x_{s_1},\ldots,x_{s_\alpha})|\Big)$$

$(\mu=1,2,\ldots,\alpha)$ and $\hat{C}_i$ $(i=1,2)$ are positive constants independent of ρ.

As a consequence of (40) and Assumptions I-III, we get

$$(42)\quad \max_{\nu,k}\sup_{<0,A_\nu>}|e_{\nu,k}[x^{(\nu)},(L_\mu[V(.),\hat{x}^{(\nu,k)}]),V^{\nu,*}(\hat{x}^{(\nu,k)})]|\le M_1$$

$(k=1,2,\ldots,r_\nu;\ \nu=1,2,\ldots,m)$;

$$(43)\quad \sup_P|F\{x^{(\nu,t)},(L_\mu[V(.),x^{(\nu,t)}]),V(x^{(\nu,t)})\}|\le M_2,$$

where

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$M_1 = \tilde{C}_1[K_3+K_4(M_*+(A\rho)^\kappa)+K_4'\rho^\kappa];$$
$$M_2 = \tilde{C}_2[K_1+K_2(M_*+(A\rho)^\kappa)+K_2'\rho^\kappa], \tag{44}$$

$\tilde{C}_1$, $\tilde{C}_2$ being positive constants independent of ρ.

It is easily seen (see (38), (39), (42), and (43)) that

$$\sup_P|\tilde{v}_{\nu,k}(x)| \le M_1 \sum_{\alpha=0}^{r_\nu-k} A^\alpha + M_2 A^{r_\nu-k+1} \tag{45}$$

$(k=1,2,\dots,r_\nu;\ \nu=1,2,\dots,m)$.

Bearing in mind relations (35), (36) and (45), we can conclude that $T(\mathfrak{Z})\subset\mathfrak{Z}$ if the following inequality

$$C\{A(K_1+M_*K_2)+K_3+M_*K_4+[(AK_2+K_4)A^\kappa+AK_2'+K_4']\rho^\kappa\} \le \rho \tag{46}$$

is valid, where C is a positive constant independent of ρ.

Let us distinguish the following two cases:

(a) $\kappa\in(0,1)$; (b) $\kappa = 1$.

It is clear that (46) holds good in case (a) if ρ in (36) is sufficiently large, so that

$$\max\{2C[A(K_1+M_*K_2)+K_3+M_*K_4],(2C[(AK_2+K_4)A^\kappa+AK_2'+K_4'])\} \le \rho^{1-\kappa} \tag{47}$$

In case (b) we choose ρ so large that

$$2C[A(K_1+M_*K_2)+K_3+M_*K_4] \le \rho \tag{47'}$$

and we assume that $\max(K_2,K_2',K_4,K_4')$ or $\max(K_4',A)$ are sufficiently small, so that

$$2C[(AK_2+K_4)A+AK_2'+K_4'] \le 1. \tag{48}$$

In order to apply the Schauder fixed point theorem one has to use the following lemma whose straightforward proof will be omitted.

Lemma 9. The transformation T (see (37)) is continuous.

Thus, by the Schauder fixed point theorem, we can assert that there is a fixed point $V^0=(v^0_{s,k})\in\mathfrak{Z}$ of the transformation T, whence the function (vector) V^0 is a continuous solution of

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

system (26), (27). As a consequence (see Lemma 8) the corresponding function $z^{o}_{o\,\vec{k}_{(m)}}$ (see formulas (17), (23) and (24)) is a solution of problem (*S*) and hence the following theorem is established.

Theorem 1. If assumptions I-IV are satisfied and if the inequality (46) is valid (see the discussion subsequent to (46)), then problem (*S*) has a solution.

Remark 2. Let us observe that, in case (a), we have proved the global existence of a solution of problem (S) for arbitrary values of K_i and K_j' (i=1,2,3,4; j=2,4), and without any additional conditions for the curves (surfaces) appearing in the problem. Thus, for n=2 our assumptions and results are different from those in [12] (cp.[12], Theorems 1-3 and Remark 4).

5. In this section we use the Banach fixed point theorem to prove the uniqueness of the solution of problem (*S*).

We retain Assumptions I and IV, and we replace Assumptions II and III by the following ones:

II′. The function $F:P\times \mathbb{R}^{n\sigma}\to \mathbb{R}^{n}$ is continuous with respect to x and satisfies the Lipschitz condition

$$|F[x,\xi,(\eta^{\alpha}),\zeta]-F[x,\tilde{\xi},(\tilde{\eta}^{\alpha}),\tilde{\zeta}]| \leq K_2\tilde{\Xi}_0+K_2'\tilde{\Xi}_0', \tag{49}$$

where K_2 and K_2' are the coefficients appearing in (11), while $\tilde{\Xi}_0$ and $\tilde{\Xi}_0'$ are given by

$$\tilde{\Xi}_0 = \sum_{k_1=1}^{r_1}\ldots\sum_{k_m=1}^{r_m}|\xi_{\vec{k}_{(m)}}-\tilde{\xi}_{\vec{k}_{(m)}}| + \sum_{\alpha=1}^{m-2}\sum_{1\leq k_i\leq r_i}|\eta^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}-\tilde{\eta}^{\alpha}_{\vec{s}_{(\alpha)},\vec{k}_{(m-\alpha)}}| \tag{50}$$

(where i is as in (12)) and

$$\tilde{\Xi}_0' = \sum_{s=1}^{m}\sum_{k=1}^{r_s}|\zeta_{s,k}-\tilde{\zeta}_{s,k}|, \tag{51}$$

respectively.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

III′. The functions $e_{\nu,k} : P_\nu \times \mathbb{R}^{n(\sigma-r_\nu)} \to \mathbb{R}^n$ $(k=1,2,\dots,r_\nu;\ \nu=1,2,\dots,m)$ are continuous with respect to x and satisfy the Lipschitz condition

$$|e_{\nu,k}[x^{(\nu)},\xi,(\eta^\alpha),\zeta^{\nu,*}]-e_{\nu,k}[x^{(\nu)},\tilde{\xi},(\tilde{\eta}^\alpha),\tilde{\zeta}^{\nu,*}]| \le \tag{52}$$

$$\le K_4\tilde{\Xi}_0+K_4'\tilde{\Xi}_1',$$

where K_4 and K_4' are the coefficients appearing in (14), and $\tilde{\Xi}_1$ denotes the expression $\tilde{\Xi}_0'$ with the term corresponding to $s=\nu$ being omitted.

Remark 3. Denoting

$$K_*:=\max\{\sup_P|F[x,(0)_1]|,\max_\nu\max_k\sup_{P_\nu}|e_{\nu,k}[x^{(\nu)},(0)_2]|\} \tag{53}$$

$(1\le k\le r_\nu;\ 1\le\nu\le m)$, where $(0)_1$ and $(0)_2$ denote the systems of the appropriate number of zeros (cp. the parts of Assumptions II′ and III′ concerning the domains of F and $e_{\nu,k}$, respectively), we easily conclude from (49) and (52) that the inequalities

$$|F[x,\xi,(\eta^\alpha),\zeta]| \le K_*+K_2\overset{*}{\Xi}_0+K_2'\overset{*}{\Xi}_0', \tag{54}$$

and

$$|e_{\nu,k}[x^{(\nu)},\xi(\eta^\alpha),\zeta^{\nu,*}]| \le K_*+K_4\overset{*}{\Xi}_0+K_4'\overset{*}{\Xi}_1 \tag{55}$$

$(k=1,2,\dots,r_\nu;\ \nu=1,2,\dots,m)$ hold good, where $\overset{*}{\Xi}_0$, $\overset{*}{\Xi}_0'$ and $\overset{*}{\Xi}_1$ denote the expressions Ξ_0, Ξ_0' and Ξ_1, respectively (see (12), (13) and (14)) with $\kappa=1$.

In other words, the functions F and $e_{\nu,k}$ $(k=1,2,\dots,r_\nu;\ \nu=1,2,\dots,m)$ satisfy Assumptions II and III, respectively, with $\kappa=1$ and $K_1=K_3=K_*$.

Now, let us consider again the set $\mathfrak{Z}$ and the transformation T (see (36) and (37)).

It follows from the results obtained in Section 4 and from Remark 3 that if inequality (46) with $\kappa=1$ and $K_1=K_3=K_*$ is satisfied (cp. the discussion in case (b) subsequent to (46)), then $T(\mathfrak{Z})\subset\mathfrak{Z}$ and the transformation T has a fixed point $V^*\in\mathfrak{Z}$.

Evidently, $\mathfrak{Z}$ is a metric space with the distance $d(V,V_1)=\|V-V_1\|$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Using Assumptions I, II′, III′ and IV, one can prove by an argument similar to that applied in the derivation of (46) that the transformation T is a contraction if the inequality

$$C'[A(AK_2+K_2'+AK_4)+K_4']<1 \tag{56}$$

is valid, where C′ is a positive constant.

Evidently, (56) holds good if K_ν and K_ν' ($\nu=2,4$), or K_4' and A, are sufficiently small.

Thus, under the aforementioned conditions, all the assumptions of the uniqueness part of the Banach fixed point theorem are satisfied and we can conclude that the said fixed point V^* of transformation T is unique, whence (see Lemmas 4 and 8) also the corresponding solution z_0^* of problem (S) is unique.

$\vec{K}_{(m)}$

As a consequence, we can formulate the following theorem.

Theorem 2. If assumptions I, II′, III′ and IV are satisfied, and if the inequalities (46) with $\kappa=1$; $K_1=K_3=K_*$ and (56) are valid (see the discussions subsequent to (46) and (56), respectively), then problem (S) has a unique solution.

REFERENCES

[1] **A. Borzymowski** : Concerning a Goursat problem for some partial differential equation of order 2p, Demonstratio Math. 18 (1985) 253-277.

[2] **A. Borzymowski** : A non-linear Goursat problem for a high order polyvibrating equation, Proc.Roy.Soc. Edinburgh 102A (1986) 159-172.

[3] **A. Borzymowski, M. Michalski** : A nonlinear boundary value problem for a system of hyperbolic partial differential equations, Rev.Real Acad. Cienc. Exact. Fis. Natur. Madrid - 82 (1988) 235 - 247.

[4] **A. Borzymowski, A. Sadowski** : A boundary value problem for the Mangeron polyvibrating equation, Rend.Mat. 6 (1986) 565 - 585.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] Z.Dalla : Some boundary value problems for systems of partial differential equations of hyperbolic type, Doctoral Thesis, Warsaw University of Technology 1988.

[6] Z. Denkowski, A. Pelczar : On the existence and uniqueness of solutions of some partial differential functional equations, Ann.Polon.Math. 35 (1978) 261-304.

[7] G. Hecquet : Étude de quelques problèmes d'existence globale concernant l'équation

$$\frac{\partial^{r+s} u}{\partial x^r \partial y^s} = f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\ldots,\frac{\partial^{p+q} u}{\partial x^p \partial y^q},\ldots),\ 0\le p\le r;\ 0\le q\le s;$$

$p+q<r+s$, Ann. Mat. Pura Appl. 113 (1977), 173-197.

[8] M. Michalski : A non-linear Goursat problem for a polyvibrating equation of Mangeron, Mem. Sect. Stiint. Acad. Soc. Repub. Romania, Ser.4, 8(1985) 97-112.

[9] M. Michalski : Remarks on a Goursat problem for a 4th order polyvibrating equation, Rend. Sem. Mat. Univers. Politecn. Torino 44 (1986) 361-363.

[10] M.N. Oguztöreli : Sur le problème de Goursat pour une équation de Mangeron d'ordre supérieur I, II, Acad.Roy.Belg.Cl.Sci.58 (1972), 464-471, 577-582.

[11] J. Persson : Global linear Goursat problems for functions of Gevrey-Lednev type, Ann.Scuola Norm.Sup.Pisa 23(1969) 387-412.

[12] Z. Szmydt : Sur un problème concernant un système d'équations differentielles hyperboliques d'ordre arbitraire á deux variables independentes, Bull.Acad.Pol.Sci. Cl.III, 5(1957), 577-582.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

DEPARTMENT OF MATHEMATICS, AL. FATEH UNIVERSITY, TRIPOLI, LIBYA.

Received May 5, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Andrzej Strojnowski, Włodzimierz Wojas

SOME REMARKS ON 0 - GROUPS

In this paper we present a new proof of Vinogradov theorem about the fully ordered groups. Vinogradov proved [9] that the free products of 0-groups are again 0-groups. Original proof of this theorem is difficult and complicated, in connexion with it the trials of simplification of this proof were undertaken in previous years [1], [3], [6], [8]. At present this problem is still up-to-date, the publications about this subject area bear evidence of that. Besides we also show that 0-groups are locally indicable.

Let us recall the definitions of ordered group and ordered ring.

Definition (see [4] p.9). We shall say that G is partially ordered group if :

G. 1. G is a group under multiplication

G. 2. G is a partially ordered set under a relation $\leq$

G. 3. $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for all $a,b,c \in G$.

If $\leq$ is a full order we call G fully ordered group. If G can be fully ordered group we call G 0-group.

Definition (see [4] p.105). We shall say that R is partially ordered ring if :

R. 1. R is a ring

R. 2. R is a partially ordered set under a relation $\leq$

R. 3. $a \leq b$ implies $a + c \leq b + c$ for all $c \in R$

R. 4. $a \leq b$ and $c > 0$ imply $ca \leq cb$ and $ac \leq bc$, $a,b,c \in R$.

In such rings R.4 can be replaced by the stronger condition R.4*. $a < b$ and $c > 0$ imply $ca < cb$ and $ac < bc$. If

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we replace R.4.* instead of R.4. then $\leq$ is called strict. If R_+ is fully ordered group we call R fully ordered ring. If R can be fully ordered we call R 0-ring.

Lemma (see [4] p. 106). A ring admits a strict full order if, and only if, it is an 0-ring containing no divisors of zero.

We begin the proof of Vinogradov theorem with the following example of fully ordered group of matrices.

Example (see [7] p.47) Let R be a fully ordered ring with identity. Let T be a group of all infinite upper triangular matrices over R where diagonal elements are positive and invertible in R. We define the full order relation on the set of all pairs of the indices at the matrice elements which are on and over diagonal in the following way: $(i,j) \leq (k,l)$ if, and only if, $j - i < l - k$ or $j - i = l - k$ and $i < k$ or $i = k$ and $j = l$. The set of the pairs of the indices is well-ordered in this order relation. Let $[x_{ij}]$, $[y_{ij}] \in T$. We define the full order relation on T in the following way: $[x_{ij}] \leq [y_{ij}]$ if, and only if, there exist pair (k,l) such that $x_{kl} < y_{kl}$ and $x_{ij} = y_{ij}$ for all $(i,j) < (k,l)$ or $x_{ij} = y_{ij}$ for all $(i,j) < (k,l)$ or $x_{ij} = y_{ij}$ for all (i,j). Because the set of the pairs of the indices is well-ordered therefore pair (k,l) is uniquely determined. The group T with this order relation is fully ordered group.

Theorem [Vinogradov]. A free product of two 0-groups is an 0-group.

Proof. Let G and H be a fully ordered gropups. Let $A = G \times H$ be the fully ordered group with the lexicographic ordering and let Q[A] be the group ring of A over the field of the rational numbers. As in [7 p. 48], we define the order relation in Q[A] taking combination $\sum_{i=1}^{n} r_i a_i$ where $r_i \in Q$, $a_i \in A$ as positive, if the coefficient of the least a_i, in the order in A is positive. It is easy to prove that the ring Q[A] with this order relation is a fully ordered ring with identity and this order is strict. Let T be the group of all ifinity

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

upper triangular matrices from the example, over the ring $Q[A]$. Therefore T is a fully ordered group. Let 0 and 1 be, in $Q[A]$ the neutral elements of the addition and multiplication respectively.

We consider a group homomorphism $\phi : G \bullet H \longrightarrow T$ given by:

$$\phi(g) = \begin{bmatrix} g & 1-g & & & & 0 \\ 0 & 1 & & & & \\ & & g & 1-g & & \\ & & 0 & 1 & & \\ 0 & & & & \ddots & \end{bmatrix} \quad \text{for } g \in G,$$

$$\phi(h) = \begin{bmatrix} 1 & 0 & & & & 0 \\ 0 & h & 1-h & & & \\ & 0 & 1 & & & \\ & & & h & 1-h & \\ 0 & & & 0 & 1 & \\ & & & & & \ddots \end{bmatrix} \quad \text{for } h \in H.$$

Thus, for the reduced word $x = g_1h_2g_3h_4 \ldots h_n$, $x \in G \bullet H$ $\phi(x) = \phi(g_1)\phi(h_2)\phi(g_3)\phi(h_4) \ldots \phi(h_n)$. It is easy to check that for the reduced word $x = g_1h_2g_3h_4 \ldots h_n$ the matrix $\phi(x)$ has $(1, n+1)$ entry equal $(1-g_1)(1-h_2)(1-g_3)(1-h_4) \ldots (1-h_n)$. Beacuse the order relation in $Q[A]$ is strict therefore $Q[A]$ contains no divisors of zero as it follows from of Lemma. Hence $(1-g_1)(1-h_2)(1-g_3)(1-h_4) \ldots (1-h_n) \neq 0$ and $\phi(x)$ is not unit matrix. Therefore the image of every no empty word by ϕ is not unit matrix, thus ϕ is an injection. Hence $G \bullet H$ is isomorphic with subgroup of fully ordered group, thus $G \bullet H$ is fully ordered group. The proof is completed.

Remark 1. In some cases the homomorphism ϕ is not an injection. For example, if G is the Klein group and H is nontrivial group then homomorphism ϕ is not an injection.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Remark 2. This theorem may be extended to free products of any numbers of group (see [8] p. 597).

Let us pass on to locally indicable groups now.

Definition (see [5]). The group G is called locally indicable if each finitely generated subgroup of il can be mapped on to an infinite cyclic group.

Theorem. Let G be an O-group. Then G is locally indicable.

Proof. Let $H = \langle x_1 \dots x_n \rangle$ be a finitely generated subgroup of G such that $1 < x_1 < \dots < x_n$.

Let $U = \{g \in H ; g^t < x_n$ for all integer t$\}$. It is clear that U is the unique maximal convex subgroup of H. So U is a normal subgroup of H. By the Hölder theorem [7] the group H/U is abelian and torsion-free, finitely generated. Hence we can map H onto infinite cyclic group.

The converse theorem is not true. It is appropriate to point out here that Burns and Hale have shown that locally indicable groups are right-orderable and Bergman (see [2]) has shown the example of right-orderable group which is not locally indicable.

REFERENCES

[1] **G.H. Bergman** : Ordering coproducts of groups and semigroups, Berkeley 1989, (appear in J.Algebra)

[2] **G.H. Bergman** : Right orderable groups that are not locally indicable, Berkeley 1989 (unpublished).

[3] **R. Botto Mura, A. Rhemutalla** : Orderable groups, Dekker Lecture Notes in Pure and Applied Math., 27 (1977) 33 - 35.

[4] **L. Fuchs** : Partially ordered algebraic systems, Pergamon Press, Oxford 1963.

[5] **G. Higman** : Units in group rings, Doctoral Thesis, Oxford University, 1940.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[6] R. E. Johnson : Free product of ordered semigroups, Proc. Amer. Math. Soc., 19 (1968) 607 - 700.

[7] A. J. Kokorin, W. M. Kopytov : Линеино упорядочеинные группы, Moscow 1972.

[8] D. S. Passman : The algebraic structure group rings. Wiley Interscience, New York (1977) 594 - 597.

[9] A. A. Vinogradov : On the free of product of ordered groups, Math. Sb., 25 (1949) 163 - 168.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW,
00-901 WARSZAWA

Received May 28, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
VOL. XXIII No 3 1990

Maciej J. Mączyński

A THEOREM ON REPRESENTATION OF ALGEBRAIC SYSTEMS ADMITTING A FULL SET NUMERICAL FUNCTIONALS

1. Introduction

In this paper we would like to formulate a general theorem on numerical representation of algebraic systems. This theorem will allows us to represent abstract algebraic operations by means of numerical ones. This can be useful for computer applications - having a numerical representation we can abstract operations in a numerical way. The main theorem states the possibility of such a representation provided that there is a full set of numerical functionals. The theorem has been formulated after considering the analogy between the Stone representation theorem for Boolean algebras and the Gelfand-Neimark representation theorem for commutative Banach algebras. It turns out that the proofs of both theorems can be reformulated in such a way that they become fully analogical. This analogy leads to the main theorem. By no means this theorem is to be applied to Boolean algebras and Banach algebras only - it can be also applied to more general algebraic and relational systems. We would to present the main idea for such a numerical representation - it is a dualization of numerical functionals.

This paper has been completed during the author's visit at the Department of Physics of the University of Genova. The author is deeply obliged to the University for hospitality and financial support. The author is grateful to Professor Enrico Beltrametti for discussions which have influenced this paper.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. Basic definitions

Definition 1. Let $\underline{A} = (A, o_i,\ i \in I)$ be an abstract algebra defined by a set of axioms (we admit also possible relations R_j, $j \in J$). Here o_i denotes an k_i-argument operation on A. We say that $\underline{A}$ admits a functional representation if there is a one-to-one map

$$\mu : A \xrightarrow{1-1} K^X \tag{1}$$

where K,X are non-empty sets and for each o_i there is a function of the k_i-variables

$$\varphi_i : K^{k_i} \longrightarrow K \tag{2}$$

such that all $a, a_1, \ldots a_{k_i} \in A$ we have

$$a = o_i(a_1, \ldots a_{k_i}) \iff \forall_{x \in X} \mu^{(x)}(a) = \varphi_i(\mu^{(x)}a_1), \ldots, \mu^{(x)}(a_{k_i})) \tag{3}$$

for all $i \in I$. Since for each $a \in A$ $\mu(a)$ is a function from X into K, we can evaluate this function at every $x \in X$. We write

$$\mu^{(x)}(a) = \mu(a)(x).$$

If $X = \{\alpha\}$ is a one-element set and $K = A$, then the map $\mu(a) = \bar{a}$ is clearly a functional representation (where by $\bar{a}$ we denote the function $\bar{a} : \{\alpha\} \longrightarrow A$ defined by $\bar{a}(\alpha) = a$. In the sequel we will in this case omit the bar and write a instead of $\bar{a}$.) This functional representation will be called trivial. We will of course be interested in non-trivial representations.

Definition 2. We say that the representation (1) is regular if (K, φ_i) is an algebra of the same type as $\underline{A}$ with the functions φ_i satisfying the same set of axioms as the operations o_i.

Observe that if (1) is a regular representation, then for each $x \in X$ the map $\mu^{(x)} : A \longrightarrow K$ defined by $\mu^{(x)}(a) = \mu(a)(x)$ is a homomorphism from $\underline{A}$ into (K, φ_i).

Definition 3. Let (1) be a regular representation of $\underline{A}$. We say that (1) is minimal if K is isomorphic to the smallest

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

subalgebra of $\underline{A}$ (provided it exists).

Definition 4. Let $(A,o_i,\ i \in I)$ be an abstract algebra as in Def. 1. We say that (A_0,f_i) is a numerical model for $\underline{A}$ if the following conditions hold:

(i) (A_0,f_i) is an algebra satisfying the same of axioms as (1).

(ii) $A_0 \subseteq C$ is a set of real or complex numbers.

(iii) Every operation $f_i : A_0^{k_i} \longrightarrow A_0$ can be expressed in terms of the field operations in C (or in R).

Definition 5. We say that a numerical model (A_0,f_i) is minimal if (A_0,f_i) is isomorphic to the smallest subalgebra of $\underline{A}$ (provided it exists).

Definition 6. The representation (1) is called numerical if (K,φ_i) is numerical model for (1).

Definition 7. Let $(A,o_i,\ i \in I)$ be an abstract algebra with a minimal numerical model $(A_0,f_i,\ i \in I)$. Every homomorphism from A into A_0 will be called a numerical functional on A.

Definition 8. Let $h_\alpha : A \longrightarrow A_0$, $\alpha \in S$, be a set of numerical functionals on A. We say that this set is full if

$$(4)\quad a = o_i(a_1,\dots,a_{k_i}) \iff \forall_{\alpha\in S}\ h_\alpha(a) = f_i\big(h_\alpha(a_1),\dots,h_\alpha(a_{k_i})\big)$$

for $a,a_1,\dots,a_{k_i} \in A$, $i \in I$.

Observe that the implication from left to right in (4) always holds (h_α is a homomorphism). The converse implication holds only for a full set of numerical functionals.

3. The representation theorem

We can now formulate and prove our main theorem.

Theorem 1. Let $A = (A,o_i,\ i \in I)$ be an algebra with a minimal numerical model $(A_0,f_i,\ i \in I)$. The following conditions are equivalent:

(I) $\underline{A}$ admits a minimal numerical representation.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(II) $\underline{A}$ admits a full set of numerical functionals.

Proof. (I) $\Longrightarrow$ (II). Let (1) be a minimal numerical representation. Then $K = A_0$ and $\varphi_i = f_i$ are operations in A_0 expressible in terms of the field operations. For each $\alpha \in X$ be define a numerical functional on A as follows

$$h_\alpha(a) = \mu^\alpha(a) \quad \text{for all} \quad a \in A.$$

Since h_α is a homomorphism, it is a numerical functional. To show that the set $\{h_\alpha : \alpha \in X\}$ is full, let

$$h_\alpha(a) = f_i(h_\alpha(a_1),\ldots,h_\alpha(a_{k_i})) \quad \text{for all } \alpha \in S.$$

From condition (3) it follows that

$$a = o_i(a_1,\ldots,a_{k_i}) \quad \text{for all } \alpha \in S.$$

Hence condition (4) is satisfied and $\underline{A}$ admits a full set of numerical functionals, i.e. (II) holds. Thus (I) $\Longrightarrow$ (II).

(II) $\Longrightarrow$ (I). Assume that $\{h_\alpha : \alpha \in X\}$ is a full set of numerical functionals. We will define a minimal numerical representation for $\underline{A}$ as follows. For each $a \in A$, let $\bar{a}$ denote the mapping

$$\bar{a} : X \longrightarrow A_0$$

defined as follows

$$\bar{a}(\alpha) = h_\alpha(a) \quad \text{for all } \alpha \in X,$$

so in $h_\alpha(a)$ we keep a fixed and we let α run throught all $\alpha \in S$.

Now let $\mu(a) = \bar{a}$ for all $a \in A$. Then μ is a mapping from A into A_0^X, $\mu : A \longrightarrow A_0^X$.

We show that it is a functional representation for $\underline{A}$ in the sense of Definition 1. For each o_i we have the function of k_i-variables

$$f_i : A^{k_i} \longrightarrow A_0,$$

where f_i are operations in the minimal numerical model

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(A_0,f_i). Let $a = o_i(a_1,\ldots,a_{k_i})$. Since for each $\alpha \in X$, h_α is a numerical functional, this implies

(5) $$h_\alpha(a) = f_i(h_\alpha(a_1),\ldots,h_\alpha(a_{k_i})) \quad \text{for all } \alpha \in X,$$

that is,

$$\bar{a} = \varphi_i(\bar{a}_1,\ldots,\bar{a}_{k_i}),$$

where φ_i is defined by

$$\varphi_i(\bar{a}_1,\ldots,a_{k_i}) = \overline{f_i(h\alpha_0(a_1),\ldots,h\alpha_0(a_{k_i}))}$$

for any fixed $\alpha_0 \in X$. Since (5) holds for all $\alpha \in X$, we see that φ_i is well-defined for every $i \in I$.

Hence the map μ defines a functional representation. This representation is numerical because $K = A_0$ and minimal because (A_0,f_i) is a minimal numerical model for $\underline{A}$. So (II) implies (I), and Theorem 1 holds.

Observe that Theorem 1 allows us to represent an abstract algebra $\underline{A}$ in such a way that the algebraic operations in A become expressed by field operations in A_0^X. We operate in A_0^X numerically coordinatwise. For $f_1,\ldots,f_{k_i} \in A_0^X$ the operation $o_i(f_1,f_2,\ldots,f_{k_i})$ is performed as operation composed of field operations on regal or complex numbers. We will see this on examples in the next section.

To apply Theorem 1 to a concrete algebra we have to check whether this algebra possesses a minimal numerical model and whether it has a full set of numerical functionals. This is a severe restriction on the systems considered. In any case we will show that for such systems as Boolean algebras and Banach algebras these conditions are satisfied. It turns out that for Boolean algebras Theorem 1 becomes the Stone representation theorem (see.e.g. [3]), and for commutative Banach algebras - the Gelfand-Neimark theorem (see.e.g. [4]). Since these theorems are already classical, we do not refer to the original works but to monographs where these theorems are presented in the way useful for our discussion.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Application to Boolean algebras and to commutative Banach algebras

I. Boolean algebras

A Boolean algebra is defined as an abstract algebra $\underline{A} = (A, \circ, ')$ with one binary operation $\circ$ and one unary operation $'$. With the help of these operations we can define the usual Boolean operations:

$$a \cap b = a \circ b$$

$$a \cup b = \big((a') \circ (b')\big)'$$

$$-a = a'.$$

The set of axioms for Boolean algebras can taken conveniently as that given e.g. in the book of Sikorski [3]:

$$a \cup b = b \cup a, \qquad a \cap b = b \cap a,$$

$$a \cup (b \cup c) = (a \cup b) \cup c, \qquad a \cap (b \cap c) = (a \cap b) \cap c,$$

$$(a \cap b) \cup b = b, \qquad (a \cup b) \cap b = b,$$

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \qquad a \cup (b \cap c) = (a \cup b) \cap (a \cup c),$$

$$(a \cap -a) \cup b = b, \qquad (a \cup -a) \cap b = b.$$

We can define now $0 = a \cap -a$ and $1 = 0'$.

We obtain a minimal numerical model for $\underline{A}$ by taking

$$A_0 = \{0,1\} \qquad \text{(the real numbers 0 and 1)},$$

and defining f_i as follows

$$f_1(a,b) = ab,$$

$$f_2(a) = 1 - a.$$

The operation f_1 corresponds to $o_1(a,b) = a \circ b$, and f_2 corresponds to $o_2(a) = a'$. This model is minimal, because $(A_0, f_i,\ i=1,2)$ is isomorphic to the smallest subalgebra of $\underline{A}$, namely to $\{0,1\}$. Now every numerical functional is a homomorphism

$$h : A \longrightarrow A_0$$

such that $h(a \circ b) = h(a)h(b)$ and $h(a') = 1 - h(a)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is clear that every numerical functional is a two-valued measure on $\underline{A}$, $h = m$. From the theory of Boolean algebras it is known that every two-valued measure is determined by a maximal ideal in $\underline{A}$ (see [3]), and the set of all two-valued measures is full, because it is homeomorphic to the Stone space of $\underline{A}$ (with the suitable topology). Hence every Boolean algebra satisfies the assumptions of Theorem 1. Namely it has the minimal numerical model $A_0 = \{0,1\}$ with the operations f_i expressed by the field operations multiplication and subtraction from 1 in R. Also $\underline{A}$ admits a full set of numerical functionals. Consequently, by Theorem 1 it admits a minimal numerical representation. We obtain this representation by taking for each $a \in A$, $\bar{a}(m) = m(a)$ for all $m \in M$. In this case $X = M$ the set of all numerical functionals on $\underline{A}$, $K = A_0$ In this numerical model we operate on elements $\bar{a}$, $a \in A$, as follows

$$\bar{a} \circ \bar{b} = \overline{ab}$$

$$\bar{a}' = 1 - \bar{a}.$$

So the Boolean operations $\circ$ and $'$ are represented by numerical operations multiplication and subtraction from 1. Using the definitions of $\cup$ and $\cap$ we can represent these operations numerically as well. Observe that the representation of Boolean algebras obtained above is a special case of the representation discussed in [1]. Note also that the numerical functionals $m \in M$ satisfy the condition $m(a \circ b) = m(a)m(b)$, so they are multiplicative functionals on $\underline{A}$. This will also repeated in the case of Banach algebras discussed next.

II. Commutative Banach algebras

We define a commutative Banach algebra as an algebra $\underline{A} = (A,+,\alpha\cdot,\cdot,\|\cdot\|)$ such that $(A,+,\cdot,\|\cdot\|)$ is a Banach space and $(A,+,\alpha\cdot,\cdot)$ is an algebra with unit satisfying the properties:

(1)

(2) $$\|ab\| \leq \|a\| \, \|b\|$$

$$\|1\| = 1.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We assume that $ab = ba$ for all $a,b \in A$. In order that Definition 1 comprise the case of Banach algebra we would have to extend it in such a way that as algebraic operations there are admitted also multiplication by numbers $\alpha\cdot$, $\alpha \in C$, and valuation in the non-negative real numbers $a \longrightarrow \|a\|$. Since such a generalization is obvious and Theorem 1 also holds in this generalized case, we omit it and use Theorem 1 for Banach algebras without further discussion.

If $\underline{A}$ is a commutative Banach algebra, then the smallest subalgebra of $\underline{A}$ has the following form $\{\alpha\cdot 1 : \alpha \in C\}$. It is isomorphic to the algebra of complex numbers. Hence to obtain a minimal numerical model for $\underline{A}$ we take $A_0 = C$ the complex numbers and the operations in A_0 are defined by

$$f_1(a,b) = a + b,$$

$$f_\alpha(a) = \alpha \cdot a \quad \text{for} \quad \alpha \in C,$$

$$f_2(a,b) = ab,$$

$$f_3(a) = |a|.$$

It is clear that $(A_0, f_1, f_2, f_3, f_\alpha, \alpha \in C)$ is a commutative Banach algebra isomorphic to the smallest subalgebra of $\underline{A}$.

A numerical functional on $\underline{A}$ is a map

$$m : A \longrightarrow C$$

such that m is a homomorphism, i.e.

$$m(a + b) = m(a) + m(b), \qquad m(0) = 0,$$
$$m(\lambda a) = \lambda m(a), \quad \lambda \in C \qquad m(1) = 1,$$
$$m(ab) = m(a)m(b), \qquad |m(a)| \leq \|a\|.$$

Hence every numerical functional on $\underline{A}$ is a multiplicative functional on $\underline{A}$. In the theory of Banach algebras it is proved that every multiplicative functional m on commutative Banach algebra $\underline{A}$ is determined by a maximal ideal $\Delta \in M$ in $\underline{A}$ (M denotes the set of all maximal ideals in $\underline{A}$), so we have $m = m_\Delta$. The mapping associating with every maximal ideal Δ a multiplicative functional m_Δ is called the Gelfand mapping (see [4]). From the Gelfand-Neimark theorem it follows that the set of all multiplicative functionals on $\underline{A}$ is full. Hence the assump-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

tion (II) of Theorem 1 is satisfied and we can infer that $\underline{A}$ admits a minimal numerical representation. Following the proof of Theorem 1 we can define this representation as follows: for every $a \in A$ we define $\bar{a}$ by

$$\bar{a}(m_\Delta) = m_\Delta(a) \quad \text{for all} \quad \Delta \in M.$$

Hence taking $X = \{ m_\Delta : \Delta \in M \}$ and $K = C$ we see that the mapping $\mu(a) = \bar{a}$ is a functional representation of $\underline{A}$ into K^X, $\mu : \underline{A} \longrightarrow K^X$. From the discussion above it follows that this representation is numerical and minimal, as postulated in Theorem 1.

Again in this representation, the operations in a Banach algebra are represented by the field operations in complex numbers. Thus we have shown that the Stone representation theorem and the Gelfand-Naimark theorem can be considered as special cases of a general theorem. In this setting the proofs of both theorems become fully analogical, although they concern so diverse algebraic systems.

Theorem 1 can be also applied to other algebraic systems (orthomodular posets, Post algebras, quantum logics, see e.g. [2]). But in our opinion the cases of Boolean algebras and commutative Banach algebras are specially illuminating.

REFERENCES

[1] M. Mączyński: On some numerical characterization of Boolean algebras, Colloq. Math. 27 (1973) 207-210.

[2] M. Mączyński, T. Traczyk: A characterization of orthomodular partially ordered set admitting a full set of states, Bull.Acad.Polon.Sci.Ser.Sci.Math.Astronom.Phys. 21 (1973) 3 - 8.

[3] R. Sikorski: Boolean algebras. Springer-Verlag, Berlin-Göttingen-Heidelberg-New York 1964.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[4] B. Simmons: Introduction to topology and modern analysis, Addison-Wesley 1978.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received June 15, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Danuta Stachowiak - Gnilka

CLASSES AND SPACES OF UL'YANOV TYPE
II. TOPOLOGICAL PROPERTIES OF SPACES $\varphi(l)$

1. Preliminaries

Let φ be a function which is finite, even on $(-\infty,\infty)$ and non-decreasing on $<0,\infty)$ such that $\varphi(0)=0$ and $\varphi(u)>0$ for $u>0$.

Denote by $\varphi(l)$ the class of all real sequences $(\xi_k)_{k\geq 1}$ for which $\sum_{k=1}^{\infty}\varphi(\xi_k)<\infty$.

In the sequel we denote by $x,y,z,\ldots$ the sequences $(\xi_k)_{k\geq 1},(\eta_k)_{k\geq 1},(\zeta_k)_{k\geq 1},\ldots$ respectively, and by x_n the sequences $(\xi_k^n)_{k\geq 1}$ for $n\geq 1$.

The set $\overline{\varphi(l)}\equiv\{y\in s: x+y\in\varphi(l) \text{ for all } x\in\varphi(l)\}$, where s is the set of all real sequences, is called the additive adjoint to the class $\varphi(l)$ (see [1], [6]).

If $x,y\in\varphi(l)$, then the number

$$d_\varphi(x,y)=\sum_{k=1}^{\infty}\varphi(\xi_k-\eta_k)$$

is called the φ-distance between x and y.

Now, we introduce in the class $\varphi(l)$ a topology $\mathcal{T}_\varphi$. Let $x\in\varphi(l)$ be arbitrarily chosen. For each $\varepsilon>0$ we denote by $A_\varphi(x,\varepsilon)$ the ε-neighbourhood of x in the sense of the φ-distance, that is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$A_\varphi(x,\varepsilon) \equiv \{y \in \varphi(l): d_\varphi(x,y)<\varepsilon\}.$$

By $\mathcal{T}_\varphi$ we denote the topology generated by the subbase $\{A_\varphi(x,\varepsilon)\}_{\substack{x\in\varphi(L)\\ \varepsilon>0}}$. The topological space obtained in this manner is denoted by $(\varphi(l),\mathcal{T}_\varphi)$.

If $\varphi(+0) = 0$, then by $\varphi^*(l)$ we denote the modular space (definition: see [3]) of those $x\in s$ for which $\sum_{k=1}^{\infty}\varphi(\lambda\,\xi_k) \longrightarrow 0$ as $\lambda \longrightarrow 0$. In this space we can define an F - norm:

$$\|\,x\,\| = \inf\{a > 0: \sum_{k=1}^{\infty}\varphi(\tfrac{1}{a}\,\xi_k) \le a\}.$$

By $\mathcal{T}_{\varphi^*}$ we denote the topology generated by this F-norm restricted to $\varphi(l)\subset\varphi^*(l)$.

We say that a sequence $(x_n)_{n\ge1}$, $x_n\in\varphi(l)$ for $n\ge1$, is convergent to $x\in\varphi(l)$ in the sense of the φ-distance if for every $\varepsilon>0$ there exists a natural number $N(\varepsilon)$ such that $d_\varphi(x_n,x)<\varepsilon$ for all $n>N(\varepsilon)$.

We say that a function φ satisfies the condition (Δ_2) if there are constants $K>0$, $u_0>0$ such that $\varphi(2u)\le K\varphi(u)$ for $0<u\le u_0$.

Later we denote by W the class of all real sequences $(\xi_k)_{k\ge1}$ for which every ξ_k is rational and $\xi_k=0$ for almost k.

2. Topological space $(\varphi(l),\mathcal{T}_\varphi)$

First, we shall prove that the space $(\varphi(l),\mathcal{T}_{\varphi^*})$ is separable.

Theorem 2.1. Let $\varphi(+0)=0$. The space $(\varphi(l),\mathcal{T}_{\varphi^*})$ is separable if and only if φ satisfies the condition (Δ_2).

Proof. Sufficiency. Let $\varepsilon>0$ be an arbitrary real, let m be a natural number such that $\varepsilon\ge2^{-m+1}$ and let $x\in\varphi(l)$. There is a natural number k_0 such that $\sum_{k=k_0+1}^{\infty}\varphi(\xi_k) < \min\left(\frac{\varepsilon}{4K^m},\varphi(u_0)\right)$, where

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$K>0$, $u_0>0$ are constants as in condition (Δ_2). There is a $\delta>0$ such that $\varphi\left(\frac{2\delta}{\varepsilon}\right) < \frac{\varepsilon}{4k_0}$. For every ξ_k $(1\le k\le k_0)$ there exists a rational number w_k such that $|\xi_k - w_k|<\delta$. Now, we define $w=(\bar{w}_k)_{k\ge1}$ as follows

$$\bar{w}_k = \begin{cases} w_k & \text{if } 1\le k\le k_0, \\ 0 & \text{if } k>k_0. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} \varphi\left(\frac{2}{\varepsilon}\left(\xi_k - \bar{w}_k\right)\right) < \frac{\varepsilon}{4} + K^m \sum_{k=k_0+1}^{\infty} \varphi\left(\xi_k\right) < \frac{\varepsilon}{2}.$$

This implies that $\|x-w\|<\varepsilon$.

Necessity. If φ does not satisfy the condition (Δ_2), then there is a sequence $u_n \downarrow 0$ as $n\to\infty$ such that $\varphi(u_n)< \frac{1}{2^n}$ and $\varphi(2u_n)>2^n\varphi(u_n)$ for $n\ge1$. Further, there are finite, pairwise disjoint sets F_n of indices j for which

$$\frac{1}{2^n} \le \sum_{j\in F_n} \varphi(u_n) < \frac{1}{2^{n-1}} \quad \text{for } n\ge1$$

(compare [6], the proof of Lemma 1.2).

Let $\alpha=(\alpha_n)_{n\ge1}$ be an arbitrary sequence such that every α_n is equal to 0 or 1. Now, we define $x_\alpha=(\xi_k^\alpha)_{k\ge1}$ as follows: $\xi_k^\alpha = \alpha_n u_n$ if $k\in F_n$. Clearly $x_\alpha\in\varphi(l)$. Let $\alpha'=(\alpha'_n)_{n\ge1}$ and $\alpha''=(\alpha''_n)_{n\ge1}$ be such as above and let $\alpha'\ne\alpha''$. Then there is a natural number m such that $\alpha'\ne\alpha''$. Then $d_\varphi(2x_{\alpha'}, 2x_{\alpha''}) \ge \sum_{j\in F_m} \varphi(2u_m) > \frac{1}{2}$ and hence $\|x_{\alpha'} - x_{\alpha''}\| > \frac{1}{2}$. Thus the space $(\varphi(l), \mathcal{T}_{\varphi^*})$ is non-separable.

Remark 2.1. Suppose that φ is a continuous on $<0,\infty)$ and convex function for which $\lim_{u\to0} \frac{\varphi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\varphi(u)}{u} = +\infty$. Then $\varphi^*(l)$ is equal to the Orlicz space l^φ (see [4]). It is evident that Theorem 2.1 generalizes the theorem obtained by Orlicz in [4], p.104.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, we shall prove that the separability of the space $(\varphi(l),\mathcal{T}_\varphi)$ can be proved without the condition (Δ_2).

Theorem 2.2. Let $\varphi(u+0)=\varphi(u)$ for $u>0$. The space $(\varphi(l),\mathcal{T}_\varphi)$ is separable if and only if $\varphi(+0)=0$.

Proof. Sufficiency. It is sufficient to show that if $x_j\in\varphi(l)$, $x_j=(\xi_k^j)_{k\geq 1}$ for $1\leq j\leq n$ and if for $x\in\varphi(l)$

$$\alpha_j \equiv d_\varphi(x,x_j) < \varepsilon_j \quad (1\leq j\leq n),$$

where ε_j $(1\leq j\leq n)$ are positive numbers, then there is $w\in W$ such that $d_\varphi(x_j,w)<\varepsilon_j$ $(1\leq j\leq n)$.

Let $\varepsilon = \min\limits_{1\leq j\leq n} \frac{\varepsilon_j-\alpha_j}{2}$. Then there is a natural number k_0 such that $\sum\limits_{k=k_0+1}^{\infty} \varphi\left(\xi_k^j\right) < \varepsilon$ for $1\leq j\leq n$. Further, there is a $\delta>0$ such that

$$\sum_{k=1}^{k_0} \varphi\left(|\xi_k^j - \xi_k| + \delta\right) < \varepsilon + \sum_{k=1}^{k_0} \varphi\left(\xi_k^j - \xi_k\right) \quad \text{for } 1\leq j\leq n.$$

For every ξ_k $(1\leq k\leq k_0)$ there exists a rational number w_k such that $|\xi_k-w_k|<\delta$. Now, we define $w=(\bar{w}_k)_{k\geq 1}$ as follows

$$\bar{w}_k = \begin{cases} w_k & \text{if } 1\leq k\leq k_0 \\ 0 & \text{if } k>k_0. \end{cases}$$

Then $w\in W$ and $d_\varphi(x_j,w)<2\varepsilon + \alpha_j<\varepsilon_j$ for $1\leq j\leq n$.

The necessity follows from this fact that the space $(\varphi(l),\mathcal{T}_\varphi)$ is discrete if $\varphi(+0)>0$.

Theorem 2.3. The space $(\varphi(l),\mathcal{T}_\varphi)$ is a T_2-space.

Proof. Let $x,y\in\varphi(l)$ be arbitrary and let $x\neq y$. Observe that $A_\varphi(x,r)\cap A_\varphi(y,r)=\emptyset$ for some $r>0$. In fact, suppose that $A_\varphi(x,r)\cap A_\varphi(y,r)\neq\emptyset$ for each $r>0$. Let $r_n\to 0$ for $n\to\infty$. Then there exists a sequence $x_n\in\varphi(l)$ for $n\geq 1$ such that $x_n\in A_\varphi(x,r_n)\cap A_\varphi(y,r_n)$. Hence $d_\varphi(x_n,x)\to 0$ and $d_\varphi(x_n,y)\to 0$ for $n\to\infty$. Clearly $x=y$, a contradiction.

One can easily prove the following fact.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proposition 2.1. The family $\{A_\varphi(x,\varepsilon)\}_{\varepsilon>0}$ is a base for the topology $\mathcal{T}_\varphi$ at the point x for every $x\in\varphi(l)$ if and only if the following property holds:

(2.1) for each $x\in\varphi(l)$, for each $\varepsilon>0$ and for each $y\in A_\varphi(x,\varepsilon)$ there is a $\delta>0$ such that $A_\varphi(y,\delta)\subset A_\varphi(x,\varepsilon)$.

Observe that the condition $\varphi(+0)>0$ implies the property (2.1). Now, we shall investigate sufficient and necessary conditions under which the property (2.1) is fulfilled in the case $\varphi(+0)=0$.

Lemma 2.1. If $\varphi(u+0)=\varphi(u)$ for $u\geq 0$ and φ satisfies the condition (Δ_2), then the property (2.1) is fulfilled.

Proof. Let $\varepsilon>0$ and, let $x\in\varphi(l)$. Let $y\in A_\varphi(x,\varepsilon)$ be arbitrarily chosen. Then there is a real number $M>0$ such that $|\xi_k-\eta_k|\leq M$ for $k\geq 1$. We can choose a constant $K_M>0$ such that $\varphi(u+v)\leq K_M\left[\varphi(u)+\varphi(v)\right]$ for $0<\max(u,v)\leq M$. Let $\alpha\equiv d_\varphi(x,y)$ and $\beta=\frac{\varepsilon-\alpha}{3}$. There is a natural number k_0 such that $\sum_{k=k_0+1}^{\infty}\varphi(\xi_k-\eta_k)<\frac{\beta}{K_M}$. Let $\gamma>0$ be such that

$$\sum_{k=1}^{k_0}\varphi(|\xi_k-\eta_k|+\gamma)<\beta+\sum_{k=1}^{k_0}\varphi(\xi_k-\eta_k).$$

Now, we choose a $\delta>0$ such that $\delta<\min\left(\varphi(M),\varphi(\gamma),\frac{\beta}{K_M}\right)$. We shall prove that $A_\varphi(y,\delta)\subset A_\varphi(x,\varepsilon)$. Let $z\in A_\varphi(y,\delta)$: Then

$$d_\varphi(x,z)\leq\sum_{k=1}^{k_0}\varphi\left(|\xi_k-\eta_k|+\gamma\right)+$$

$$+K_M\left(\sum_{k=k_0+1}^{\infty}\varphi\left(\xi_k-\eta_k\right)+\sum_{k=k_0+1}^{\infty}\varphi\left(\zeta_k-n_k\right)\right)<3\beta+\alpha=\varepsilon.$$

This implies that $z\in A_\varphi(x,\varepsilon)$.

Lemma 2.2. If $\varphi(+0)=0$ and the property (2.1) is fulfilled, then φ satisfies the condition (Δ_2).

Proof. Let $x\in\varphi(l)$ and $\varepsilon>0$ be arbitrarily chosen. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

there is a natural number k_1 such that $\sum_{k=k_1+1}^{\infty} \varphi(\xi_k)<\varepsilon$. Let

$$\eta_k = \begin{cases} \xi_k & \text{if } 1\le k\le k_1, \\ 0 & \text{if } k>k_1. \end{cases}$$

Then $d_\varphi(x,y)<\varepsilon$. Hence $A_\varphi(y,\delta)\subset A_\varphi(x,\varepsilon)$ for some $\delta>0$. There is a natural number k_2 such that $\sum_{k=k_2+1}^{\infty} \varphi(\xi_k)<\delta$. We denote $k'=\min(k_1,k_2)$, $k''=\max(k_1,k_2)$ and we define

$$\zeta_k = \begin{cases} \xi_k & \text{if } 1\le k\le k', \\ 0 & \text{if } k'<k\le k'', \\ -\xi_k & \text{if } k>k''. \end{cases}$$

It is easily seen that $z\in\varphi(1)$ and $z\in A_\varphi(y,\delta)$. Hence $z\in A_\varphi(x,\varepsilon)$, that is $\varepsilon>d_\varphi(x,z)\ge \sum_{k=k''+1}^{\infty} \varphi(2\xi_k)$. It follows that $2x\in\varphi(1)$. By Lemma 1.2 in [6] we obtain that the function φ satisfies the condition (Δ_2).

Lemma 2.3. Let $\varphi(+0)=0$. If $\varphi(a+0)>\varphi(a)$ for some $a<0$, then the property (2.1) is not fulfilled.

Proof. We put $\varepsilon = \frac{\varphi(a+0)+\varphi(a)}{2}$. Clearly $0<\varphi(a)<\varepsilon<\varphi(a+0)$. Let $\delta>0$, let $x = (0,0,\dots)$, $y = (a,0,\dots)$ and $z_\delta = (a+\gamma,0,\dots)$, where $\gamma>0$ is such that $\varphi(\gamma)<\delta$. Then $d_\varphi(x,y)<\varepsilon$, $d_\varphi(y,z_\delta)<\delta$ but $d_\varphi(x,z_\delta)\ge\varphi(a+0)>\varepsilon$. Hence $A_\varphi(y,\delta)\setminus A_\varphi(x,\varepsilon)\neq\emptyset$ for all $\delta>0$. This implies that the property (2.1) is not fulfilled.

From Lemmas 2.1 - 2.3 there follows immediately.

Theorem 2.4. Let $\varphi(+0)=0$. The property (2.1) is fulfilled if and only if $\varphi(u+0)=\varphi(u)$ for $u>0$ and φ satisfies the condition (Δ_2).

3. Convergence in the space $(\varphi(1),\mathcal{T}_\varphi)$

Now we shall examine the notion of convergence of sequences in the $\mathcal{T}_\varphi$ topology.

Later we shall need the following lemmas.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 3.1. Let φ be a function continuous on $<0,\infty)$, let $x\in\varphi(1)$ and $x_n\in\varphi(1)$ for $n\geq 1$. If

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi\left(\xi_k^n-\eta_k\right)=\sum_{k=1}^{\infty}\varphi\left(\xi_k-\eta_k\right)$$

for all $y\in\varphi(1)$ such that $x-y\in\varphi(1)$, then the sequence $(x_n)_{n\geq 1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

Proof. Let $\varepsilon>0$ be arbitrary and let $y\in\varphi(1)$ be such that $x\in A_\varphi(y,\varepsilon)$. Hence $x-y\in\varphi(1)$ and $\sum_{k=1}^{\infty}\varphi\left(\xi_k^n-\eta_k\right)<\varepsilon$ for all $n>N$ and some natural number N. Thus $x_n\in A_\varphi(y,\varepsilon)$ for $n>N$ and so the sequence $(x_n)_{n\geq 1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

Lemma 3.2. Let φ be a function continuous on $<0,\infty)$, and let the sequence $(x_n)_{n\geq 1}$, $x_n\in\varphi(1)$ for $n\geq 1$, be convergent to $x\in\varphi(1)$ in the $\mathcal{T}_\varphi$ topology. If $y\in\varphi(1)$ is such that $x-y\in\varphi(1)$ than

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi\left(\xi_k^n-\eta_k\right)=\sum_{k=1}^{\infty}\varphi\left(\xi_k-\eta_k\right).$$

Proof. Let $y\in\varphi(1)$ be such that $x-y\in\varphi(1)$ and let $\varepsilon>0$ be arbitrary real. There is a natural number k_0 such that $\sum_{k=k_0+1}^{\infty}\varphi\left(\xi_k-\eta_k\right)<\frac{\varepsilon}{3}$. Now, we define

$$\zeta_k=\begin{cases}\xi_k & \text{if } 1\leq k\leq k_0,\\ \eta_k & \text{if } k>k_0.\end{cases}$$

Then $z\in\varphi(1)$. Let $U=A_\varphi(z,\frac{\varepsilon}{3})$. Clearly $x\in U$. Then $x_n\in U$ for $n>N_0$, where N_0 is some natural number. Hence $\sum_{k=k_0+1}^{\infty}\varphi\left(\xi_k^n-\eta_k\right)<\frac{\varepsilon}{3}$ for $n>N_0$. On the other hand the sequence $(x_n)_{n\geq 1}$ is convergent to x in the sense of the φ-distance and so $\lim_{n\to\infty}\xi_k^n=\xi_k$ for $k\geq 1$. This implies that $\lim_{n\to\infty}\varphi\left(\xi_k^n-\eta_k\right)=\varphi\left(\xi_k-\eta_k\right)$ for $k\geq 1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For every $1\le k\le k_0$, there is a natural number N_k such that $|\varphi(\xi_k^n-\eta_k)-\varphi(\xi_k-\eta_k)|<\frac{\varepsilon}{3k_0}$ for $n>N_k$. Let $N=\max_{0\le k\le k_0} N_k$, then

$$|\sum_{k=1}^{\infty}\varphi(\xi_k^n-\eta_k)-\sum_{k=1}^{\infty}\varphi(\xi_k-\eta_k)|\le\sum_{k=1}^{k_0}|\varphi(\xi_k^n-\eta_k)-\varphi(\xi_k-\eta_k)|+$$

$$+\sum_{k=k_0+1}^{\infty}\varphi(\xi_k^n-\eta_k)+\sum_{k=k_0+1}^{\infty}\varphi(\xi_k-\eta_k)<\varepsilon \text{ for } n>N.$$

This implies that

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi(\xi_k^n-\eta_k)=\sum_{k=1}^{\infty}\varphi(\xi_k-\eta_k).$$

From the above lemmas there follows immediately

Theorem 3.1. Let φ be a continuous on $<0,\infty)$ function. Then a sequence $(x_n)_{n\ge1}$, $x_n\in\varphi(1)$ for $n\ge1$, is convergent to $x\in\varphi(1)$ in the $\mathcal{T}_\varphi$ topology if and only if $\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi(\xi_k^n-\eta_k)=\sum_{k=1}^{\infty}\varphi(\xi_k-\eta_k)$ for all $y\in\varphi(1)$ such that $x-y\in\varphi(1)$.

The following theorem gives a construction of convergent sequences in $\mathcal{T}_\varphi$ topology.

Theorem 3.2. Let φ be a function continuous on $<0,\infty)$, let $x\in\varphi(1)$ and $x_n\in\varphi(1)$ for $n\ge1$ be such that $\lim_{n\to\infty}\xi_k^n=\xi_k$ for $k\ge1$, and let $|\xi_k^n|\le\eta_k$ for all $k,n\ge1$, where $y\in\overline{\varphi(1)}$. Then the sequence $(x_n)_{n\ge1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

Proof. Let $z\in\varphi(1)$ be arbitrarily chosen. Then $y+|z|\in\varphi(1)$. We have that $\lim_{n\to\infty}\varphi(\xi_k^n+\zeta_k)=\varphi(\xi_k+\zeta_k)$ and $\varphi(\xi_k^n+\zeta_k)\le\varphi(\eta_k+|\zeta_k|)$ for $k,n\ge1$. Hence (see[5],p.317) we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi(\xi_k^n+\zeta_k)=\sum_{k=1}^{\infty}\varphi(\xi_k+\zeta_k).$$

This implies (see Theorem 3.1) that the sequence $(x_n)_{n\ge1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

Lemma 3.3. Let φ be a function continuous on $<0,\infty)$, let E

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

be a subset of the set of all natural numbers, let $x \in s$ be such that $\sum_{k\in E} \varphi(\xi_k) < \infty$. If a sequence $(x_n)_{n\geq 1}$ is such that $\sum_{k\in E} \varphi(\xi_k^n) < \infty$ for $n \geq 1$, $0 \leq \xi_k^n \leq \xi_k$ for $k \in E$, $n \geq 1$ and $\lim_{n\to\infty} \xi_k^n = \xi_k$ for $k \in E$, then

$$\lim_{n\to\infty} \sum_{k\in E} \varphi(\xi_k^n + \eta_k) = \sum_{k\in E} \varphi(\xi_k + \eta_k)$$

for all $y \in s$ such that $\sum_{k\in E} \varphi(\eta_k) < \infty$ and $\sum_{k\in E} \varphi(\xi_k + \eta_k) < \infty$.

Proof. If E is a finite set then the assertion of the lemma is obvious. Consequently we suppose that E is an infinite set. Let $E_1 = \{k \in E: \eta_k > 0\}$, $E_2 = \{k \in E: -\xi_k \leq \eta_k \leq 0\}$ and $E_3 = \{k \in E: \eta_k < -\xi_k\}$. These sets are pairwise disjoint and $E_1 \cup E_2 \cup E_3 = E$. Define

$$\zeta_k = \begin{cases} \xi_k + \eta_k & \text{if } k \in E_1, \\ \xi_k & \text{if } k \in E_2, \\ |\eta_k| & \text{if } k \in E_3. \end{cases}$$

Clearly $\sum_{k\in E} \varphi(\zeta_k) < \infty$ and so for an arbitrary $\varepsilon > 0$ there is a natural number k_0 such that $\sum_{k\in E_{k_0}} \varphi(\zeta_k) < \varepsilon$, where $E_{k_0} = E \cap \{k_0+1, k_0+2, \ldots\}$. Further, $|\xi_k^n + \eta_k| \leq \zeta_k$ for $k \in E$ and $n \geq 1$. Thus $\sum_{k\in E_{k_0}} \varphi(\xi_k^n + \eta_k) < \varepsilon$ for $n \geq 1$. This implies that the series $\sum_{k\in E} \varphi(\xi_k^n + \eta_k)$ is uniformly convergent. Then

$$\lim_{n\to\infty} \sum_{k\in E} \varphi(\xi_k^n + \eta_k) = \sum_{k\in E} \varphi(\xi_k + \eta_k).$$

The lemma is proved.

Denote for an arbitrary $x \in \varphi(l)$

$$E_0 = \{k \geq 1: \xi_k = 0\}, \quad E_+ = \{k \geq 1: \xi_k > 0\}, \quad E_- = \{k \geq 1: \xi_k < 0\}.$$

Theorem 3.3. Let φ be a function continuous on $<0, \infty)$, let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$x \in \varphi(l)$ and $x_n \in \varphi(l)$ for $n \geq 1$ be such that

1° $\xi_k^n = 0$ for $k \in E_0$ and $n \geq 1$,

2° $0 \leq \xi_k^n \leq \xi_k$ for $k \in E_+$ and $n \geq 1$,

3° $\xi_k \leq \xi_k^n \leq 0$ for $k \in E_-$ and $n \geq 1$,

4° $\lim_{n\to\infty} \xi_k^n = \xi_k$ for $k \geq 1$.

Then the sequence $(x_n)_{n\geq 1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

Proof. If $y \in s$ is such that $\sum_{k\in E_0} \varphi(\eta_k) < \infty$, then

$$\lim_{n\to\infty} \sum_{k\in E_0} \varphi(\xi_k^n + \eta_k) = \sum_{k\in E_0} \varphi(\xi_k + \eta_k).$$

From Lemma 3.3 it follows that

$$\lim_{n\to\infty} \sum_{k\in E_+} \varphi(\xi_k^n + \eta_k) = \sum_{k\in E_+} \varphi(\xi_k + \eta_k)$$

for all $y \in s$ such that $\sum_{k\in E_+} \varphi(\eta_k) < \infty$ and $\sum_{k\in E_+} \varphi(\xi_k + \eta_k) < \infty$.

Lemma 3.3 applied to the sequence $(-x_n)_{n\geq 1}$ and $-x$ gives

$\lim_{n\to\infty} \sum_{k\in E_-} \varphi(\xi_k^n + \eta_k) = \sum_{k\in E_-} \varphi(\xi_k + \eta_k)$ for all $y \in s$ such that $\sum_{k\in E_-} \varphi(\eta_k) < \infty$ and $\sum_{k\in E_-} \varphi(\xi_k + \eta_k) < \infty$. Thus

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi(\xi_k^n + \eta_k) = \sum_{k=1}^{\infty} \varphi(\xi_k + \eta_k)$$

for all $y \in \varphi(l)$ such that $x+y \in \varphi(l)$. From Theorem 3.1 we obtain that the sequence $(x_n)_{n\geq 1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology.

4. Connections between several notions of convergences in $\varphi(l)$

We denote by D_{d_φ}, $D_{\mathcal{T}_\varphi}$, $D_{\mathcal{T}_{\varphi^*}}$ the classes of convergent sequences in $\varphi(l)$ in the sense of the φ-distance, in the $\mathcal{T}_\varphi$ topology and in the $\mathcal{T}_{\varphi^*}$ topology respectively.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

We now prove a number of lemmas.

Lemma 4.1. Let $\varphi(+0)=0$. If φ does not satisfy the condition (Δ_2), then there is $x\in\varphi(1)$ such that $\gamma x\notin\varphi(1)$ for each $\gamma>1$.

Proof. If φ does not satisfy the condition (Δ_2), the there is a sequence $u_n\downarrow 0$ as $n\to\infty$ such that $u_n>0$ for $n\geq 1$, $\varphi(u_n)<\frac{1}{2n^2}$ and $\varphi((1+\frac{1}{n})u_n)>2^n\varphi(u_n)$ for $n\geq 1$. Further, there are finite, pairwise disjoint sets F_n of indices j for which

$$\frac{1}{2n^2}\leq\sum_{j\in F_n}\varphi(u_n)<\frac{1}{n^2}\quad\text{for } n\geq 1$$

(compare [6], the proof of Lemma 1.2.).

Now, we define $\xi_k=u_n$ if $k\in F_n$ $(n\geq 1)$. Clearly $x\in\varphi(1)$. On the other hand, for each $\gamma>1$ there is a natural number m such that $\gamma<1+\frac{1}{m}$ and so

$$\sum_{k=1}^{\infty}\varphi(\gamma\xi_k)\geq\sum_{n=m}^{\infty}\sum_{j\in F_n}\varphi((1+\frac{1}{n})u_n)=\infty.$$

Lemma 4.2. Let $\varphi(+0)=0$. If φ does not satisfy the condition (Δ_2), then there are $x\in\varphi(1)$, $x_n\in\varphi(1)$ for $n\geq 1$ and $\alpha>0$ such that $\lim_{n\to\infty} d_\varphi(x_n,x)=0$ and

$$\sum_{k=1}^{\infty}\varphi(\xi_k^n)>\alpha+\sum_{k=1}^{\infty}\varphi(\xi_k)\quad\text{for } n\geq 1.$$

Proof. Let $x\in\varphi(1)$ be such as in the proof of Lemma 4.1. Put for every fixed $n\geq 1$,

$$\xi_k^n=\begin{cases}2u_n & \text{if } k\in F_n,\\ u_i & \text{if } k\in F_i,\ i\neq n.\end{cases}$$

Clearly, for every fixed $n\geq 1$, we have $x_n\in\varphi(1)$. On the other hand

$$\sum_{j\in F_n}^{\infty}\varphi(\xi_k^n)>\frac{2^n}{2n^2}\sum_{\substack{i=1\\ i\neq n}}^{\infty}\sum_{j\in F_i}\varphi(u_i)>\frac{3}{8}+\sum_{k=1}^{\infty}\varphi(\xi_k)\quad\text{for } n\geq 1.$$

Further, $d_\varphi(x_n,x)<\frac{1}{n^2}\to 0$ as $n\to\infty$.

Lemma 4.3. Let $\varphi(+0)=0$. If $\varphi(a+0)>\varphi(a)$ for some real

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

number $a>0$, then there are $x\in\varphi(l)$, $x_n\in\varphi(l)$ for $n\geq 1$, and $\alpha>0$ such that $\lim_{n\to\infty} d_\varphi(x_n,x)=0$ and

$$\sum_{k=1}^{\infty}\varphi(\xi_k^n) \geq \alpha + \sum_{k=1}^{\infty}\varphi(\xi_k) \quad \text{for } n\geq 1.$$

Proof. Let $x=(a,0,\dots)$ and $x_n=(a+\frac{1}{n},0,\dots)$ for $n\geq 1$. Clearly $x,x_n\in\varphi(l)$ for $n\geq 1$. Further, $d_\varphi(x_n,x)=\varphi\left(\frac{1}{n}\right)\to 0$ as $n\to\infty$ and

$$\sum_{k=1}^{\infty}\varphi(\xi_k^n) \geq \varphi(a+0) - \varphi(a)+\sum_{k=1}^{\infty}\varphi(\xi_k).$$

Theorem 4.1. Let $\varphi(+0)=0$. Then $D_{d_\varphi} = D_{\mathcal{T}_\varphi}$ if and only if the following conditions are satisfied

(a) $\varphi(u+0)=\varphi(u)$ for $u>0$,

(b) φ satisfies the condition (Δ_2).

Proof. Sufficiency. It is clear that $D_{\mathcal{T}_\varphi} \subset D_{d_\varphi}$. It suffices to show that $D_{d_\varphi} \subset D_{\mathcal{T}_\varphi}$. Let a sequence $(x_n)_{n\geq 1}$, $x_n\in\varphi(l)$ for $n\geq 1$, be convergent to $x\in\varphi(l)$ in the sense of the φ-distance. Let $\varepsilon>0$ be arbitrary real and let $y\in\varphi(l)$ be such that $x\in A_\varphi(y,\varepsilon)$. By Theorem 2.4, there is a $\delta>0$ such that $x\in A_\varphi(x,\delta)\subset A_\varphi(y,\varepsilon)$. Since $\lim_{n\to\infty} d_\varphi(x_n,x)=0$ we obtain that $x_n\in A_\varphi(y,\varepsilon)$ for $n>N(\delta)$ and so $(x_n)_{n\geq 1}$ is convergent to x in $\mathcal{T}_\varphi$ topology.

Necessity. If the condition (a) or (b) does not hold, then (see Lemmas 4.2 and 4.3.) there exist $x\in\varphi(l)$, $x_n\in\varphi(l)$ for $n\geq 1$ and $\alpha>0$ such that $\lim_{n\to\infty} d_\varphi(x_n,x)=0$ and

$$\sum_{k=1}^{\infty}\varphi(\xi_k^n) > \alpha + \sum_{k=1}^{\infty}\varphi(\xi_k) \equiv \alpha+S \quad \text{for } n\geq 1.$$

We shall prove that $(x_n)_{n\geq 1}$ is not convergent in the $\mathcal{T}_\varphi$ topology. Clearly $x\in A_\varphi(0,\alpha+S)$. This set is a neighbourhood of x such that $x_n\notin A_\varphi(0,\alpha+S)$ for any $n\geq 1$ and so $(x_n)_{n\geq 1}$ is not convergent to x in the $\mathcal{T}_\varphi$ topology. Let us suppose that it is convergent to $y\neq x$, $y\in\varphi(l)$ in the $\mathcal{T}_\varphi$ topology. Then $d_\varphi(x_n,y)\to 0$ as $n\to\infty$ and $y=x$, a contradiction.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 4.2. Let $\varphi(+0)=0$. Then $D_{d_\varphi} = D_{\mathcal{J}_\varphi}$ if and only if φ satisfies the condition (Δ_2).

Proof. Sufficiency follows immediately from 1.82 in [2].

Necessity. Let φ do not satisfy the condition (Δ_2). We define $x\in\varphi(1)$ and $x_n\in\varphi(1)$ for $n\geq 1$ as in the proof of Lemma 4.1. Then $\lim_{n\to\infty} d_\varphi(x_n,x)=0$. Let us suppose that $\lim_{n\to\infty} \|x_n-x\| = 0$. Then (see [3], Theorem 1.6.) $\lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi(2(\xi_k^n - \xi_k))=0$. On the other hand

$$\sum_{k=1}^{\infty} \varphi(2(\xi_k^n - \xi_k)) = \sum_{j\in F_n} \varphi(2u_n) > \frac{2^n}{2n^2} \longrightarrow \infty \text{ as } n\to\infty,$$

a contradiction.

From the above theorems there follows immediately

Corollary 4.1. Let $\varphi(+0)=0$.

1^0 If $\varphi(u+0)=\varphi(u)$ for $u>0$ and φ satisfies the condition (Δ_2), then $D_{\mathcal{J}_\varphi} = D_{d_\varphi} = D_{\mathcal{J}_{\varphi^*}}$.

2^0 If $\varphi(a+0)>\varphi(a)$ for some real number $a>0$ and φ satisfies the condition (Δ_2), then $D_{\mathcal{J}_\varphi} \subsetneq D_{d_\varphi} = D_{\mathcal{J}_{\varphi^*}}$.

3^0 If φ does not satisfy the condition (Δ_2), then $D_{\mathcal{J}_\varphi} \subsetneq D_{d_\varphi}$ and $D_{\mathcal{J}_{\varphi^*}} \subsetneq D_{d_\varphi}$.

Theorem 4.3. If φ is a function continuous on $\langle 0,\infty)$ and φ does not satisfy the condition (Δ_2), then $D_{\mathcal{J}_\varphi} \setminus D_{\mathcal{J}_{\varphi^*}} \neq \emptyset$ and $D_{\mathcal{J}_{\varphi^*}} \setminus D_{\mathcal{J}_\varphi} \neq \emptyset$.

Proof. First, we shall prove that $D_{\mathcal{J}_\varphi} \setminus D_{\mathcal{J}_{\varphi^*}} \neq \emptyset$. Let $x\in\varphi(1)$ be such as in the proof of Lemma 4.1 and let for every fixed $n\geq 1$

$$\xi_k^n = \begin{cases} \xi_k & \text{if } 1\leq k\leq n, \\ u_i & \text{if } k>n. \end{cases}$$

Then $x_n\in\varphi(1)$ for every fixed $n\geq 1$, $0\leq\xi_k^n\leq\xi_k$ for $k,n\geq 1$ and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\lim_{n\to\infty} \xi_k^n = \xi_k$ for $k\geq 1$. By Theorem 3.3, the sequence $(x_n)_{n\geq 1}$ is convergent to x in the $\mathcal{T}_\varphi$ topology. Let us suppose that $\lim_{n\to\infty} \|x_n - x\| = 0$. Then $\lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi(2(\xi_k^n - \xi_k)) = 0$. On the other hand

$$\sum_{k=1}^{\infty} \varphi(2(\xi_k^n - \xi_k)) = \sum_{k=n+1}^{\infty} \varphi(2(\xi_k) = \infty, \text{ contradiction.}$$

Now, we shall prove that $D_{\mathcal{T}_{\varphi^*}} \setminus D_{\mathcal{T}_\varphi} \neq \emptyset$. Let $x \in \varphi(1)$ be such as in Lemma 4.1 and let $y = \frac{1}{2}x$, $x_n = (\frac{1}{2} + \frac{1}{n})x$ for $n\geq 1$. Let $\lambda > 0$ be arbitrary real. There is a natural number N such that $\lambda < \sqrt{n}$ if $n > N$. Then $d_\varphi(\lambda x_n, \lambda y) \leq \sum_{k=1}^{\infty} \varphi\left(\frac{1}{\sqrt{n}}\xi_k\right) \longrightarrow 0$ as $n\to\infty$. This implies (see [3], Theorem 1.6) that $\lim_{n\to\infty} \|x_n - y\| = 0$. Let us suppose that the sequence $(x_n)_{n\geq 1}$ is convergent to y in the $\mathcal{T}_\varphi$ topology. Then for $z = -\frac{1}{2}x$, $y - z \in \varphi(1)$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi(\xi_k^n - \zeta_k) = \sum_{k=1}^{\infty} \varphi(\eta_k - \zeta_k) < \infty$$

(see Theorem 3.1). On the other hand $d_\varphi(x_n, z) = \sum_{k=1}^{\infty} \varphi((1+\frac{1}{n})\xi_k) = \infty$ for all $n\geq 1$. This gives a contradiction.

Finally, we shall prove the following theorem.

Theorem 4.4. Let $\varphi(+0) = 0$. The convergence of the sequence $(x_n)_{n\geq 1}$, $x_n \in \varphi(1)$ for $n\geq 1$, to $x \in \varphi(1)$ in the sense of the φ-distance implies that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \varphi(\xi_k^n) = \sum_{k=1}^{\infty} \varphi(\xi_k)$$

if and only if the following conditions are satisfied:

(a) φ is a function continuous on $(0,\infty)$,

(b) φ satisfies the condition (Δ_2).

Proof. Sufficiency follows immediately from Theorems 4.1 and 3.1 applied to $y = (0,0,\ldots)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Necessity. It suffices to show that if $\varphi(a-0)<\varphi(a)$ for some real number $a>0$, then there are $x\in\varphi(1)$, $x_n\in\varphi(1)$ for $n\geq 1$ and $\alpha>0$ such that $\lim_{n\to\infty} d_\varphi(x_n,x)=0$ and $\lim_{n\to\infty}\sum_{k=1}^{\infty}\varphi(\xi_k^n)\leq -\alpha+\sum_{k=1}^{\infty}\varphi(\xi_k)$ for $n\geq 1$. Let n_0 be a natural number such that $a>\frac{1}{n_0}$, let $x=(a,0,\dots)$ and $x_n=\left(a-\frac{1}{n+n_0},0,\dots\right)$ for $n\geq 1$. Clearly $x\in\varphi(1)$ and $x_n\in\varphi(1)$ for $n\geq 1$. Further, $d_\varphi(x_n,x)=\varphi\left(\frac{1}{n+n_0}\right)\longrightarrow 0$ as $n\to\infty$ and $\sum_{k=1}^{\infty}\varphi(\xi_k^n)\leq\varphi(a-0)-\varphi(a)+\sum_{k=1}^{\infty}\varphi(\xi_k)$. Now we apply Lemmas 4.2 and 4.3.

REFERENCES

[1] S. Gnilka, D. Stachowiak-Gnilka; Remarks on additive adjoints (I), Funct. Approximatio Comment. Math. (1980), 123-125.

[2] S. Mazur, W. Orlicz; On some classes of linear spaces, Studia Math. 17(1958), 97-119.

[3] J. Musielak; Przestrzenie modularne, UAM Poznań 1978 (in Polish)

[4] W. Orlicz; Über Räume (L^M), Bull. Inst. Acad. Polon. Sci. Ser. A (1936), 93-107.

[5] R. Sikorski; Funkcje rzeczywiste, t.I. PWN Warszawa 1958 (in Polish).

[6] D. Stachowiak-Gnilka; Remarks on additive adjoints (II), Funct. Approximatio Comment. Math. 11 (1981), 83-89.

INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY,
60-769 POZNAŃ, POLAND

Received July 27, 1988; received version July 30,1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Khalil I. Tabash

AUTOMORPHISMS OF P_oP-LATTICES WITH DISTINGUISHED CHAIN BASE

The concept of P_0-lattice was first introduced by T. Traczyk [6]. P_0-lattices which are P-algebras (called P_0P-lattices) are examined in [3]. P_2-lattices and P_1-lattices are examined in [2].

The automorphism group of a finite P_2-lattice was given in [5]. The relation between finite P_2-lattice and P_0-lattice is given by Theorem 4.11 of [2]. It is known that, the chain base of a P_2-lattice is unique but of a P_0-lattice is not necessarily unique. The aim of this paper is to describe the automorphisms of a finite P_0P-lattice $\langle B, e_0, \ldots, e_{n-1} \rangle$ that make the chain of constants $e_0, \ldots, e_{n-1}$ fixed, where the chain satisfies the condition:

(*) If a is an atom of B and $0 \neq ae_i$, then $0 \neq ae_j$ for all $1 \leq j \leq i \leq n-2$.

1. Preliminaries

We adopt the notations of [3] here. A P_0-lattice is a bounded distributive lattice P which is generated by its center B and a finite subchain $0 = e_0 < \ldots < e_{n-1} = 1$ containing 0 and 1. It is denoted by $P = \langle B, e_0, \ldots, e_{n-1} \rangle$.

(i) $e_0, \ldots, e_{n-1}$ is called a chain base of P.

(ii) A P_0-lattice P is of order n, $(n \geq 2)$, if n is the smallest integer such that P has a chain base with n-terms.

(iii) Every element $x \in P$ can be written in the form: $x = d_1e_1 \vee \ldots \vee d_{n-1}e_{n-1} = \bigvee_{i=1}^{n-1} d_ie_i$, where $d_i \in B$, $i=1,\ldots,n-1$ and $d_1 \geq d_2 \geq \ldots \geq d_{n-1}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Such a representation is called a monotonic representation (mon. rep.) of X.

Let P be a bounded distributive lattice with center B. Let the largest element $z \in P$ $(z \in B)$ such that $zx \le y$, if it exists, be denoted by $x \longrightarrow y$ $(x \Longrightarrow y)$. Let $\neg x = x \longrightarrow 0$ and $!x = 1 \Longrightarrow x$. If $x \longrightarrow y$ $(x \Longrightarrow y)$ exists for $x,y \in P$, then P is called a Heyting algebra (a B - algebra). It is clear that $x \Longrightarrow y = !(x \longrightarrow y)$.

An element $x \in P$ is called dense if $\neg x = 0$. A P-algebra is a B-algebra satisfying $(x \Longrightarrow y) \vee (y \Longrightarrow x) = 1$ for all $x,y \in P$. If P is a P-algebra [1], then:

$!(xy) = !x!y$ for all $x,y \in P$,

$x \longrightarrow (y \vee z) = (x \longrightarrow y) \vee (x \longrightarrow z)$,

$yz \longrightarrow x = (y \longrightarrow x) \vee (z \longrightarrow x)$,

$x \longrightarrow yz = (x \longrightarrow y)(x \longrightarrow z)$ and

$(y \vee z) \longrightarrow x = (y \longrightarrow x)(z \longrightarrow x)$.

P_0-lattices which are P-algebras are called P_0P-lattices [3]. Let the least Boolean element greater than or equal to x (if it exists) be denoted by x!. It was proved in [3] that, in a P_0P-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$,

(i) $(x \Longrightarrow y)(y \Longrightarrow z) \le (x \Longrightarrow z)$,

(ii) $x! = \overline{x \Longrightarrow 0}$ ($\overline{x \Longrightarrow 0}$ is the complement of $x \Longrightarrow 0$),

(iii) $(x \vee y)! = x! \vee y!$,

(iv) Every element $x \in P$ can be written in the form

$$x = \bigvee_{i=1}^{n-1} D_i(x) e_i \quad \text{where} \quad D_i(x) = x!(e_i \Longrightarrow x),$$

$i = 1, \ldots, n-1$ and the following properties hold:

(a) $D_1(x) \ge \ldots \ge D_{n-1}(x)$,

(b) $D_i(x \vee y) = D_i(x) \vee D_i(y)$,

(c) $D_i(xy) = D_i(x) D_i(y)$,

(d) $D_i(b) = b$ for $b \in B$,

(e) $D_i(e_j) = e_j!$ for $i \le j$ and $D_i(e_j) = e_j!\,(e_i \Longrightarrow e_j)$ for $i > j$ and in particular $D_{n-1}(e_j) = !e_j$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. P_0P-lattice automorphisms

Each P_0P-lattice $P = \langle B, e_0, \ldots, e_{n-1}\rangle$ used in this section is finite and the chain base $e_0, \ldots, e_{n-1}$ satisfies (*).

Definition 2.1. Let $P = \langle B, e_0, \ldots, e_{n-1}\rangle$ and $P' = \langle B', e'_0, \ldots, e'_{m-1}\rangle$ be two P_0P-lattices of orders n and m respectively, $m \geq n$. Then a lattice homomorphism h of P into P' is called a P_0-homomorphism provided that

(i) $h|B$ is a Boolean homomorphism,

(ii) $h(e_1), \ldots, h(e_{n-2})$ is a chain in $P' - B'$.

If P and P' are P_0P-lattices then h is said to be a P_0P-homomorphism, provided that

$$h(x \Longrightarrow y) = h(x) \Longrightarrow h(y) .$$

A one-to-one P_0P-homomorphism of a P_0P-lattice P onto itself is called a P_0P-automorphism. Hence, if h is an automorphism of a P_0P-lattice $P = \langle B, e_0, \ldots, e_{n-1}\rangle$ then $h(B) = B$ and $h(e_0), \ldots, h(e_{n-1})$ is a chain base of P.

Definition 2.2. Let $P = \langle B, e_0, \ldots, e_{n-1}\rangle$ be a P_0P-lattice of order n. Let

(i) $D_{e_i} = \{a : a \text{ is an atom of } B \text{ and } ae_i = a\}$, $i=1,\ldots,n-1$,

(ii) $F_{e_i} = \{a : a \text{ is an atom of } B \text{ and } ae_i = 0\}$, $i=1,\ldots,n-1$,

(iii) $P_{e_i} = \{a : a \text{ is an atom of } B \text{ and } 0 < ae_i < a\}$, $i=1,\ldots,n-1$ and

(iv) $P_{e_i}^{s(j)} = \left\{a : a \in P_{e_i},\ ae_{L-1} = ae_1 \longleftrightarrow L \in s(j) \subseteq \{2,\ldots i\}\right\}$,

where $s : \{1,\ldots,2^{i-1}\} \xrightarrow[\text{onto}]{\text{one-to-one}} 2^{\{2,\ldots,i\}}$, $j=1,\ldots,2^{i-1}$.

(If $i = 1$ then $s(j) = \varnothing$ and $P_{e_1}^{s(j)} = P_{e_1}$).

If A is the set of all atoms in B then it is clear that

(t_1) $A = D_{e_i} \cup P_{e_i} \cup P_{e_i}$ for all $i = 1,\ldots,n-1$,

(t_2) $D_{e_1} \subseteq D_{e_2} \subseteq \ldots \subseteq D_{e_{n-1}}$.

Using (*) it is easy to prove that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(f_3) $F_{e_1} = \dots F_{e_{n-2}}$, hence

(f_4) $P_{e_1} \supseteq P_{e_2} \supseteq \dots \supseteq P_{e_{n-1}}$.

Lemma 2.1. $P_{e_i}^{s(j_1)} \cap P_{e_i}^{s(j_2)} = \emptyset$ for $1 \le j_1 \ne j_2 \le 2^{i-1}$, $i = 2,\dots,n-2$.

Proof. Since s is one-to-one, $j_1 \ne j_2$ implies $s(j_1) \ne s(j_2)$. Hence, there exist i_0, $1 < i_0 \le i$ such that $i_0 \in s(j_1)$ and $i_0 \notin s(j_1)$ or $i_0 \notin s(j_1)$ and $i_0 \in s(j_2)$. In the first case, let $a \in P_{e_i}^{s(j_1)} \cap P_{e_i}^{s(j_2)}$. Then

(i) $a \in P_{e_i}^{s(j_1)}$ and $i_0 \in s(j_1)$ implies $ae_{i_0-1} = ae_{i_0}$ and

(ii) $a \in P_{e_i}^{s(j_2)}$ and $i_0 \notin s(j_2)$ implies $ae_{i_0-1} < ae_{i_0}$

which contradicts (i). We arrive at the contradiction in the second case. Hence $P_{e_i}^{s(j_1)} \cap P_{e_i}^{s(j_2)} = \emptyset$.

Lemma 2.2. $P_{e_i} = \bigcup_{j=1}^{2^{i-1}} P_{e_i}^{s(j)}$ (some of the terms may be empty).

Proof. Let $a \in P_{e_i}$. Consider two cases.

(i) $ae_{L-1} \ne ae_L$ for all $2 \le L \le i$. Then $a \in P_{e_i}^{s(j)}$, where $s(j) = \emptyset$.

(ii) $ae_{L_1-1} = ae_{L_1}, \dots, ae_{L_{i_0}-1} = ae_{L_{i_0}}$ for some $2 \le L_1, \dots, L_{i_0} \le i$. Let $s(j_0) = \{L_1,\dots,L_{i_0}\}$. Then $s(j_0) \subseteq \{2,\dots,i\}$ and $L \in s(j_0)$ iff $ae_{L-1} = ae_{L'}$ which implies $a \in P_{e_i}^{s(j_0)}$. From (i) and (ii), $P_{e_i} \subseteq \bigcup_{j=1}^{2^{i-1}} P_{e_i}^{s(j)}$.

The converse inclusion is obvious from the definition.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Using Lemmas 2.1 and 2.2 it is clear that,

(I) $P_{e_i} - P_{e_{i+1}} = \left(\bigcup_{j=1}^{2^{i-1}} P_{e_i}^{s(j)} \right) - P_{e_{i+1}} = \bigcup_{j=1}^{2^{i-1}} \left(P_{e_i}^{s(j)} - P_{e_{i+1}} \right)$

(some of the terms may be empty), $i=1,\ldots,n-2$.

Since $P_{e_{n-1}} = \emptyset$, $P_{e_1} = (P_{e_1} - P_{e_2}) \cup \ldots \cup (P_{e_{n-2}} - P_{e_{n-1}}) =$

$= \bigcup_{i=1}^{n-2} (P_{e_i} - P_{e_{i+1}}) = \bigcup_{i=1}^{n-2} \left(\bigcup_{j=1}^{2^{i-1}} (P_{e_i}^{s(j)} - P_{e_{i+1}}) \right)$, which implies

(II) $P_{e_1} = \bigcup_{i=1}^{n-2} \left(\bigcup_{j=1}^{2^{i-1}} (P_{e_i}^{s(j)} - P_{e_{i+1}}) \right)$. Some of the terms may be empty.

If A is the set of all atoms of B, then using (f_1), $A = D_{e_1} \cup F_{e_1} \cup P_{e_1}$. Hence, using (II),

(III) $A = D_{e_1} \cup F_{e_1} \cup \bigcup_{i=1}^{n-2} \left(\bigcup_{j=1}^{2^{i-1}} (P_{e_i}^{s(j)} - P_{e_{i+1}}) \right)$. Some of the terms may be empty.

Lemma 2.3. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a P_0P-lattice of order n. Let h be an automorphism of P. Then

(i) $d(D_{e_i}) = D_{h(e_i)}$, $i = 1,\ldots,n-2$,

(ii) $h(F_{e_i}) = F_{h(e_i)}$, $i = 1,\ldots,n-2$ and

(iii) $h(P_{e_i}^{s(j)}) = P_{h(e_i)}^{s(j)}$, $i = 1,\ldots,n-2$, $j = 1,\ldots,2^{i-1}$.

Proof. (i) $ae_i = a$ iff $h(a)h(e_i) = h(a)$. Hence, $a \in D_{e_i}$ iff $h(a) \in D_{h(e_i)}$, i.e. (i) holds.

The proof of (ii) is similar.

(iii) Let $a \in P_{e_i}^{s(j)}$. Then $a \in P_{e_i}$ and $ae_{L-1} = ae_L$ iff $L \in s(j)$. Hence $0 < ae_i < a$ and $ae_{L-1} = ae_L$ iff $L \in s(j)$ and this implies $0 < h(a)h(e_i) < h(a)$ and $h(a)h(e_{L-1}) =$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$= h(a)h(e_L)$ iff $L \in s(j)$. Hence, $h(a) \in P^{s(j)}_{h(e_j)}$. The converse implication is similar.

Corollary 2.1. If $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ is a P_0P-lattice of order n and if h is an automorphism of P with $h(e_i) = e_i$, $i=1,\ldots,n-1$, then h permutes each of the sets $F_{e_1} = \ldots F_{e_{n-2}}$, D_{e_i}, $i=1,\ldots,n-2$ and $P^{s(j)}_{e_i}$, $i=1,\ldots,n-2$, $j=1,\ldots,2^{i-1}$.

Lemma 2.4. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a P_0P-lattice of order n. Let A be the set of all atoms in B. Then (i) $e_i! = \bigvee (A - F_{e_i})$ and (ii) for $i > j$, $e_i \Longrightarrow e_j = \bigvee (F_{e_i} \cup D_{e_j} \cup \bigcup \{ P^{s(L)}_{e_i} : s(L) \supseteq \{j+1,\ldots,i\}\})$.

Proof. (i): Let $a \in F_{e_i}$, then $ae_i = 0$, hence $a \leq e_i \Longrightarrow 0$ which implies $\bigvee F_{e_i} \leq e_i \Longrightarrow 0$. Now, $\bigvee (A - F_{e_i}) = \overline{(\bigvee F_{e_i})} \geq \overline{e_i \Longrightarrow 0} = e_i!$. Let a be an atom of B and let $a \leq \bigvee (A - F_{e_i})$. Then $0 < ae_i \leq a$. Hence $ae_i! = a$, i.e. $a \leq e_i!$, which implies that $\bigvee (A - F_{e_i}) \leq e_i!$ so (i) holds.

(ii): Let $a \leq \bigvee (F_{e_i} \cup D_{e_j} \cup \bigcup \{ P^{s(L)}_{e_i} : s(L) \supseteq \{j+1,\ldots,i\}\})$, $i > j$. Then, either $a \in F_{e_i}$ (then $ae_i = 0 < e_j$; which implies $a \leq e_i \Longrightarrow e_j$), $a \in D_{e_j}$ (then $ae_i = a = ae_j \leq e_j$ which implies $a \leq e_i \Longrightarrow e_j$) or $a \in P^{s(L)}_{e_i}$, where $s(L) \supseteq \{j+1,\ldots,i\}$. Hence, $0 < ae_i = ae_{i-1} = \ldots = ae_{j+1} = ae_j \leq e_j$ which implies $a \leq e_i \Longrightarrow e_j$. Thus, $\bigvee (F_{e_i} \cup D_{e_j} \cup \bigcup \{ P^{s(L)}_{e_i} : s(L) \supseteq \{j+1,\ldots,i\}\}) \leq e_i \Longrightarrow e_j$, $i>j$.

Suppose equality does not hold and let $a \in A$, $a \leq e_i \Longrightarrow e_j$ and $a \notin (F_{e_i} \cup D_{e_j} \cup \bigcup \{P^{s(L)}_{e_i} : s(L) \supseteq \{j+1,\ldots,i\}\})$, $i > j$. Hence, $ae_i \leq e_j$ and $0 < ae_i \neq ae_j < a$. Now, $0 < ae_i \neq ae_j < a$ and $i < j$ implies $ae_i \neq e_j$ and consequently $a \not\leq e_i \Longrightarrow e_j$, a contradiction.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 2.5. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a P_0P-lattice of order n. Then, for any $x \in P$ and $i > j$, $D_j(x)(e_i \Longrightarrow e_j) \leq D_i(x)$, where $D_i(x) = x!(e_i \Longrightarrow x)$.

Proof. Let $a \leq D_j(x)(e_i \Longrightarrow e_j)$. Then $a \leq x!$, $a \leq e_j \Longrightarrow x$ and $a \in \left(F_{e_i} \cup D_{e_j} \cup \bigcup \{ P^{s(L)}_{e_i} : s(L) \supseteq \{j+1, \ldots, i\}\}\right)$.

(i) If $a \in F_{e_i}$, then $ae_i = 0 < x$ and $a \leq e_i \Longrightarrow x$. Hence $a \leq D_i(x)$.

(ii) If $a \in D_{e_j}$, then $ae_i = ae_j < x$. Hence, $a \leq e_i \Longrightarrow x$ and $a \leq D_i(x)$.

(iii) If $a \in P^{s(L)}_{e_i}$, $s(L) \supseteq \{j+1, \ldots, i\}$, then $ae_i = ae_{i-1} = \ldots = ae_{j+1} = ae_j \leq x$. Hence, $a \leq e_i \Longrightarrow x$ and $a \leq D_i(x)$.

Theorem 2.1. If $h_0 : B \longrightarrow B$ is an automorphism of a Boolean algebra B (where B is finite) and $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ is a P_0P-lattice having another chain base $0 = e'_0 < \ldots < e'_{n-1} = 1$, then there exists an automorphism h of P with $h|B = h_0$ and $h(e_i) = e'_i$, $i = 1, \ldots, n-1$ if and only if

(**) $h_0(D_{e_1}) = D_{e'_1}$ and $h(P^{s(j)}_{e'_i}) = P^{s(j)}_{e'_i}$, $i = 1, \ldots, n-2$, $j = 1, \ldots, 2^{i-1}$.

Proof. Suppose h_0 satisfies (**) and let $x, y \in P$. Then x and y have a mon. reps. $x = \bigvee_{i=1}^{n-1} D_i(x)e_i$ and $y = \bigvee_{i=1}^{n-1} D_i(y)e_i$ where

$$D_i(x) = x!(e_i \Longrightarrow x) \quad \text{and} \quad D_i(y) = y!(e_i \Longrightarrow y).$$

Let $h : P \longrightarrow P$ be defined by

$$h(x) = \bigvee_{i=1}^{n-1} h_0(D_i(x))e'_i..$$

Then

(1) $$h(D_i(x)) = \bigvee_{j=1}^{n-1} h_0(D_j(D_i(x)))e'_i = \bigvee_{j=1}^{n-1} h_0(D_i(x))e'_j =$$

$$= h_0(D_i(x)).$$

Since $D_i(x)$ is a Boolean element, $h|B = h_0$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) We shall prove the following:

(a) $h_o(e_i \Longrightarrow e_j) = e'_i \Longrightarrow e'_j$,

(b) $h(e_i!) = e'_i!$,

(c) $\bigvee_{j=1}^{n-1} h_o(D_j(x))e'_j! = h_0(D_i(x))$ and

(d) $h_0(D_j(x))(e'_i \Longrightarrow e'_j) = h_0(D_i(x))$ for $i > j$.

The proof of (a): It is clear for $i \le j$. Let $i > j$. Then $h_0(e_i \Longrightarrow e_j) = h_0\big(\bigvee (F_{e_i} \cup D_{e_j} \cup \bigcup\{ P_{e_i}^{s(L)} : s(L) \supseteq \{j+1,\ldots,i\}\}\big)$ (by Lemma 2.4 (ii)) $= \bigvee \big(h_0(F_{e_i}) \cup h_0(D_{e_j}) \cup \bigcup\{ h_0(P_{e_i}^{s(L)}) : s(L) \supseteq \{j+1,\ldots,i\}\}\big) = \bigvee \big(F_{e'_i} \cup D_{e'_j} \cup \bigcup\{ P_{e'_i}^{s(L)} : s(L) \supseteq \{j+1,\ldots,i\}\}\big) = e'_i \Longrightarrow e'_j$.

The proof of (b): $h_0(e_i!) = h_0(A - F_{e_i})$ (by Lemma 2.4) $= \bigvee (A - h_0(F_{e_i})) = A - F_{e'_i} = e'_i!$

The proof of (c): Since $i \le j$, $h_0(D_j(x))e'_j! \le h_0(D_j(x)) \le h_0(D_i(x))$. Hence, $\bigvee_{j=i}^{n-1} h_0(D_j(x))e'_j! \le h_0(D_i(x))$.

Let $h_0(a) \le h_0(D_i(x))$, where a is an atom of B. Then, either (i) $0 \ne h_0(a)e'_i \le h_0(a)$ or (ii) $h_0(a)e'_i = 0$. If (i) holds, then $h_0(a) \le e'_i!$. Hence, $h_0(a) \le h_0(D_i(x))e'_i! \le \bigvee_{j=i}^{n-1} h_0(D_j(x))e'_j!$ and (c) holds.

If (ii) holds, then $ae_i = 0$ implies $ae_j = 0$ for all $1 \le j \le n-2$. Hence, $a \le e_j \Longrightarrow 0$ which implies $ae_j! = 0$ for all $1 \le j \le n-2$ and consequently $a \le D_{n-1}(x) = D_{n-1}(x)e_{n-1}!$. Hence $h_0(a) \le h_0(D_{n-1}(x))e'_{n-1}! \bigvee_{j=i}^{n-1} h_0(D_j(x))e'_{n-1}!$ and again (c) holds.

The proof of (d): The proof follows by Lemma 2.5 and part (a) of (2).

(3) We shall prove that $D'_i(h(x)) = h_0(D_i(x))$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$D'_i(h(x)) = D'\Big(\bigvee_{j=1}^{n-1} h_o(D_j(x))e'_j\Big) = \bigvee_{j=1}^{n-1} D'_i(h_o(D_j(x)))D'_i(e'_j) =$$

$$= \bigvee_{j=1}^{n-1} h_o(D_j(x))e'_j!(e'_i \Longrightarrow e'_j) =$$

$$= \bigvee_{j=1}^{i-1} h_o(D_j(x))e'_j!(e'_i \Longrightarrow e'_j) \vee \bigvee_{j=i}^{n-1} h_o(D_j(x))e'_j! = h_o(D_i(x))$$

by parts (d) and (c) of (2).

(4) We shall prove that $h(e_j) = e'_j$ for $j = 1,\ldots,n-1$.

$$h(e_j) = \bigvee_{i=1}^{n-1} h_o(D_i(e_j))e'_i = \bigvee_{i=1}^{n-1} h_o(e_j!(e_i \Longrightarrow e_j))e'_i =$$

$$= \bigvee_{i=1}^{n-1} h_o(e_j!)h_0(e_i \Longrightarrow e_j))e'_i = \bigvee_{i=1}^{n-1} e'_j!(e'_i \Longrightarrow e'_j)e'_i =$$

$$= \bigvee_{i=1}^{j} (e'_j!)e'_i \vee \bigvee_{i=j+1}^{n-1} e'_j!(e'_i \Longrightarrow e'_j)e'_i =$$

$$= e'_j \vee \bigvee_{i=j+1}^{n-1} e'_j!(e'_i \Longrightarrow e'_j)e'_i = e'_j \qquad \text{(because } (e'_i \Longrightarrow e'_j)e'_i \leq$$

$\leq e'_i(e'_i \to e'_j) = e'_j$ for $j < i$).

(5) $h(x \vee y) = h(x) \vee h(y)$ and $h(xy) = h(x)h(y)$ for all $x,y \in P$.

The easy proof is omitted.

(6) To prove h is one-to-one, denote $x = \bigvee_{i=1}^{n-1} D_i(x)e_i$, $y = \bigvee_{i=1}^{n-1} D_i(y)e_i$ and $h(x) = h(y)$. Since the representation is unique, $D'_i(h(x)) = D'_i(h(y))$, $i=1,\ldots,n-1$. Hence, (3) implies that $h_0(D_i(x)) = h_0(D_i(y))$, $i=1,\ldots,n-1$ and consequently $D_i(x) = D_i(y)$, $i=1,\ldots,n-1$. Hence, $x = y$.

(7) To prove h is onto, denote by $y = \bigvee_{i=1}^{n-1} D'_i(y)e'_i$ a mon. rep. of $y \in P$. Let $x = \bigvee_{i=1}^{n-1} h_0^{-1}(D'_i(y))e_i$. Using (**), it is easy to prove that, $x = \bigvee_{i=1}^{n-1} h_0^{-1}(D'_i(y))e_i$ is a mon. rep. of x and $h(x) = y$.

(8) We shall prove that, $h(x \Longrightarrow y) = h(x) \Longrightarrow h(y)$ for all

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$x,y \in P$. Since h is a lattice automorphism, h is order embedding i.e. $x \le y$ if and only if $h(x) \le h(y)$ for all $x,y \in P$ ([4], Theorem III. 1.3).

Let $b = h(x) \Longrightarrow h(y)$, hence $bh(x) \le h(y)$ which implies $h_0^{-1}(b)x \le y$ and this implies $h_0^{-1}(b) \le x \Longrightarrow y$. Hence, $b \le h_0(x \Longrightarrow y) = h(x \Longrightarrow y)$, i.e. $h(x) \Longrightarrow h(y) \le h(x \Longrightarrow y)$.

For $h(x \Longrightarrow y) \le h(x) \Longrightarrow h(y)$, the proof is simple. Thus, we have proved that h is a P_0P-automorphism of P. The converse implication follows by Lemma 2.4.

Remarks 2.1. Let A be the set of all atoms in P_0P-lattice $P = \langle B, e_0, \ldots, e_{n-1} \rangle$, then by (III) (Lemma 2.2), A has the partition

$$A = D_{e_1} \cup F_{e_1} \cup \bigcup_{i=1}^{n-2} \left(\bigcup_{j=1}^{2^{i-1}} (P_{e_i}^{s(j)} - P_{e_{i+1}}) \right),$$

some of the terms may be empty.

Hence, if $|A| = m$, $|D_{e_1}| = d$, $|F_{e_1}| = f$ and $|P_{e_i}^{s(j)} - P_{e_{i+1}}| = P_{ij}$, $i=1,\ldots,n-2$ and $j=1,\ldots,2^{i-1}$, then

$$m = d + f + \sum_{i=1}^{n-2} \left(\sum_{j=1}^{2^{i-1}} P_{ij} \right).$$

Remark 2.2. The following theorem is well-known in abstract algebra: If A is a set of m elements, then the number of permutations on A that leave a set $S \subset A$ of r elements ($r < m$) fixed is $(m-r)!r!$.

Theorem 2.2. Let $P = \langle B, e_0, \ldots, e_{n-1} \rangle$ be a P_0-lattice of order n with the center $B = 2^m$, where m is finite. Then the number of automorphisms of P that leave the chain $e_0, \ldots, e_{n-1}$ fixed is

$$\prod_{i=1}^{n-2} \prod_{j=1}^{2^{i-1}} (P_{ij})!d!f! \quad \text{where } d = |D_{e_1}|,\ f = |F_{e_1}| \text{ and } P_{ij} = |P_{e_i}^{s(j)} - P_{e_{i+1}}|,\ i=1,\ldots,n-2 \text{ and } j=1,\ldots,2^{i-1}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let A be the set of all atoms of B. Then $|A| = m$. By Remark 2.1, $m = d + f + \sum_{j=1}^{2^{i-1}} P_{ij}$, where $d = |D_{e_1}|$, $f = |F_{e_1}|$ and $P_{ij} = |P_{e_i}^{s(j)} - P_{e_{i+1}}|$, $i=1,\dots,n-2$ and $j=1,\dots,2^{i-1}$.

Hence, by Theorem 2.1 and Corollary 2.1, the only automorphisms of B that can be extended to automorphisms of P leaving the chain $e_0,\dots,e_{n-1}$ fixed are the automorphisms that leave each of the sets $P_{e_i}^{s(j)}$, $i=1,\dots,2^{i-1}$ and D_{e_1} fixed. Thus they leave each of the sets $P_{e_i}^{s(j)} - P_{e_{i+1}}$, $i=1,\dots,n-2$, $j=1,\dots,2^{i-1}$, D_{e_1} and F_{e_1} fixed.

Hence, using Remark 2.2, the number of automorphisms of P is greater than or equal to

$$\prod_{i=1}^{n-2} \prod_{j=1}^{2^{i-1}} (P_{ij})!d!f! \ .$$

Since the extension of h_0 is unique, Theorem 2.1 implies that the number is exactly equal to

$$\prod_{i=1}^{n-2} \prod_{j=1}^{2^{i-1}} (P_{ij})!d!f! \ . \ \blacksquare$$

REFERENCES

[1] G. Epstein, A. Horn: P-algebras, an abstraction from Post algebras, Algebra Universalis 4 (1974), 195-206.

[2] G. Epstein, A. Horn: Chain based lattices, Pacific J. Math. 55 (1975), 65-84.

[3] J. Klukowski, M. Zworski: On the representation of P_0-lattices being P-algebras, Demonstratio Math. 18(1985) 103-113.

[4] H. Rasiowa: An algebraic approach to nonclassical logics,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

North Holland Publishing Company, Amsterdam 1974.

[5] K.I. Tabash: Automorphism groups of P_2-lattices, Demonstratio Math. 22(1989) (to appear).

[6] T. Traczyk: Axiom and some properties of Post Algebras, Colloq. Math. 10 (1963), 193-209.

AL-FATAH UNIVERSITY, P.O.Box 13019, TRIPOLI, LIBYA

Received August 19, 1988; revised version August 20, 1990.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

P. K. Das

H^∞ INTERPOLATION INTO A CLASS OF ANALYTIC FUNCTIONS

Dedicated to the memory of Professor S.K. Bose

1. Introduction

Let $\{u_n\}_{n=0}^{\infty}$ be any fixed sequence of nonzero complex numbers satisfying

(1.1) $\liminf |u_n|^{1/n} = \alpha \quad (0<\alpha\le\infty)$.

We consider the set $X = \Big\{ f : f(z) = \sum_{n=0}^{\infty} a_n z^n, \ a_n \in \mathbb{C}$ and the sequence $\{a_n u_n\}_{n=0}^{\infty}$ is bounded$\Big\}$.

For $f,g\in X$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ define

(1.2) $\|f\| = \sup_{n\ge 0} |a_n u_n|$

and

(1.3) $(f\circ g)(z) = \sum_{n=0}^{\infty} a_n b_n u_n z^n$.

It is easily seen that X forms a vector space with respect to usual pointwise addition. Norm on X is given by (1.2). Taking multiplication of two functions in X by (1.3) it can be seen that X forms a commutative Banach algebra with identity $\Phi = \Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{u_n}$. This class of analytic functions was introduced in [3] and was further studied in [2].

Consider a function $f(z)$ analytic in the unit disc $|z|<1$. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$M_\infty(r,f) = \sup_{0\le\theta<2\pi} |f(re^{i\theta})|$$

is a monotonically increasing function of r in $[0,1)$. If we put

$$\|f\|_\infty = \lim_{r\to 1} M_\infty(r,f)$$

then the class H^∞ is defined to consist of all $f(z)$ analytic in the unit disc for which $\|f\|_\infty<\infty$. Thus H^∞ is the class of bounded analytic functions in the unit disc.

A sequence of points $\{z_k\}$ in the open unit disc is said to be an interpolating sequence if, for each bounded sequence of complex numbers $\{\omega_k\}$ there exists a function f in H^∞ such that $f(z_k) = \omega_k$ for every k. For a general discussion on the interpolating sequence one can refer to [5] and [6].

Carleson [1] studied the interpolation problem for bounded analytic functions. Shapiro and Shields [4] studied the interpolation problem for H^p $(0<p\le\infty)$ spaces. In this paper we consider the interpolation problem in X. Our result directly depends on the results obtained in [1] and [4].

A sequence $\{z_k\}$ of points in the open unit disc is said to be uniformly separated [5] if there is a number $\delta>0$ such that

$$\prod_{\substack{j=0\\ j\neq k}}^{\infty} \left|\frac{z_k - z_j}{1-\bar{z}_j z_k}\right| \ge \delta , \quad k = 0,1,2,\ldots \tag{1.4}$$

For $f\in H^\infty$, $\{f(z_k)\}$ is a bounded sequence if $\{z_k\}$ is a sequence of points in the open unit disc.

Define $T : H^\infty \longrightarrow X$ by taking

$$T(f) = \sum_{k=0}^{\infty} \frac{f(z_k)}{u_k} z^k , \tag{1.5}$$

where $\{u_k\}$ is taken as in (1.1). Then $T(f) \in X$ for all $f\in H^\infty$.

2. **Theorem:** $T(H^\infty) = X$ if and only if $\{z_k\}$ is a uniformly separated sequence.

Proof. 1) Suppose $T(H^\infty) = X$ and let

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$N^\infty = \{f \in H^\infty : f(z_k) = 0,\ k = 0,1,2,\ldots\}$$

denote the kernel of T. The quotient space H^∞/N^∞ is a Banach space under the usual norm. Since $T(H^\infty) = X$, T induces a one-one bounded linear operator $\tau : H^\infty/N^\infty$ onto X. By the open mapping theorem τ^{-1} is bounded. In other words, corresponding to each $v \in X$ there is a function $f \in H^\infty$ with

$$\sum_{k=0}^{\infty} \frac{f(z_k)}{u_k} z^k = v\ ,$$

where $\{u_k\}$ is the sequence as in (1.1) and

$$\|f\|_\infty \le A\|v\|\ ,$$

where A is a constant. In particular, for each positive integer k, there is a function $f_k \in H^\infty$ with $\|f_k\|_\infty \le A\|v\|$ and

$$f_k(z_j) = \begin{cases} 1\ , & j = k \\ 0\ , & j \ne k. \end{cases}$$

Let

$$F_{nk(z)} = f_k(z) \prod_{\substack{j=0 \\ j\ne k}}^{\infty} \frac{1-\bar{z}_j z}{z-z_j}$$

for $n>k$, then $F_{nk} \in H^\infty$ and $\|F_{nk}\|_\infty = \|f_k\|_\infty \le A$.

On the other hand, for $F \in H^\infty$, it follows that

$$|F(z)| \le c\|F\|_\infty .$$

Hence $\prod_{\substack{j=0 \\ j\ne k}}^{\infty} \left|\frac{1-\bar{z}_j z_k}{z_k - z_j}\right| = |F_{nk}(z_k)| \le c\|F_{nk}\|_\infty \le cA.$

Thus we conclude that $\{z_k\}$ is uniformly separated.

2) Given $v \in X$ and a uniformly separated sequence $\{z_k\}$ we have to show that a function $g \in H^\infty$ exists such that $T(g) = v$ where $v = \sum_{k=0}^{\infty} v_k z^k$. That is we have to show that $T(H^\infty) \supset X$.

Let

$$B_n(z) = \prod_{j=0}^{n} \frac{z-z_j}{1-\bar{z}_j z}\ ,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$B_{nk}(z) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{z-z_j}{1-\bar{z}_j z} ,$$

$$b_{nk} = B_{nk}(z_k) .$$

Define

$$f_n(z) = \sum_{k=0}^{n} \frac{v_k u_k}{b_{nk}} B_{nk}(z) ,$$

where $\{u_k\}$ is taken as in (1.1). Then $f_n(z_k) = v_k u_k$, $k = 0,1,2,\ldots,n$.

Now we define

$$k_n(v) = \inf_{\omega\in H^\infty} \|f_n+B_n\omega\|_\infty = \inf_{\omega\in H^\infty} \|F_n+\omega\|_\infty ,$$

where $F_n = f_n/B_n$. As we may identify L^∞/H^∞ with the conjugate space of the Banach space H_0', consisting of the H' functions which vanish at the origin we see, in particular, that $k_n(v)$ is the norm of F_n as a linear functional on H_0'. Thus

$$k_n(v) = \sup_h \left|\frac{1}{2\pi}\int_0^{2\pi} F_n(e^{i\theta})h(e^{i\theta})e^{i\theta}d\theta\right| , \quad h\in H' , \quad \|h\|_1\leq 1.$$

If we now apply Cauchy's integral formula to the above integral we obtain

$$k_n(v) = \sup_h \left|\sum_{k=0}^{n} \frac{v_k u_k}{b_{nk}} h(z_k)(1-|z_k|^2)\right| .$$

Thus, if we restrict v to the unit ball of X then

$$\sup_v k_n(v) = \sup_h \sum_{k=0}^{n} \frac{1}{|b_{nk}|} |h(z_k)|(1-|z_k|^2) , \quad h\in H', \quad \|h\|_1\leq 1.$$

If we now let

$$K = \sup_h \sum_{k=0}^{\infty} |h(z_k)|(1-|z_k|^2) , \quad h\in H', \quad \|h\|_1\leq 1$$

and remember that $|b_{nk}|\leq 1$, we see that

$$K < \sup_n \sup_v k_n(v) \leq K\cdot\frac{1}{\inf |b_{nk}|} .$$

As $\{z_k\}$ is uniformly separated we have $\inf |b_{nk}| \geq \delta$. Thus,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we conclude that

$$\sup_n \sup_v k_n(v) < \infty$$

if and only if $K < \infty$.

As the convergence of K is equivalent to the convergence of

$$\sum_{k=0}^{\infty} |h(z_k)|(1-|z_k|)$$

for every $h \in H'$ [6], we can find a sequence of functions $\{g_n\}$ in H^∞ such that

$$g_n(z_k) = v_k u_k , \quad 0 \le k \le n$$

and $\|g_n\| \le M$.

The sequence $\{g_n\}$ is a normal family and will have a subseqence which converges uniformly on compact subsets of the disc to a function g in H^∞ with $\|g\| \le M$ for which $T(g) = v$. Since v is arbitrary, this proves $T(H^\infty) \supset X$.

Acknowledgement. The author wishes to thank the referee for his extensive comments on the earlier version of this paper.

REFERENCES

[1] L. Carleson: An interpolation problem for bounded analytic functions, Amer. J. Math. 80(1958), 921-930.

[2] P.K. Das: Multipliers into a class of analytic functions, Demonstratio Math. 18(1985), 893-897.

[3] P.D. Srivastava, O.P. Juneja: On a Banach algebra of analytic functions, Glasnik Math. 20(40)(1985), 51-59.

[4] H.S. Shapiro, A.L. Shields: On some interpolation problems for for analytic functions, Amer. J. Math. 83(1961), 513-532

[5] P.L. Duren: Theory of H^p spaces. Academic Press, 1970.

[6] K. Hoffman: Banach spaces of analytic functions. Prentice-Hall, Inc. 1962.

STAT-MATHS. DIVISION, INDIAN STATISTICAL INSTITUTE,
CALCUTTA 700 035, INDIA.

Received August 26, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

D.I. Adu

ON A CHARACTERIZATION OF CM-HOMOMORPHISMS

1. Introduction

Terms and notation not defined here can be found in Adu, [1] and Clifford and Preston, [4]. A topological space X is a Magill space (Adu, [2]) if it is T_1 and compositions of closed relations on X are closed. Such a space was referred to by Magill and Yamamuro [6] as a C-space. It will be assumed that a Magill space is first countable and has more than one point. The symbol M[X] denotes the semigroup, under composition, of all closed binary relations on a Magill space X.

Let X and Y be Magill spaces, A CM-homomorphism from M[X] into M[Y] is one which is non-constant, preserves unions and carries symmetric relations onto symmetric relations. This notion was introduced by A.H. Clifford and D.D. Miller [3] and put in topological setting by K.D. Magill and S. Yamamuro [6]. We denote by $HOM_{CM}(M[X],M[Y])$ the set of all CM-homomorphisms from M[X] into M[Y]. In this paper, our main result is that if $\psi : M[X] \longrightarrow M[Y]$ is a union preserving homomorphism from M[X] into M[Y], then the requirement that ψ be symmetry preserving is equivalent to the apparently weaker condition that both $\psi(0)$ and $\psi(X\times X)$ be symmetric, where 0 is the empty relation.

2. CM-Homomorphisms

Let X be a Magill space and let $\alpha,\beta \in M[X]$. We define their composition $\alpha\circ\beta$ by

$$\alpha\circ\beta = \{(x,y) \in X\times X \mid (x,y) \in \beta \text{ and } (z,y) \in \alpha \text{ for some } z \in X\}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The domain of α is the set

$$D(\alpha) = \{x \in X \mid (x,y) \in \alpha \text{ for some } y \in X\}.$$

The range of α is the set

$\mathfrak{R}(\alpha) = D(\alpha^{-1})$, where $\alpha^{-1} = \{(x,y) \mid (y,x) \in \alpha\}$.
Let $\alpha \in M[X]$ and let $x \in X$, we denote by $\alpha[x]$, the set

$$\alpha[x] = \{y \in X \mid (x,y) \in \alpha\}.$$

Observe that $\alpha[x]$ is closed in X for each $x \in X$ since

$$\alpha \circ \langle x \rangle = \{x\} \times \alpha[x] \text{ is closed in } X \times X.$$

We are now in a position to state our first result.

Proposition (2.1). Let X be a Magill space, let ρ be a congruence on M[X] and let $(\alpha,\beta) \in \rho$. If $D(\alpha) \neq D(\beta)$ or $\mathfrak{R}(\alpha) \neq \mathfrak{R}(\beta)$, then $(r_1,r_2) \in \rho$ for each pair of rectangles $r_1,r_2 \in M[X]$.

Proof. Let $(\alpha,\beta) \in \rho$ and suppose that $D(\alpha) \neq D(\beta)$. We can assume that $D(\alpha) \setminus D(\beta) \neq \emptyset$. Thus let $x \in D(\alpha) \setminus D(\beta)$ and let A be any closed subset of X. Then since ρ is right compatible, we have that

$$(A \times \alpha[x], 0) = (\alpha \circ A \times \{x\}, \beta \circ A \times \{x\}) \in \rho.$$

Let $y \in \alpha[x]$ and let D be any closed subset of X. Then, since ρ is left compatible, we have that

$$(2.1.1) \qquad (A \times D, 0) = (\{y\} \times D \circ A \times \alpha[x], 0) \in \rho.$$

Since every rectangle of M[X] is of the form $A \times D$, where A and D are closed subsets of X, we have by (2.1.1) that $(r,0) \in \rho$ for all rectangles $r \in M[X]$. The result follows in this case from the transitivity of ρ. The proof of the case where $\mathfrak{R}(\alpha) \neq \mathfrak{R}(\beta)$ is similar.

Proposition (2.2). Let ρ be a congruence on M[X] and let $(\alpha,\beta) \in \rho$ for some $\alpha \neq \beta$. Then $(r_1,r_2) \in \rho$ for each pair of rectangles $r_1,r_2 \in M[X]$.

Proof. We can assume that $\alpha \setminus \beta \neq \emptyset$. Also, by Proposition (2.1), we can assume that $D(\alpha) = D(\beta)$ and $\mathfrak{R}(\alpha) = \mathfrak{R}(\beta)$. Thus let $(x,y) \in \alpha \setminus \beta$. Then

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.2.1) $(\alpha^{-1}[y]\times\{x\},\ \beta^{-1}[y]\times\{x\}) = (\langle y,x\rangle\circ\alpha,\ \langle y,x\rangle\circ\beta)\in\rho.$

Since $x\in\alpha^{-1}[y]$ and $x\notin\beta^{-1}[y]$, we have that

$$D(\alpha^{-1}[y]\times\{x\}) = \alpha^{-1}[y] \neq \beta^{-1}[y] = D(\beta^{-1}[y]\times\{x\}).$$

The result now follows from (2.2.1) and Proposition (2.1).

Let A be a subset of a set X, we denote by Δ_A, the diagonal (or identity) relation on A, that is,

$$\Delta_A = \{(x,x) \mid x\in A\}.$$

The following lemma will be useful.

Lemma (2.3). Let X and Y be two Magill spaces, let $\psi : M[X] \longrightarrow M[Y]$ be a homomorphism from M[X] into M[Y] which is union preserving and let $x,y,u,v \in X$, $\alpha,\beta \in M[X]$ and A, B be closed nonempty subsets of X. Then the following statements are true:

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

(2.3.1) $D(\psi(\langle x\rangle)) = D(\psi(\{x\}\times A))$,

(2.3.2) $\mathfrak{R}(\psi(\langle y\rangle)) = \mathfrak{R}(\psi(B\times\{y\}))$,

(2.3.3) $D(\alpha) = D(\beta)$ implies $D(\psi(\alpha)) = D(\psi(\beta))$,

(2.3.4) $\mathfrak{R}(\alpha) = \mathfrak{R}(\beta)$ implies $\mathfrak{R}(\psi(\alpha)) = \mathfrak{R}(\psi(\beta))$.

If, in addition, ψ is zero preserving and

$$D(\psi(X\times X)) = \mathfrak{R}(\psi(X\times X)),$$

then

(2.3.5) $D(\psi(\langle x\rangle)) = \mathfrak{R}(\psi(\langle x\rangle))$,

(2.3.6) $D(\psi(\langle x,y\rangle)) \cap D(\psi(\langle u,v\rangle)) \neq \emptyset$ implies $x = u$,

(2.3.7) $\mathfrak{R}(\psi(\langle x,y\rangle)) \cap \mathfrak{R}(\psi(\langle u,v\rangle)) \neq \emptyset$ implies $v = y$,

(2.3.8) $D(\alpha) = \mathfrak{R}(\alpha)$ implies $D(\psi(\alpha)) = \mathfrak{R}(\psi(\alpha))$.

Proof. (2.3.1) From $\langle x\rangle = A\times\{x\}\circ\{x\}\times A$, we have that

Thus $\psi(\langle x\rangle) = \psi(A\times\{x\}\circ\{x\}\times A) = \psi(A\times\{x\})\circ\psi(\{x\}\times A)$.

$$D(\psi(\langle x\rangle)) \subseteq D(\psi(\{x\}\times A)).$$

Similarly, we obtain the reverse inclusion from the equation

$$\psi(\{x\}\times A) = \psi(\{x\}\times A)\circ\psi(\langle x\rangle)$$

and (2.3.1) follows.

The proof of (2.3.2) is similar to that of (2.3.1).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.3.3) $\alpha = \bigcup_{x\in D(\alpha)} (\{x\}\times\alpha[x]) = \bigcup_{y\in \Re(\alpha)} (\alpha^{-1}[y]\times\{y\})$.

Thus by applying (2.3.1) and hypothesis that ψ is union preserving, we obtain

$$D(\psi(\Delta_{D(\alpha)})) = D\Big(\psi\Big(\bigcup_{x\in D(\alpha)} <x>\Big)\Big) = D\Big(\bigcup_{x\in D(\alpha)} \psi(<x>)\Big) =$$

$$= D\Big(\bigcup_{x\in D(\alpha)} \psi(\{x\}\times\alpha[x])\Big) = D\Big(\psi\Big(\bigcup_{x\in D(\alpha)} (\{x\}\times\alpha[x])\Big)\Big) = D(\psi(\alpha)).$$

Hence if $D(\alpha) = D(\beta)$ we obtain

$$D(\psi(\alpha)) = D(\psi(\Delta_{D(\alpha)})) = D(\psi(\Delta_{D(\beta)})) = D(\psi(\beta)).$$

The proof of (2.3.4) is similar to that of (2.3.3).

(2.3.5) Let $a \in D(\psi(<x>))$. Then since ψ is union preserving and therefore order preserving, we have that

$$a \in D(\psi(X\times X)) = \Re(\psi(X\times X)).$$

By (2.3.2) and the fact that ψ is union preserving we obtain

$$a\in\Re(\psi(<u,v>)) = \Re(\psi(<v>)) \quad \text{for some } u,v\in X.$$

It follows that

$$a \in D(\psi(<x>)) \cap \Re(\psi(<v>)) \neq \emptyset.$$

Thus

$$0 \neq \psi(<x>)\circ\psi(<v>) = \psi(<x>\circ<v>).$$

Since ψ is zero-preserving, we have that $<x>\circ<v> \neq 0$. Consequently, $x = v$ and $a\in\Re(\psi(<x>))$. This shows that

$$D(\psi(<x>)) \subseteq \Re(\psi(<x>)).$$

Similarly we obtain

$$\Re(\psi(<x>)) \subseteq D(\psi(<x>)).$$

The result follows.

(2.3.6) Suppose that $x\neq u$. Then

$$0 = \psi(0) = \psi(<u,v>\circ<y,x>) = \psi(<u,v>)\circ\psi(<y,x>).$$

Thus by (2.3.2), (2.3.5) and (2.3.1), we obtain

$$\emptyset = \Re(\psi(<y,x>)) \cap D(\psi(<u,v>)) = \Re(\psi(<x>)) \cap D(\psi(<u,v>)) =$$

$$= D(\psi(<x>)) \cap D(\psi(<u,v>)) = D(\psi(<x,y>)) \cap D(\psi(<u,v>)).$$

The result follows.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.3.7) The proof here is similar to that of (2.3.6).

(2.3.8) Suppose that $D(\alpha) = \mathfrak{R}(\alpha)$. Then since ψ is union preserving, we have, by (2.3.3) and (2.3.4), that

$$D(\psi(\alpha)) = D(\psi(\Delta_{D(\alpha)})) = D\Big(\psi\Big(\bigcup_{x\in D(\alpha)} \langle x\rangle\Big)\Big) = D\Big(\bigcup_{x\in D(\alpha)} \psi(\langle x\rangle)\Big) =$$

$$= \bigcup_{x\in D(\alpha)} D(\psi(\langle x\rangle)) = \bigcup_{x\in D(\alpha)} \mathfrak{R}(\psi(\langle x\rangle)) = \mathfrak{R}\Big(\bigcup_{x\in \mathfrak{R}(\alpha)} \psi(\langle x\rangle)\Big) =$$

$$= \mathfrak{R}\Big(\psi\Big(\bigcup_{x\in \mathfrak{R}(\alpha)} \langle x\rangle\Big)\Big) = \mathfrak{R}(\psi(\Delta_{\mathfrak{R}(\alpha)})) = \mathfrak{R}(\psi(\alpha)).$$

We are now in a position to state the following result which shows that the assumptions on a zero-preserving CM-homomorphism can be weakened.

Theorem (2.4). Let X and Y be any two Magill spaces and let $\psi : M[X] \longrightarrow M[Y]$ be a non-constant union preserving and zero-preserving homomorphism from M[X] into M[Y]. Then the following statements are equivalent:

(2.4.1) ψ is a CM-homomorphism,

(2.4.2) $\psi(X\times X)$ is symmetric.

Proof. (2.4.1) implies (2.4.2), trivially. To prove that (2.4.2) implies (2.4.1), we only have to show that ψ takes symmetric relations onto symmetric relations. To this end, let $\alpha \in M[X]$ be symmetric and let $(a,b) \in \psi(\alpha) \subseteq \psi(X \times X)$. Then since ψ is union preserving, $(a,b) \in \psi(\langle x,y\rangle)$ for some $(x,y) \in \alpha$. Since $\psi(X \times X)$ is symmetric, we have that $(b,a)\in\psi(X \times X)$. This, in turn, implies that $(b,a)\in\psi(\langle u,v\rangle)$ for some $(u,v) \in X \times X$. By applying (2.3.2) and (2.3.5) we have that

$$b \in D(\psi(\langle u,v\rangle)) \cap \mathfrak{R}(\psi(\langle x,y\rangle)) = D(\psi(\langle u,v\rangle)) \cap \mathfrak{R}(\psi(\langle y\rangle)) =$$

That is,

$$= D(\psi(\langle u,v\rangle)) \cap D(\psi(\langle y\rangle)).$$

$$D(\psi(\langle u,v\rangle)) \cap D(\psi(\langle y\rangle)) \neq \emptyset.$$

It then follows from (2.3.6) that $u=y$. Since

$$a \in \mathfrak{R}(\psi(\langle u,v\rangle)) \cap D(\psi(\langle x,y\rangle)) = \mathfrak{R}(\psi(\langle u,v\rangle)) \cap D(\psi(\langle x\rangle)) =$$

$$= \mathfrak{R}(\psi(\langle u,v\rangle)) \cap \mathfrak{R}(\psi(\langle x\rangle)),$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

we have, by (2.3.7) that $v = x$. We have thus proved that $(b,a) \in \psi(\langle y,x\rangle)$. Since ψ is union preserving and α is symmetric, we have that $\psi(\langle y,x\rangle) \leq \psi(\alpha)$. Consequently, $(b,a) \in \psi(\alpha)$ and $\psi(\alpha)$ is symmetric.

A.H. Clifford and D.D. Miller [3], proved the following result for the case where X is a discrete space and ψ is an endomorphism. Appropriate modifications of their proof results in the proof we give here.

Theorem (2.5). Let X and Y be two Magill spaces, let $\zeta \in M[Y]$ be a partial equivalence on Y with domain F and let ψ be a homomorphism from M[X] into M[Y] mapping 0 onto 0 and having the property that there exists a subset E of Y, disjoint from F such that $\psi(\alpha) \leq \langle E\rangle$ for all α in M[X], Define $\psi' : M[X] \longrightarrow M[Y]$ by

(2.5.1) $\psi'(\alpha) = \psi(\alpha) \cup \zeta$ for all α in M[X].

Then ψ' is a homomorphism from M[X] into M[Y] mapping 0 onto ζ. If ψ preserves unions and carries $X \times X$ onto a symmetric element, so does ψ'. Moreover, every homomorphism from M[X] into M[Y] that preserves unions and carries $X \times X$ and 0 onto symmetric elements is obtained in this way.

Proof. Let $\alpha,\beta \in M[X]$. Then since $\langle E\rangle$ and $\langle F\rangle$ are disjoint and ψ is a homomorphism, we have that

$$\psi'(\alpha)\circ\psi'(\beta) = (\psi(\alpha) \cup \zeta)\circ(\psi(\beta) \cup \zeta) =$$
$$= (\psi(\alpha)\circ\psi(\beta)) \cup (\psi(\alpha)\circ\zeta) \cup (\zeta\circ\psi(\beta)) \cup \zeta =$$
$$= (\psi(\alpha)\circ\psi(\beta)) \cup \zeta = \psi(\alpha\circ\beta) \cup \zeta = \psi'(\alpha\circ\beta).$$

Thus ψ' is a homomorphism. $\psi'(0) = \psi(0) \cup \zeta = \zeta$ and it is obvious that if ψ preserves unions and carries $X\times X$ onto a symmetric element, so does ψ'.

Conversely, let $\psi' : M[X] \longrightarrow M[Y]$ be a homomorphism from M[X] into M[Y] which preserves unions and carries $X\times X$ and 0 onto symmetric elements. Let $\zeta = \psi'(0)$. Then

(2.5.2) $\psi'(\alpha) = \bigcup \{\psi'(\langle x,y\rangle) \mid (x,y) \in \alpha\}$ for all α in M[X].

Now, $\zeta\circ\zeta = \zeta$ and ζ is symmetric since $\psi'(0) = \zeta$. Hence ζ is a partial equivalence on Y. Put $F = D(\zeta)$. If $\zeta = 0$, there is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

nothing to prove. Hence we may assume that $\zeta \neq C$, and hence $F \neq \emptyset$. Put $E'_x = D(\psi'(\langle x\rangle))$. Then $\zeta \subseteq \psi'(\langle x\rangle)$ for all x in X. In particular, $F \subseteq E'_x$. We also have

$$\psi'(\langle u,v\rangle)\circ\psi'(\langle x,y\rangle) = \begin{cases} \psi'(\langle x,v\rangle) & \text{if } y=u \\ \zeta & \text{if } y\neq u \end{cases} \tag{2.5.3}$$

and

$$\psi'(\langle x,y\rangle)\circ\zeta = \zeta\circ\psi'(\langle x,y\rangle) = \zeta. \tag{2.5.4}$$

By (2.3.1), we obtain $D(\psi'(\langle x,y\rangle)) = D(\psi'(\langle x\rangle))$ and by (2.3.2) we obtain, $R(\psi'(\langle x,y\rangle)) = R(\psi'(\langle y\rangle))$. Observe that if $s \in F$ and $(s,t) \in \psi'(\langle x,y\rangle)$, then $(s,t) \in \zeta$. For since $F = D(\zeta)$ and ζ is a partial equivalence, we have that $\langle s\rangle \subset \zeta = \psi'(0)$. Hence $\langle s,t\rangle = \langle s,t\rangle\circ\langle s\rangle \subset \psi'(\langle x,y\rangle)\circ\zeta = \zeta$ by (2.5.4). Similarly, $t\in F$ and $(s,t) \in \psi'(\langle x,y\rangle)$ imply $(s,t)\in \zeta$. Consequently, if $(s,t) \in \psi'(\langle x,y\rangle)$, then either $(s,t)\in \zeta$ or $(s,t) \in E_x\times E^y$, where $E_x = E'_x\backslash F$, $E^y = R(\psi'(\langle y\rangle)\backslash F$. This means that if we set

$$\psi(\langle x,y\rangle) = \psi'(\langle x,y\rangle) \cap (E_x\times E^y), \tag{2.5.5}$$

then

$$\psi'(\langle x,y\rangle) = \psi(\langle x,y\rangle) \cup \zeta. \tag{2.5.6}$$

Since both the domain and the ranges of $\psi(\langle x,y\rangle)$ and ζ are disjoint, we have that

$$\psi'(\langle u,v\rangle)\circ\psi'(\langle x,y\rangle) = (\psi(\langle u,v\rangle)\circ\psi(\langle x,y\rangle)) \cup \zeta. \tag{2.5.7}$$

If $y=u$, from (2.5.3) and (2.5.6), we have that the left side of (2.5.7) is equal to $\psi(\langle x,v\rangle) \cup \zeta$ and since the domain and ranges of $\psi(\langle x,y\rangle)$ and ζ are disjoint for each $(x,y) \in X\times X$ we obtain

$$\psi(\langle u,v\rangle)\circ\psi(\langle x,y\rangle) = \psi(\langle x,v\rangle). \tag{2.5.8}$$

If $y\neq u$, then the left side of (2.5.7) is equal to ζ and we infer that $\psi(\langle u,v\rangle)\circ\psi(\langle x,y\rangle) = 0$.

Now, define $\psi : M[X] \longrightarrow M[Y]$ by

$$\psi(\alpha) = \bigcup \{\langle x,y\rangle \mid (x,y)\in\alpha\} \quad \text{and} \quad \psi(0) = 0. \tag{2.5.9}$$

We show that ψ is a homomorphism. Let $\alpha,\beta \in M[X]$, and let $(u,v) \in \psi(\alpha)\circ\psi(\beta)$. Then there exists $\omega \in X$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$(u,\omega) \in \psi(\beta)$ and $(\omega,v) \in \psi(\alpha)$. This implies that there exist x, y, s, t in X such that

$$(x,y)\in\beta,\ (u,\omega)\in\psi(<x,y>),\ (s,t)\in\alpha,\ (\omega,v)\in\psi(<s,t>).$$

Hence

$$(u,v)\in\psi(<s,t>)\circ\psi(<x,y>)$$

which implies that $y=s$ and then

$$(u,v)\in\psi(<x,t>).$$

Since $y=s$ also implies $(x,t) \in \alpha\circ\beta$, we have that $(u,v) \in \psi(\alpha\circ\beta)$. We have thus proved that $\psi(\alpha)\circ\psi(\beta) \subseteq \psi(\alpha\circ\beta)$.

To prove the reverse inclusion, suppose that $(u,v) \in \psi(\alpha\circ\beta)$. Then $(u,v) \in \psi(<x,t>)$ for some $(x,t) \in \alpha\circ\beta$. Thus there exists y in X such that $(x,y) \in \beta$ and $(y,t) \in \alpha$. Since

$$(u,v) \in \psi(<x,t>) = \psi(<y,t>)\circ\psi(<x,y>),$$

there exists ω in Y such that

$$(u,\omega)\in\psi(<x,y>) \quad \text{and} \quad (\omega,v)\in\psi(<y,t>).$$

Hence

$$(u,\omega)\in\psi(\beta) \quad \text{and} \quad (\omega,v)\in\psi(\alpha)$$

so that

$$(u,v)\in\psi(\alpha)\circ\psi(\beta).$$

Consequently

$$\psi(\alpha\circ\beta) \subseteq \psi(\alpha)\circ\psi(\beta)$$

and ψ is a homomorphism.

Let $\{\alpha_i \mid i\in I\}$ be any subset of M[X]. Then

$$\psi\Big(\bigcup_i \alpha_i\Big) = \bigcup_{x,y} \{\, \psi(<x,y>) \mid (x,y) \in \bigcup_i \alpha_i\} =$$

$$= \bigcup_i \bigcup_{x,y} \{\psi(<x,y>) \mid (x,y) \in \alpha_i\} = \bigcup_i \psi(\alpha_i).$$

Thus ψ is union preserving. For every α in M[X],

$$\psi(\alpha) \subseteq <E>, \quad \text{where} \quad E = \bigcup \{E_x \mid x\in X\}.$$

Since $E_x \cap F = \emptyset$ for each E_x, we have that $E \cap F = \emptyset$. Hence (2.5.1) follows from (2.5.2), (2.5.6) and (2.5.9) for $\alpha \neq 0$. Since $\psi'(0) = \zeta$ and $\psi(0)=0$, we have that (2.5.1) holds for $\alpha=0$. This completes the proof.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Combining Theorems (2.4) and (2.5), we have our main result, which is also a partial solution to the question raised by A.H. Clifford and D.D. Miller, [3] as to whether a union preserving endomorphism on β_X is necessarily symmetry preserving.

Theorem (2.6). Let X and Y be any two Magill spaces and let

$$\psi : M[X] \longrightarrow M[Y]$$

be a non-constant and union preserving homomorphism from M[X] into M[Y]. Then the following statements are equivalent:

(2.6.1) ψ is a CM-homomorphism,

(2.6.2) $\psi(0)$ and $\psi(X\times X)$ are symmetric.

REFERENCES

[1] D.I. Adu: The partially ordered family of CM-homomorphism. Semigroup Forum 13(1), (1976), 1-17.

[2] D.I. Adu: Connected CM-homomorphisms into semigroups of closed binary relations on hereditarily locally connected continua, Semigroup Forum 16, (1978), 403-425.

[3] A.H. Clifford, D.D. Miller: Union and symmetry preserving endomorphisms of the semigroup of all binary relations on a set, Czech. Math. J. 20 (1970), 303-314.

[4] A.H. Clifford, G.B. Preston: The algebraic theory of semigroups, Vol. I, Amer. Math. Soc. Math. Surveys 7(1961).

[5] K.D. Magill, Jr.: Isomorphisms of triform semigroups, J. Austr. Math. Soc. 10 (1969), 185-193.

[6] K.D. Magill, Jr., S. Yamamuro: Union and symmetry preserving homomorphisms of semigroups of closed relations, Proc. London Math. Soc. Third Series 24 (1972), 59-78.

[7] D.B. McAlister: Homomorphisms of semigroups of binary relations, Semigroup Forum 3 (1978), 175-188.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LAGOS, LAGOS, NIGERIA

Received December 13, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

0. In

Quasi
theorem f
Banach sp

1.1.
with conj
1) B is b
2) (A*g*)

1.2.
= g*(Af)

If B
x*By = li
in d(A)
In partic
(A*g*)By

1.3.
A : H →
tinuous q
ously ext
y∈(A) th
Define th
for all (
= 0 for

1.4.
[·,·], an

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Paul Andrew

QUASI-INVERSE FORMS AND EXTENDED OPERATORS

0. Introduction

Quasi-inverse forms, extended operators, a closed graph theorem for general operators and Picard's theorem: X,Y are Banach spaces.

1.1. **Definition.** A densely defined linear operator $A:X\longrightarrow Y$ with conjugate A^* is said to have a quasi-inverse form B if
1) B is bilinear form on $\overline{r(A^*)}\times\overline{r(A)}$.
2) $(A^*g^*)B(Af) = g^*(Af) = (A^*g^*)f$ for all $(g^*,f)\in d(A^*)\times d(A)$.

1.2. **Remark.** The bilinear form $(A^*g^*,Af)\longrightarrow(A^*g^*)B(Af) = g^*(Af) = (A^*g^*)f$ is well defined on $r(A^*)\times r(A)$.

If B is defined and is continuous on $\overline{r(A^*)}\times\overline{r(A)}$ then $x^*By = \lim_{n,m} g_m^*(Af_n) = \lim_{m,n}(A^*g_m^*)f_n$ for any sequences (f_n) in $d(A)$ and (g_m^*) in $d(A^*)$ with $Af_n\longrightarrow y$ and $A^*g_m^*\longrightarrow x^*$. In particular, $x^*B(Af) = x^*f$ for all $(x^*,f)\in\overline{r(A^*)}\times d(A)$ and $(A^*g^*)By = g^*y$ for all $(g^*,y)\in d(A^*)\times\overline{r(A)}$.

1.3. **Example.** Let H be a Hilbert space and suppose $A : H \longrightarrow H$ is a densely defined linear operator with a continuous quasi-inverse form B on $\overline{r(A^*)}\times\overline{r(A)}$. B can be continuously extended to a bilinear form $\overline{B}$ on $H^*\times\overline{r(A)}$, and for each $y\in\overline{r(A)}$ there is an $x\in H$ with $x^*\overline{B}y = x^*x$ for all $x^*\in H^*$. Define the linear operator $\overline{B} : \overline{r(A)} \longrightarrow H$ by $x^*\overline{B}y = x^*(\overline{B}y)$ for all $(x^*,y)\in H^*\times\overline{r(A)}$. $\overline{B}$ is continuous and, as $A^*g^*(\overline{B}Af-f) = 0$ for all $g^*\in d(A^*)$, $\overline{B}Af = f+h$ for some $h \in {}^{\perp}r(A^*) \supseteq k(A)$.

1.4. **Example.** Let H be a Hilbert space with inner product $[\cdot,\cdot]$, and suppose $T : H \longrightarrow H$ is any compact endomorphism.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For example, one such operator is $T : L_2[0,1] \longrightarrow L_2[0,1]$ defined by $Tx = \int_{[0,1]} k(s,t)x(t)dt$ for all $x \in L_2[0,1]$ when $k \in L_2([0,1]\times[0,1])$.

The adjoint operator $T' : H \longrightarrow H$ is also a compact endomorphism, so $T'T : H \longrightarrow H$ is compact and self adjoint. Clearly $k(T'T) = k(T)$.

If λ is an eigenvalue of $T'T$ with normalized eigenvector u, then $\lambda = [T'Tu,u] = \|Tu\|^2$. Let $\lambda_1,\lambda_2,\ldots$ be a list of all positive eigenvalues of $T'T$, with multiplicities included, and let $u_1,u_2,\ldots$ be the corresponding set of orthonormal eigenvectors, then the spectral theorem implies that $T'Th = \sum_{1\le n<\infty} \lambda_n[h,u_n]u_n$ for each $h \in H$ and it follows that $h = \sum_{1\le n<\infty} [h,u_n]u_n = h_0$ for each $h \in H$, where $h_0 \in k(T'T) = k(T)$. Clearly $H = \overline{r(T'T)} \oplus k(T)$.

Set $v_n = \lambda_n^{-1/2} Tu_n$ for $n \ge 1$, then $v_1,v_2,\ldots$ is an orthonormal set in H and $u_n = w_n^{-1/2} T'v_n$. Also $H = \overline{r(T)} \oplus k(T')$ where $\overline{r(T)}$ is the closure of the span of $v_1,v_2,\ldots$ in H.

Consider the linear operator $S : H \longrightarrow H$, defined by $d(S) = r(T) \oplus k(T')$ and $Sh = \sum_{1\le n<\infty} \lambda_n^{-1/2}[h,v_n]u_n$ for all $h \in d(S)$. This is well defined as $S(Th) = \sum_{1\le n<\infty} \lambda_n^{-1/2}[Th,v_n]u_n = \sum_{1\le n<\infty} [h,u_n]u_n = h$ whenever $Th \in r(T)$ with $h \in \overline{r(T'T)}$, and $S(k') = 0$ whenever $k' \in k(T')$. Note that $r(S) = \overline{r(T'T)}$.

The adjoint $S' : H \longrightarrow H$ has domain $d(S') = \{h' \in H : h'' \in H, [h',STh] = [h'',Th]$ for all $h \in \overline{r(T'T)}$ and $[h',0] = [h'',k']$ for all $k' \in k(T')\} = \{h' \in H : h'' \in \overline{r(T)}, [h',STh] = [h'',Th]$ for all $h \in \overline{r(T'T)}\} = \{h' \in H : h'' \in \overline{r(T)}, [h',h] = [h'',Th]$ for all $h \in \overline{r(T'T)}\} = T'(\overline{r(T)}) \oplus k(T)$, with $S'(T'h) = h$ for all $h \in \overline{r(T)}$ and $S'(k) = 0$ for all $k \in k(T)$. Note that $r(S') = \overline{r(T)}$.

For convenience the adjoint operator S' will be used interchangeably with the conjugate operator S^*.

Define B by $h'Bh = [h',Th] = [T'h',h]$ for all $(h',h) \in \overline{r(S')} \times \overline{r(S)} = \overline{r(T)} = \overline{r(T'T)}$. As $S'(T'h+k)Bh'' = [T'h+k,h'']$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for all $(T'h+k,h'')\in d(S')\times r(S)$ and $h'BS(Th+k') = [h',Th+k']$ for all $(h',Th+k')\in r(S')\times d(S)$, it follows that B is a continuous quasi-inverse of S.

1.5. **Theorem.** If $A : X \longrightarrow Y$ is a densely defined linear operator and if B is a bilinear form on $r(A^*)\times r(A)$ for which $(A^*g^*)B(Af) = g^*(Af) = (A^*g^*)f$, then $|x^*By| \le c\|x^*\|\cdot\|y\|$ for some constant c and all $(x^*,y)\in r(A^*)\times r(A)$ if and only if either $\sup\{|x^*f| : x^*\in r(A^*) \text{ and } \|x^*\|\le 1\} \le c\|Af\|$ for all $f\in d(A)$ or $\sup\{|g^*y| : y\in r(A) \text{ and } \|y\|\le 1\} \le c\|A^*g^*\|$ for all $g^*\in d(A^*)$.

Proof. If $|x^*By| \le c\|x^*\|\cdot\|y\|$ for all $(x^*,y)\in r(A^*)\times r(A)$, then $|(A^*g^*)f| = |(A^*g^*)B(Af)| \le c\|A^*g^*\|\cdot\|Af\|$ for all $(g^*,f)\in d(A^*)\times d(A)$, which implies that $\sup\{|x^*f| : x^*\in r(A^*) \text{ and } |x^*|\le 1\} \le c\|Af\|$. If $\sup\{|x^*f| : x^*\in r(A^*) \text{ and } |x^*|\le 1\} \le c\|Af\|$ for all $f\in d(A)$, then $|(A^*g^*)B(Af)| = |(A^*g^*)f| \le \|A^*g^*\|\cdot\sup\{|x^*f| : x^*\in r(A^*) \text{ and } |x^*|\le 1\} \le c\|A^*g^*\|\cdot\|Af\|$ for all $(g^*,f)\in d(A^*)\times d(A)$.

The remainder of the theorem is proved similarly.

2.1. **Definition.** If $A : X \longrightarrow Y$ is a densely defined linear operator, then the extended operator associated with A is the linear operator $A^{\#} : X \longrightarrow \overline{r(A)}/K$, defined by $d(A^{\#}) = \{x\in X : y\in\overline{r(A)}$ and $(A^*a^*)x = a^*u$ for all $a^*\in d(A^*)\}$ and $Ax = u+K$ whenever $x\in d(A^{\#})$ and $(A^*g^*)x = g^*y$ for all $g^*\in d(A^*)$, where $K = \{y\in\overline{r(A)} : g^*y = 0$ for all $g^*\in d(A^*)\}$.

2.2. **Remark.** As K is a closed subspace, the quotient space $\overline{r(A)}/K$ is well defined and is a Banach space, furthermore $K = \{0\}$ if and only if $d(A^*)$ is total over $\overline{r(A)}$ if and only if A is closeable (see [1], II.2.11).

If $(A^*g^*)x = g^*y$ for all $g^*\in d(A^*)$, then $y\in {}^{\perp}k(A^*) = \overline{r(A)}$, so $A^{\#}$ is well defined and $r(A^{\#})\le\overline{r(A)}/K$. Clearly $d(A^{\#})\supseteq d(A)$, $k(A^{\#}) = {}^{\perp}r(A^*)$ and $A^{\#}$ is closed.

2.3. **Theorem.** If $A : X \longrightarrow Y$ is a densely defined linear operator, then $A^{\#\#} = A^{\#}$.

Proof. As $d(A^{\#})\supseteq d(A)$ and $A^{\#}$ is closed, $A^{\#\#} : X \longrightarrow \overline{r(A^{\#})}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

is defined by $d(A^{\#\#}) = \{x \in X : y+K \in \overline{r(A^{\#})}$ and $(A^{\#*}g^{\#*})x = g^{\#*}(y+K)$ for all $g^{\#*} \in d(A^{\#*})\}$ and $A^{\#\#}x = y+K$ whenever $x \in d(A^{\#\#})$ and $(A^{\#*}g^{\#*})x = g^{\#*}(y+K)$ for all $g^{\#*} \in d(A^{\#*})$.

For each $g^* \in d(A^*)$, define $g^{\#*} \in \overline{r(A^{\#})}^*$ by $g^{\#*}(y+K) = g^*y$ for all $y+K \in \overline{r(A^{\#})}$. As $g^{\#*}A^{\#}x = g^*y = (A^*g^*)x$ whenever $A^{\#}x = y+K$, it follows that $g^{\#*} \in d(A^{\#*})$ and $[A^{\#*}g^{\#*} - A^*x^*]x = 0$ for all $x \in d(A^{\#})$, so $A^{\#*}g^{\#*} = A^*g^*$.

If $A^{\#\#}x = y+K$, then $(A^{\#*}g^{\#*})x = g^{\#*}(y+K)$ for all $g^{\#*} \in d(A^{\#*})$ and this implies that $(A^*g^*)x = g^*y$ for all $g^* \in d(A^*)$, so $d(A^{\#}) \supseteq d(A^{\#\#})$. As $d(A^{\#\#}) \supseteq d(A^{\#})$ and as $A^{\#\#}|_{d(A^{\#})} = A^{\#}$, it follows that $A^{\#\#} = A^{\#}$.

2.4. Theorem. If $A : X \longrightarrow Y$ is a linear operator with domain X and $K = \{y \in \overline{r(A)} : g^*y = 0$ for all $g^* \in d(A^*)\}$, then $A^{\#} : X \longrightarrow \overline{r(A)}/K$ is continuous.

Proof. By the closed graph theorem as $A^{\#}$ is closed.

3.1. Theorem. If $A : X \longrightarrow Y$ is a densely defined linear operator with continuous quasi-inverse form B, then $h \in d(A^{\#})$ and $A^{\#}h = g+K$ if and only if $g \in \overline{r(A)}$ and $x^*Bg = x^*h$ for all $x^* \in \overline{r(A^*)}$.

Proof. If $h \in d(A^{\#})$ and $A^{\#}h = g+K$, then $g \in \overline{r(A)}$ and $(A^*g^*)Bg = g^*g = (A^*g^*)h$ for all $g^* \in d(A^*)$ by 1.2, so $x^*Bg = x^*h$ for all $x^* \in \overline{r(A^*)}$.

If $g \in \overline{r(A)}$ and $x^*Bg = x^*h$ for all $x^* \in \overline{r(A^*)}$ then, as $(A^*g^*)h = (A^*g^*)Bg = g^*g$ for all $g^* \in d(A^*)$, it follows that $h \in d(A^{\#})$ and $A^{\#}h = g+K$.

3.2. Corollary. $K = \{g \in \overline{r(A)} : x^*Bg = 0$ for all $x^* \in \overline{r(A^*)}\}$.

Proof. As $k(A^{\#}) = {}^{\perp}r(A^*)$ by 2.2.

3.3. Example. Continuing example 1.4, S' is densely defined, so 2.2 implies that $K = \{0\}$ and $S^{\#} : H \longrightarrow \overline{r(S)}$ is defined by $d(S^{\#}) = \{h \in H : g \in \overline{r(S)}$ and $[S'h', h] = [h', g]$ for all $h' \in d(S')\}$ and $S^{\#}h = g$.

It follows from 3.1 that $h \in d(S^{\#})$ and $S^{\#}h = g$ if and only if $g \in \overline{r(S)} = \overline{r(T'T)}$ and $h'Bg = [h', Tg] = h'h$ for all $h' \in \overline{r(S')} = \overline{r(T)}$. If $Tg = h$ is soluble then, as $H = \overline{r(T'T)}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\in k(T)$, it can be assumed that $g \in \overline{r(T'T)}$ and it follows that $h \in d(S^{\#})$. If $h \in \overline{r(T)}$ and $h \in d(S^{\#})$, then setting $S^{\#}h = g$, it follows that $[h', Tg-h] = 0$ for all $h' \in \overline{r(T)}$ which implies that $Tg = h$. So $Tg = h$ is soluble if and only if $h \in \overline{r(T)} \cap d(S^{\#})$.

As $\bar{h} = \sum_{1 \le n < \infty} [\bar{h}, v_n] v_n$ for all $\bar{h} \in \overline{r(T)}$, $[S'h', h] = \sum_{1 \le n < \infty} \lambda^{-1/2} [h', u_n][v_n, h]$ for all $(h', h) \in d(S') \times H$ and it follows that $d(S^{\#}) = \{h \in H : g \in \overline{r(S)} = \overline{r(T'T)}$ and $\sum_{1 \le n < \infty} \lambda_n^{-1/2} [h', u_n][v_n, h] = [h'; g]$ for all $h' \in d(S')\}$.

Clearly $d(S^{\#}) \supseteq \{h \in H : \sum_{1 \le n < \infty} \lambda_n^{-1} |[v_n, h]|^2 < \infty\}$.

If $h \in d(S^{\#})$ and $S^{\#}h = g$, then $[h', Tg] = h'Bg = [h', h]$ for all $h' \in \overline{r(S')} = \overline{r(T)}$ and

$$\sum_{1 \le n < N} \lambda_n^{-1} |[v_n, h]|^2 = \sum_{1 \le n < N} \lambda_n^{-1} |[v_n, Tg]|^2 =$$

$$= \sum_{1 \le n < N} \lambda_n^{-1} |[T'v_n, g]|^2 = \sum_{1 \le n < N} |[u_n, g]|^2 \le \|g\|^2,$$

so $d(S^{\#}) = \{h \in H : \sum_{1 \le n < \infty} \lambda_n^{-1} |[v_n, h]|^2 < \infty\}$.

Picard's theorem on the existence of solutions of integral equations of the first kind follows, namely $Tg = h$ is soluble if and only if $h \in \overline{r(T)}$ and $\sum_{1 \le n < \infty} \lambda_n^{-1} |[v_n, h]|^2 < \infty$.

3.4. Theorem. If X is reflexive and $A : X \longrightarrow Y$ is a densely defined linear operator with continuous quasi-inverse form B, then $A^{\#}h = g+K$ has a solution if and only if $|g^*g| \le c\|A^*g^*\|$ for all $g^* \in d(A^*)$, and $\|h\| \le c$.

Proof. Suppose $A^{\#}h = g+K$, then $(A^*g^*)h = g^*g$ for all $g^* \in d(A^*)$, so $|g^*g| \le \|A^*g^*\| \cdot \|h\|$ for all $g^* \in d(A^*)$.

Suppose $|g^*g| \le c\|A^*g^*\|$ for all $g^* \in d(A^*)$, then $g \in {}^{\perp}k(A^*) = \overline{r(A)}$ and $|(A^*g^*)Bg| = |g^*g| \le c\|A^*g^*\|$ for all $A^*g^* \in r(A^*)$. As Bg is continuous on $\overline{r(A^*)}$ and X is reflexive, $x^*Bg = x^*h$ for some $h \in X$ with $\|h\| \le c$ for all $x^* \in \overline{r(A^*)}$, and the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

theorem follows from 3.1.

3.5. Corollary. If X is reflexive, $A : X \to Y$ is a densely defined operator for which $\sup\{|g^*y| : y \in r(A) \text{ and } \|y\| \le 1\} \le c\|A^*g^*\|$ for all $g^* \in d(A^*)$, then $A^{\#}h = g+K$ has a solution for each $g \in \overline{r(A)}$.

Proof. A has a continuous quasi-inverse form B by 1.5, 1.2 and 1.1, so the corollary follows from 3.4.

REFERENCES

[1] **S. Goldberg:** Unbounded linear operators. Dover 1966.

[2] **C.W. Groetsch:** Elements of applicable functional analysis. Dekker, 1980.

Mailing address:
10 Grace St., Goodwood 5034, AUSTRALIA

Received January 27, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

G.P. Kapoor, O.P. Juneja, J. Patel

ANALYTIC FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where defining Taylor series has radius of convergence $R \geq 1$. For a non-decreasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers, the Gelfond-Leontev derivative (GLD) Df of f is defined as [7]

$$Df(z) = \sum_{n=1}^{\infty} d_n a_n z^{n-1}. \tag{1.1}$$

The higher order GLD $D^p f$, $p = 2,3,\ldots$, of f are given by

$$D^p f(z) = \sum_{n=1}^{\infty} d_n \ldots d_{n-p+1}\, a_n z^{n-p}. \tag{1.2}$$

Wherever needed, for uniformity in notations, we shall write $D^1 f$ for Df.

It is readily seen that if $d_n \equiv n$ for $n = 1,2,\ldots$, then GLD corresponds to the ordinary differentiation of f; whereas, if $d_n \equiv 1$, $n = 1,2,\ldots$, then GLD is the shift operator L which transforms $f(z) = \sum_{n=0}^{\infty} a_n z^n$ into $Lf(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$, GLD are used by Kazmin [9] for investigating interpolation problems. Buckholtz and Frank [3], [4] seem to have started the use of these derivatives in connection with the study of problems in univalent function theory. The work in this direction is further extended by Juneja and Shah [10] who found relationships of radius of convergence of the Taylor

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

series of f with radii of univalence or radii of convexity of GLD's $D^p f$, $p = 1,2,\ldots$. Thus, if ρ_p and $\rho_p(c)$, $p = 1,2,\ldots$, are respectively the radii of univalence and radii of convexity of $D^p f$, i.e. if ρ_p and $\rho_p(c)$ are largest numbers such that $D^p f(\rho_p z)$ and $d^p f(\rho_p(c) z)$ are analytic and respectively univalent and convex in $\Delta = \{z : |z| < 1\}$, then

$$\text{(1.3)} \qquad \liminf_{p \to \infty} d_p \rho_p \leq 2 d_2 R$$

and

$$\text{(1.4)} \qquad \liminf_{p \to \infty} d_p \rho_p(c) \leq d_2 R.$$

Let H_R be the class of Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ having radius of convergence $R \geq 1$. For a strictly increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers and strictly increasing sequence $\{d_n\}_{n=1}^{\infty}$ of positive numbers, we set

$$\text{(1.5)} \qquad D^{n_p} f(z) = \sum_{k=0}^{\infty} \left(d_{n_p+k} \cdots d_{k+1}\right) a_{n_p+k} z^k =$$

$$= \left(d_{n_p} \cdots d_1\right) a_{n_p} + \left(d_{n_p+1} \cdots d_2\right) a_{n_p+1} z + \cdots$$

and define the classes of functions

$$E(\{n_p\};R) = \{f \in H_R : D^{n_p} f \text{ is analytic and univalent in } \Delta, \text{ for } p = 1,2,\ldots\},$$

$$E_c(\{n_p\};R) = \{f \in H_R : D^{n_p} f \text{ is analytic, univalent and convex in } \Delta \text{ for } p=1,2,\ldots\}.$$

Obviously, $E_c(\{n_p\};) \subset E(\{n_p\};R)$.

If $f \in E(\{p\};R)$ or $E_c(\{p\};R)$, then $\rho_p \equiv 1$ or $\rho_p(c) \equiv 1$, respectively. In these cases, (1.3) or (1.4) clearly imply that $R = \infty$; i.e. f is an entire function. However, if infinitely many ρ_p or $\rho_p(c)$ are zero, then (1.3) or (1.4) does not give any non-trivial information about the radius of convergence of f. This inadequacy in relations (1.3) and (1.4) can be overcome if instead of assuming f to be in the class $E(\{p\};R)$ or $E_c(\{p\};R)$, we consider the function f to be in the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

class $E(\{n_p\};R)$ or $E_c(\{n_p\};R)$, for some strictly increasing sub-sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers and find the relations of radius of convergence of the Taylor series of f with radii of univalence or radii of convexity of its GLD's $D^{n_p}f$. In the present paper we make an endeavor in this direction. The corollaries of our results give sufficient conditions on the sequence $\{n_p\}_{p=1}^{\infty}$ that force the function f to be entire.

To simplify the notations, we shall frequently write $a(n_p)$ for a_{n_p}, $d(n_p)$ for d_{n_p}, $E(n_p;R)$ for $E(\{n_p\};R)$ and $E_c(n_p;R)$ for $E_c(\{n_p\};R)$. We shall also assume throughout in the sequel that $d_1 \geq 1$ and $d_p \longrightarrow \infty$ as $p \longrightarrow \infty$.

Theorem 1. Let f be in $E(n_p;R)$ and the sequence $\{d_n\}_{n=1}^{\infty}$ in (1.5) satisfy

$$\frac{d(n_p+k)}{d(k)} \geq \frac{k}{k-1}, \quad k = 2,3,\ldots; \quad p \geq p_0. \tag{1.6}$$

Then,

$$\liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}} \leq \tag{1.7}$$

$$\leq R \limsup_{p \to \infty} (2d_2)^{p/n_p} \leq 2d_2R.$$

Corollary 1. Let f be in $E(n_p;R)$ and the sequence $\{d_n\}_{n=1}^{\infty}$ satisfy (1.6). If $n_{p+1} - n_p = o(\log d(n_p))$, then f is entire.

For $f \in E_c(n_p;R)$, our next theorem shows that a better lower bound than in (1.7) on the radius of convergence R of f is obtained even without using the condition (1.6).

Theorem 2. Let f be in $E_c(n_p;R)$. Then,

$$\liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots(dn_p)\}^{1/n_{p+1}} \leq \tag{1.8}$$

$$\leq R \limsup_{p \to \infty} d_2^{p/n_p} \leq d_2R.$$

Corollary 2. Let f be in $E_c(n_p;R)$. If $n_{p+1} - n_p =$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$= o(\log d(n_p))$, then f is entire.

Remarks: 1. We note that, besides many other choices of the sequence $\{d_n\}_{n=1}^{\infty}$, (1.6) holds if $d_n = n^{\alpha}$, $\alpha \geq 1$, $n = 1,2,\ldots$. In general, (1.6) is satisfied if $d(k+2)/d(k) \geq k/(k-1)$, $k = 2,3,\ldots$. This inequality is easier to check that the inequality (1.6) and it also generates a number of simple admissible sequences $\{d_n\}_{n=1}^{\infty}$ for Theorem 1.

2. Let $\lim_{n\to\infty} d_n^{1/n} = a$, $0 < a < \infty$. Then, the inequality (1.7) and (1.8) can be written as

$$\text{(1.9)} \qquad \liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots(dn_p)\}^{1/n_p} \leq \leq a R \limsup_{p\to\infty} (2d_2)^{p/n_p} \leq 2ad_2R,$$

$$\text{(1.10)} \qquad \liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots(dn_p)\}^{1/n_p} \leq \leq a R \limsup_{p\to\infty} d_2^{p/n_p} \leq ad_2R.$$

From (1.9), a result of Shah and Trimble [12, Theorem 3] follows by taking $d_n = n$, $n = 1,2,\ldots$. Similarly, for $d_n = n$, $n = 1,2,\ldots$, (1.10) gives the result of Campbell [5, Theorem 10].

If $f \in E(n_p;R)$, d_n's in (1.5) satisfy (1.6) and $\limsup_{p\to\infty} (n_{p+1} - n_p) < \infty$, then it follows from Corollary 1 that f is entire. Similarly, if $\limsup_{p\to\infty} (n_{p+1} - n_p) < \infty$ and $f \in E_C(n_p;R)$, Corollary 2 gives f is entire. The following theorems are in the other direction.

Theorem 3. Let f be in $E(n_p;R)$ and the sequence $\{d_n\}_{n=1}^{\infty}$ in (1.5) satisfy (1.6). If there is a positive integer $M > 1$ such that $\liminf_{p\to\infty} (n_{p+1} - n_p) \geq M$, then

$$\text{(1.11)} \qquad \liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{(M-1)/n_{p+1}} \geq R.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Corollary 3. Let f be in $E(n_p;R)$ and the sequence $\{d_n\}_{n=1}^{\infty}$ satisfy (1.6). If $\lim_{p\to\infty}(n_{p+1} - n_p) = \infty$ and

$$\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{1/n_{p+1}} > 1 \tag{1.12}$$

then, f is entire.

If $f \in E_c(n_p;R)$, the following theorem shows that the condition (1.6) is not needed to deduce (1.11).

Theorem 4. Let f be in $E_c(n_p;R)$. If there is a positive integer $M > 1$ such that $\liminf_{p\to\infty}(n_{p+1} - n_p) \geq M$, then (1.11) holds.

Corollary 4. Let f be in $E_c(n_p;R)$. If $\lim_{p\to\infty}(n_{p+1} - n_p) = \infty$ and (1.12) holds, then f is entire.

Remarks: 1. Example 1 given at the end of Section 3 shows that in Corollary 3, the conditions (1.6) and (1.12) are independent.

2. Example 2 given at the end of Section 3 shows that the conditions on the sequences $\{d_n\}_{n=1}^{\infty}$ and $\{n_p\}_{p=1}^{\infty}$ in Corollary 1 (respectively Corollary 2) and Corollary 3 (respectively Corollary 4) are independent.

3. If $\lim_{n\to\infty} d_n^{1/n} = a$, $0 < a < \infty$, then the inequality (1.11) can be written as

$$\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{(M-1)/n_p} \leq a^{2(M-1)}R. \tag{1.13}$$

For $f \in E(n_p;R)$ and $d_n = n$, $n = 1,2,\ldots$, a result of Shah and Trimble [12, Theorem 4] follows from Theorem 3 and (1.13). Moreover, for $f \in E_c(n_p;R)$ Theorem 4, with (1.11) replaced by (1.13), gives a new result even for $d_n = n$, $n = 1,2,\ldots$.

A function $S(x)$, continuous on $[B,\infty)$ for some positive number B, is said to be slowly oscillating (S.O.) if for every positive constant $C > 0$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\lim_{x\to\infty} \frac{S(Cx)}{S(x)} = 1.$$

A function $\sigma(n)$ is said to be the restriction of a S.O. function $S(x)$ if $\sigma(n) = S(n)$ for every positive integer n. It is known [9] that, as $k \longrightarrow \infty$

$$(1.14) \qquad \sum_{i=1}^{k} \sigma(i) \sim k\sigma(k).$$

In the following theorem we find that by restricting the growth of the sequence $\{(n_p - n_{p-1})\}_{p=2}^{\infty}$ by S.O. functions, a function f in $E(n_p;R)$ can be forced to be an entire function of specific growth rate.

Theorem 5. Let f be in $E(n_p;R)$. Let $d_n's$ in (1.5) be the restriction of a continuously differentiable function $d(x)$ on positive integers such that $xd'(x)/d(x)$ is a non-increasing function of x, where $d'(x)$ denotes the derivative of $d(x)$ with respect to x. Suppose φ and θ are S.O. functions such that $\theta(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ and, for $p = 1,2,\ldots,$

$$(1.15) \qquad 1 \le \varphi(p) \le n_p - n_{p-1} \le \theta(p).$$

If,

$$(1.16) \qquad \alpha \equiv \liminf_{p\to\infty} \frac{\log d(n_p)}{\log n_p},$$

$$(1.17) \qquad \beta \equiv \limsup_{p\to\infty} \frac{\theta(p)\log d[\theta(p)]}{\varphi(p)\log p},$$

and $\alpha > \beta$, then f is an entire function of order at most $1/(\alpha-\beta)$. In (1.17), $[\theta(p)]$ denote the greatest integer not greater than $\theta(p)$.

2. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. Since $f \in E(n_p;R)$, we have that GLD's $D^{n_p}f$ are univalent in Δ. In view of Bieberbach-Branges theorem [2] this implies that for $p = 1,2,\ldots,$ and $k = 2,3,\ldots,$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$|a(n_p+k)| \le \frac{kd(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+k)\ldots d(k+1)} . \tag{2.1}$$

Due to (1.6), the function $k/(d(n_p+k)\ldots d(k+1))$ decreases with k for every $p \ge p_0$. Therefore, by (2.1), for $k = 2,3,..$ and $p \ge p_0$

$$|a(n_p+k)| \le \frac{2d(n_p+1)\ldots d(2)|a(n_p+1)|}{d(n_p+2)\ldots d(3)} < \frac{2d(2)|a(n_p+1)|}{d(n_p)} . \tag{2.2}$$

Using induction on p in (2.2), we get for $p > p_0$ and $1 \le k \le n_{p+1} - n_p + 1$,

$$|a(n_p+k)| < \frac{(2d(2))^{p-p_0+1}|a(n_{p_0}+1)|}{d(n_{p_0})\ldots d(n_p)} = \frac{A(2d(2))^p}{d(n_1)d(n_2)\ldots d(n_p)}$$

where A is a absolute constant. From the above inequality, we get

$$\limsup_{k\to\infty} |a_k|^{1/k} =$$

$$= \limsup_{k\to\infty} \{|a(n_p+k)|^{1/(n_p+k)} : 2 \le k \le n_{p+1} - n_p + 1;\ p > p_0\} \le$$

$$\le \frac{\limsup_{p\to\infty} (2d_2)^{p/n_p}}{\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}} .$$

Since, $\limsup_{k\to\infty} |a_k|^{1/k} = \frac{1}{R}$, the first inequality in (1.7) follows. The second inequality in (1.7) is obvious.

Proof of Corollary 2. Since,

$$\liminf_{p\to\infty} \frac{\log d(n_p)}{(n_{p+1} - n)} \le \liminf_{p\to\infty} \frac{\sum_{i=1}^{n_p} \log d(i)}{n_{p+1}} \tag{2.3}$$

it follows from the condition $n_{p+1} - n_p = o(\log d(n_p))$ and Theorem 1 that $R = \infty$. Thus, f is an entire function.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof of Theorem 2. Since $f \in E_c(n_p;R)$, GLD's $D^{n_p}f$ are convex in Δ. Therefore [6], we get from (1.5) that for $p = 1,2,\ldots,$ and $k = 2,3,\ldots$

$$|a(n_p+k)| \leq \frac{d(n_p+1)\ldots d(2)\,|a(n_p+1)|}{d(n_p+k)\ldots d(k+1)} . \tag{2.4}$$

But, the function $1/(d(n_p+k)\ldots d(k+1))$ decreases with k for every p, so that (2.4) gives for $p=1,2,\ldots,$ and $k=2,3,\ldots$.

$$|a(n_p+k)| \leq \frac{d(n_p+1)\ldots d(2)\,|a(n_p+1)|}{d(n_p+2)\ldots d(3)} < \frac{d(2)\,|a(n_p+1)|}{d(n_p)} .$$

Now, following the lines of proof of Theorem 1, we get

$$\frac{1}{R} \leq \frac{\limsup_{p\to\infty} d_2^{p/n_p}}{\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_p)\}^{1/n_{p+1}}} .$$

This proves the first inequality in (1.8). The second inequality in (1.8) is obvious.

Proof of Corollary 2. Since $n_{p+1} - n_p = o(\log d(n_p))$, Corollary 2 follows easily from Theorem 2 and (2.3).

3. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. From (2.1), we get for $p = 1,2,\ldots,$ and $k = 2,3,\ldots,$

$$|a(n_p+k)| \leq \frac{kd(n_p+M)\ldots d(2)\,|a(n_p+1)|}{(d(n_p+M)\ldots d(n_p+2))\,(d(n_p+k)\ldots d(k+1))} . \tag{3.1}$$

Since, by (1.6), $k/d(n_p+k)\ldots d(k+1)$ decreases with k for all $p \geq p_0$, for $k \geq M+1$, there is a positive integer Q_1 such that for $p \geq Q_1$,

$$\frac{kd(n_p+M)\ldots d(2)}{d(n_p+k)\ldots d(k+1)} < 1.$$

Thus, for $p \geq \max\{p_0,Q_1\}$ and $k \geq M+1$, (3.1) implies that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$|a(n_p+k)| \leq \frac{|a(n_p+1)|}{d(n_p+2)\ldots d(n_p+M)} < \frac{|a(n_p+1)|}{d(n_p)^{M-1}} . \tag{3.2}$$

Also, there is a positive integer Q_2 such that, for $p \geq Q_2$ and $k \geq 2$

$$\frac{kd(n_p+1)\ldots d(2)}{d(n_p+k)\ldots d(k+1)} < 1.$$

Using (2.1) again, it follows that for $p \geq Q_2$ and $k \geq 2$,

$$|a(n_p+k)| < |a(n_p+1)|. \tag{3.3}$$

Since $\liminf_{p \to \infty} (n_{p+1} - n_p) \geq M$, there is a positive integer Q_3 such that for $p \geq Q_3$, $(n_{p+1} - n_p) \geq M$. Let $Q = \max\{p_0, Q_1, Q_2, Q_3\}$ and suppose $p \geq Q$. Then, $(n_{p+1} - n_p + 1) \geq M+1$ and hence (3.2) implies that for $p \geq Q$,

$$|a(n_{p+1}+1)| < \frac{|a(n_p+1)|}{d(n_p)^{M-1}} .$$

An inductive argument on p shows that for $p > Q$

$$|a(n_p+1)| < \frac{|a(n_Q+1)|}{\{d(n_Q)\ldots d(n_{p-1})\}^{(M-1)}} = \frac{C_1}{\{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{M-1}}$$

where C_1 is an absolute constant. If $2 \leq k \leq n_{p+1} - n_p + 1$, then using the above inequality (3.3), gives for $p > Q$,

$$|a(n_p+k)| < \frac{C_1}{\{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{M-1}} .$$

Thus,

$$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k} =$$

$$= \limsup_{k \to \infty} \{|a(n_p+k)|^{1/(n_p+k)} : 2 \leq k \leq n_{p+1} - n_p + 1;\ p > 0\} \leq$$

$$\leq \frac{1}{\liminf_{p \to \infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{(M-1)/n_{p+1}}} .$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

This proves (1.11),

Proof of Corollary 3: In view of $\lim_{n\to\infty}(n_{p+1}-n_p)=\infty$, it follows obviously from (1.11) that $R=\infty$; i.e., f is entire.

Proof of Theorem 4: From (2.4), we get

$$|a(n_p+k)| \le \frac{d(n_p+M)\ldots d(2)\,|a(n_p+1)|}{(d(n_p+M)\ldots d(n_p+2))(d(n_p+k)\ldots d(k+1))}.$$

The proof of this theorem is easily constructed by noting that $1/(d(n_p+k)\ldots d(k+1))$ is a decreasing function of k for all p, and by following the lines of proof of Theorem 3.

The proof of Corollary 4 is obvious and we omit it.

Example 1. We note that conditions (1.6) and (1.12) are independent. Let $n_p = 1+[p \log\log (p+c)]$, $c=10^{10}$, $p=1,2,\ldots$, and $d_n = \log(n+1)$, $n=1,2,\ldots$. It is easily seen that $\lim_{p\to\infty}(n_{p+1}-n_p)=\infty$ and

$$\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{1/n_{p+1}} \ge 0 > 1,$$

but (1.6) is not satisfied. Thus, (1.12) does not imply (1.6).

To see that (1.6) does not imply (1.12), we may take $n_p = [e^p]$, $p=1,2,\ldots$, and $d_n = n$, $n=1,2,\ldots$, so that (1.6) is clearly satisfied and $\lim_{p\to\infty}(n_{p+1}-n_p)=\infty$. But, in this case,

$$\liminf_{p\to\infty} \{d(n_1)d(n_2)\ldots d(n_{p-1})\}^{1/n_{p+1}} = 1.$$

Example 2. The conditions on the sequence $\{d_n\}_{n=1}^{\infty}$ and the sequence $\{n_p\}_{p=1}^{\infty}$ in Corollary 1 and Corollary 3 are independent. First we show that the condition in Corollary 1 is not followed by the condition in Corollary 3. Let $n_p = 1 + [p \log p]$, and $d_n = n^{\alpha}$, $\alpha \ge 1$, $n=1,2,\ldots$. Clearly, the sequence $\{d_n\}$ satisfies (1.6) for some $p > p_1$. Further, $(n_{p+1}-n_p) > \frac{1}{2}\log p$ for large p; so that $\lim_{p\to\infty}(n_{p+1}-n_p)=\infty$. However, $(n_{p+1}-n_p) \ne o(\log d(n_p))$.

Conversely, the assertion and conditions in Corollary 1 does not imply conditions in Corollary 3 can be seen by taking any sequence $\{d_n\}_{n=1}^{\infty}$ that satisfies (1.6) and (1.12) and by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

choosing the sequence $\{n_p\}_{p=1}^{\infty}$ such that $\limsup_{p \to \infty}(n_{p+1} - n_p) = \infty$.

The above choices of the sequence $\{d_n\}$ and $\{n_p\}$ also show that the conditions in Corollary 2 and Corollary 4 are independent. Here, even the choice $d_n = n^{\alpha}$, $\alpha > 0$ is admissible, since (1.6) is not needed in either of these Corollaries.

4. Proof of Theorem 5

We need the following lemmas:

Lemma 1. [13]. Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of non-negative integers. Then, for each $p \geq 2$,

$$\prod_{i=2}^{p} (n_i - n_{i-1} + 1)^{1/(n_p+2)} \leq 2. \tag{4.1}$$

Lemma 2. Let f be in $E(n_p;R)$ and d_n's in (1.5) be the restriction of a continuously differentiable function $d(x)$ on positive integers such that $xd'(x)/d(x)$ is a non-increasing function of x. Suppose θ is a real-valued function defined on the set of positive integers such that $\theta(p) \longrightarrow \infty$ as $p \longrightarrow \infty$. If $(n_{p+1} - n_p) \leq \theta(p)$ and $1 \leq k \leq n_{p+1} - n_p$, for $p \geq 2$ then, for sufficiently large p,

$$\frac{\log |a(n_p+k)|}{(n_p+k)} < \log n_{p+1}\left\{o(1) - \frac{\log d(n_{p+1})}{\log n_{p+1}} + \frac{\theta(p+1)\log d[\theta(p+1)]}{n_{p+1} \log n_{p+1}} + \frac{1}{n_p \log n_p} \sum_{i=2}^{p} \theta(i) \log d[\theta(i)+1]\right\}. \tag{4.2}$$

Proof. The inequality (2.1), with $k = n_{p+1} - n_p + 1$ gives

$$|a(n_{p+1}+1)| < \frac{(n_{p+1}-n_p+1)(d(n_{p+1}-n_p+1)\ldots d(2))(d(n_p+1)\ldots d(2))|a(n_p+1)|}{d(n_{p+1}+1)\ldots d(2)}.$$

An inductive argument on p in the above inequality shows that for $p \geq 2$

$$|a(n_{p+1}+1)| < \tag{4.3}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$< \frac{d(n_1+1)\ldots d(2)\,|a(n_1+1)|}{d(n_p+1)\ldots d(2)} \prod_{i=2}^{p} (n_i-n_{i-1}+1)\,d(n_i-n_{i-1}+1)\ldots,d(2).$$

Using (4.3) in (2.1), we get for $p \geq 2$ and $1 \leq k \leq n_{p+1}-n_p$,

$$|a(n_p+k)| < \frac{A_1 k\, d(k)\ldots d(2)}{d(n_p+k)\ldots d(2)} \prod_{i=2}^{p} (n_i-n_{i-1}+1)\,d(n_i-n_{i-1}+1)\ldots,d(2)$$

where $A_1 = d(n_1+1)\ldots d(2)\ |a(n_1+1)$, is a constant.

Thus, for sufficiently large p,

$$(4.4)\quad \log|a(n_p+k)| < O(1) + \log k + \sum_{i=2}^{k} \log d(i) +$$

$$+ \sum_{i=2}^{p} \log(n_i-n_{i-1}+1) + \sum_{i=2}^{p} \log(d(n_i-n_{i-1}+1)\ldots d(2) -$$

$$- \sum_{i=2}^{n_p+k} \log d(i).$$

Since $xd'(x)/d(x)$ is a non-decreasing function of x, $xd'(x)/d(x)$ tends to λ, $0 \leq \lambda < \infty$, as $x \longrightarrow \infty$. This implies [11, pp. 7] that, $x^{-\lambda}d(x)$ is a slowly oscillating function. Now, applying the definition of a S.O. function, it follows that $\log d(x)$ is a S.O. function. Therefore, by (1.14),

$$\sum_{i=2}^{n_p+k} \log d(i) \sim (n_p+k) \log d(n_p+k).$$

Using this asymptotic relation, (4.4) becomes, for sufficiently large p,

$$(4.5)\quad \frac{\log|a(n_p+k)|}{(n_p+k)} < O(1) +$$

$$\frac{k \log d(k)}{(n_p+k)} - \log d(n_p+k) + \frac{1}{n_p+2} \sum_{i=2}^{p} \log(n_i-n_{i-1}+1) +$$

$$+ \frac{1}{n_p} \sum_{i=2}^{p} (n_i-n_{i-1}) \log d(n_i-n_{i-1}+1)\ .$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now consider the function

$$g(x) = \frac{x \log d(x)}{(n_p+x)} - \log d(n_p+x).$$

Since $xd'(x)/d(x)$ is a non-increasing function of x,

$$g'(x) = \frac{n_p \log d(x) + (n_p+x)\left\{x \frac{d'(x)}{d(x)} - (n_p+x) \frac{d'(n_p+x)}{d(n_p+x)}\right\}}{(n_p+x)^2} \geq 0,$$

therefore, g(x) is a non-creasing function of x, so that, for $1 \leq k \leq n_{p+1} - n_p$

$$\frac{k \log d(k)}{(n_p+k)} - \log d(n_p+k) \leq \tag{4.6}$$

$$\leq \frac{(n_{p+1}-n_p)\log d(n_{p+1}-n_p)}{n_{p+1}} - \log d(n_{p+1}).$$

The estimate (4.5) of $\log |a(n_p+k)|/(n_p+k)$ together with (4.1) and (4.6) now becomes

$$\frac{\log|a(n_p+k)|}{(n_p+k)} < \log n_{p+1} \left\{o(1) - \frac{\log d(n_{p+1})}{\log n_{p+1}} + \right.$$

$$\left. + \frac{\theta(p+1)\log d[\theta(p+1)]}{n_{p+1} \log n_{p+1}} + \frac{1}{n_p \log n_p} \sum_{i=2}^{p} \theta(i) \log d[\theta(i)+1]\right\}.$$

This proves (4.2).

Proof of Theorem 5. Using (1.15), we have $n_p \geq \sum_{i=2}^{p} \varphi(i)+n_1$. Since φ is S.O., (1.14) gives as $p \longrightarrow \infty$,

$$\sum_{i=2}^{p} \varphi(i) \sim p\, \varphi(p).$$

Thus, as $p \longrightarrow \infty$,

$$n_p \log n_p \geq 1 + o(1)\; p\, \varphi(p) \log p. \tag{4.7}$$

The definition of α in (1.16) and (4.7) give, for sufficiently large p,

$$\frac{\theta(p+1) \log d\,[\theta(p+1)]}{n_{p+1} \log n_{p+1}} = o(1). \tag{4.8}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

As in the proof of Lemma 2, the hypothesis $xd'(x)/d(x)$ is a non-increasing function of x implies that $\log d(x)$ is an S.O. function. Since $\theta(x)$ is S.O. and $\theta(x) \longrightarrow$ as $x \longrightarrow \infty$, the function $\log d[\theta(p)]$ is a restriction of an S.O. function on positive integers. Therefore, $\sum_{i=2}^{p} \theta(i) \log d[\theta(i)+1]$ is a restriction of an S.O. function on positive integers. Consequently, by (1.14), as $p \longrightarrow \infty$,

$$\sum_{i=2}^{p} \theta(i) \log d[\theta(i)+1] \sim p\,\theta(p) \log d[\theta(p)+1] .$$

$$\sim p\,\theta(p) \log d[\theta(p)] .$$

Using this asymptotic relation, (4.7) and (4.8), we get, for sufficiently large p,

$$\frac{1}{n_p \log n_p} \sum_{i=2}^{p} \theta(i) \log d[\theta(i)+1] + \frac{\theta(p+1) \log d[\theta(i)+1]}{n_{p+1} \log n_{p+1}} \leq$$

$$\leq \frac{(1+o(1))\; p\,\theta(p) \log d[\theta(p)]}{p\,\varphi(p) \log p} + o(1) \leq \frac{\theta(p) \log d[\theta(p)]}{\varphi(p) \log p} + o(1).$$

Lemma 2, together with the above estimate shows that, if $p \geq 2$ and $2 \leq k \leq n_{p+1} - n_p + 1$,

$$(4.9) \qquad \frac{\log |a(n_p+k)|}{(n_p+k)} <$$

$$< \log n_{p+1} \left\{ o(1) - \frac{\log d(n_p)}{\log n_p} + \frac{\theta(p) \log d[\theta(p)]}{\varphi(p) \log p} \right\}.$$

Since $\alpha > \beta$, we get from the above inequality that $\limsup_{k \to \infty} (\log|a_k|/k) = -\infty$ and therefore, f is an entire function.

Further, if ρ is the order of f, then it is known [1, p. 11] that

$$\rho = \limsup_{k \to \infty} \frac{k \log k}{-\log |a_k|} .$$

The above coefficient characterization of ρ and (4.9) easily give $\rho \leq 1/(\alpha-\beta)$. This completes the proof of Theorem 5.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] R.P. Boas: Entire Functions, Academic Press, New York and London (1954).

[2] L. De. Branges: A proof of Biberbach conjecture, Acta Math. 154(1985), 137-154.

[3] J.D. Buckholtz, J.L. Frank: Whittaker constants, Proc. London Math. Soc. (3) 23 (1971), 248-270.

[4] J.D. Buckholtz, J.L. Frank: Whittaker constants II, J.Approx. Theory 10 (1974), 112-122.

[5] D.M. Campbell: Locally univalent functions with locally univalent derivatives, Trans. Amer. Math. Soc. 162 (1971), 395-409.

[6] P.L. Duren: Univalent Functions. Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, (1983).

[7] A.O. Gelfond, A.F. Leontev: On a generalization of Fourier series, Math. Sb. (N.S.) 29(71), (1951), 477-500.

[8] G.H. Hardy, W. Rogosinski: Notes on Fourier series III, Asymptotic formulae for the sums of certain trigonometric series, Quart. J. Math., Oxford Series, 16 (1945), 49-58.

[9] Yu.A. Kazmin: On an interpolation problem, I, Sibirsk. Mat. Z. (1967), 293-312.

[10] O.P. Juneja, S.M. Shah: Univalent functions with univalent Gelfond-Leontev derivatives, Bull. Austr. Math. Soc. 29(1984), 329-348.

[11] E. Seneta: Regularly varying functions, Vol. 508, Springer-Verlag, Berlin-Heidelberg-New York (1976).

[12] S.M. Shah, S.Y. Trimble: Univalent functions with univalent derivatives III, J. Math. Mech. 19(1969), 451-460.

[13] S.M. Shah, S.Y. Trimble: Entire function with some derivatives univalent, Can. J. Math. 16(1974), 207-213.

G.P. Kapoor and O.P. Juneja: DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, KANPUR-208016, INDIA;
J. Patel: DEPARTMENT OF MATHEMATICS, UTKAL UNIVERSITY BHUBNESWAR (ORISSA), INDIA.

Received February 10, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

SOME

1.

Let
f of th

in K an

The
$|a_k|$ =

Thi
Cartan

Let
let f
expansi

$$= \sum_{m=0}^{\infty} P_m$$

forms D
what ty
equalit

Thi
the poly

$\ldots, p_{k_n}^{(n)}$
coordina

It
polycyl

$\ldots, z_n)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Eugeniusz Janiec

SOME UNIQUENESS THEOREMS CONCERNING HOLOMORPHIC MAPPINGS

1. Introduction

Let $K = \{z \in C: |z| < 1\}$. It is known that, if a function f of the form $f(z) = a_0 + a_1 z + \ldots$, $z \in K$, is holomorphic in K and $f : K \longrightarrow K$, then $\sum_{n=0}^{\infty} |a_n|^2 \leq 1$.

Theorem A. If the above assumptions are satisfied and $|a_k| = 1$ for some $k \in N \cup \{0\}$, then $f(z) \equiv a_k z^k$.

This theorem and the well-known uniqueness theorem of Cartan suggest the following problem.

Let D and G be arbitrary domains in C^n, $n \geq 1$, $0 \in D$ and let $f : D \longrightarrow G$ be holomorphic. Assume that the Taylor expansion of f with centre at the point 0 has the form $f(z) = \sum_{m=0}^{\infty} P_m(z)$, and that for some $k \in N$ the polynomial P_k transforms D onto G. For what n, D, G, k, P_k and, possibly, for what type of function f do the above assumptions imply the equality $f = P_k$?

This problem can be extended a little by replacing in it the polynomial P_k by a mapping h of the form $h = \left(P^{(1)}_{k_1}, P^{(2)}_{k_2}, \ldots, P^{(n)}_{k_n}\right)$, where $k_1, \ldots, k_n \in N$, and $P^{(s)}_{k_s}$ stands for the s-th coordinate of the polynomial P_{k_s}.

It is easiest to solve the above problems, when D is a polycylinder with centre at 0, and h is of the form $h(z_1, \ldots, z_n) = \left(\varepsilon_1 z_1^{k_1}, \ldots, \varepsilon_n z_n^{k_n}\right)$, where $0 \neq \varepsilon_i \in C$ for $i = 1, \ldots, n$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Confining ourselves to a polycylinder T_n with centre at 0 and radius 1, and to the case when $\varepsilon_1=\ldots=\varepsilon_n=1$, we have the following result.

Theorem 1. Let $n \in N$, $k_1,\ldots,k_n \in N$. Assume that a mapping $f = (f_1,\ldots,f_n) : T_n \longrightarrow T_n$ is holomorphic and its expansion into a Taylor series with centre at 0 has the form

$$f_s(z_1,\ldots,z_n) = \sum_{m=0}^{\infty} P_m^{(s)}(z_1,\ldots,z_n), \quad s = 1,\ldots,n,$$

where $P_m^{(s)}$ are homogeneous polynomials of degree m and $P_{k_s}^{(s)}(z_1,\ldots,z_n) \equiv z_s^{k_s}$ for $s=1,\ldots,n$. Then $f(z_1,\ldots,z_n) \equiv \left(z_1^{k_1},\ldots,z_n^{k_n}\right)$.

Proof. Let $1 \le s \le n$, $|\varepsilon_1| = \ldots = |\varepsilon_n| = 1$, $\varepsilon := (\varepsilon_1,\ldots,\varepsilon_n)$, $g_\varepsilon^{(s)}(z) := f_s(\varepsilon z)$ for $z \in K$. Of course, $g_\varepsilon^{(s)} : K \to K$ and $g_\varepsilon^{(s)}(z) = \sum_{m=0}^{\infty} a_m z^m$ for $z \in K$, where $a_m = P_m^{(s)}(\varepsilon_1,\ldots,\varepsilon_n)$ for $m \in N \cup \{0\}$, and $|a_{k_s}| = 1$. So, from Theorem A it follows that $P_m^{(s)}(\varepsilon_1,\ldots,\varepsilon_n) = 0$ for $m \in N \cup \{0\}$, $m \neq k_s$. Hence and from the arbitrariness of $\varepsilon_1,\ldots,\varepsilon_n$, it follows that $P_m^{(s)} = 0$ for $m \in N \cup \{0\}$, $m \neq k_s$, which ends the proof.

In the present paper we shall deal with the problem formulated above in the case when D is a (unit) ball or, more generally, a set of the form $\left\{(z_1,\ldots,z_n) : \gamma_1|z_1|^{\alpha_1} + \ldots + \gamma_n|z_n|^{\alpha_n} < 1\right\}$, where $\alpha_1,\gamma_1,\ldots,\alpha_n,\gamma_n > 0$. In order to make our considerations clearer, we shall assume, without loss of generality, that the parameters $\gamma_1,\ldots,\gamma_n$ are equal to 1.

2. Let $n \in N$, $\alpha_1,\ldots,\alpha_n > 0$, $k_1,\ldots,k_n \in N \cup \{0\}$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $k = (k_1,\ldots,k_n)$, $M,R > 0$. Denote by $Q(\alpha,k,R,M)$ the class of all functions $f = (f_1,\ldots,f_n) : K_R \longrightarrow C^n$, with

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$K_R = \{z \in C : |z| < R\}$, where

$$f_s(z) = a_{k_s}^{(s)} z^{k_s} + a_{k_s+1}^{(s)} z^{k_s+1} + \dots, \qquad 1 \le s \le n, \tag{1}$$

are holomorphic in K_R and satisfy the condition

$$\sum_{s=1}^{n} |f_s(z)|^{\alpha_s} \le M, \qquad z \in K_R. \tag{2}$$

Theorem 2. If $f = (f_1,\dots,f_n) \in Q(\alpha,k,R,M)$, then

$$\sum_{s=1}^{n} \left| R^{k_s} \frac{f_s(z)}{z^{k_s}} \right|^{\alpha_s} \le M, \qquad z \in K_R. \tag{3}$$

In particular, for $z = 0$, we have

$$\sum_{s=1}^{n} \left| R^{k_s} a_{k_s}^{(s)} \right|^{\alpha_s} \le M. \tag{4}$$

If, in estimate (3), the equality holds at some point $z_0 \in K_R$, then $f_s(z) = a_{k_s}^{(s)} z^{k_s}$, $z \in K_R$, $s = 1,\dots,n$, and $\sum_{s=1}^{n} \left| a_{k_s}^{(s)} R^{k_s} \right| = M$.

Proof. Without loss of generality we may assume that $M = R = 1$, so $f = (f_1,\dots,f_n) \in Q(\alpha,k,1,1)$. Put $U = \{1 \le s \le n : f_s \not\equiv 0\}$. We may assume that $U \neq \emptyset$, since in the contrary case the theorem is evident. Put $p_s^{(1)}(z) = f_s(z)/z^{k_s}$, $z \in K$, $s \in U$. Obviously, $|p_s^{(1)}(z)| \le 1$ for $z \in K$, $s \in U$.

Let $z_0 \in K$. Put

$$p_s^{(2)}(z) = p_s^{(1)}\left(\frac{z+z_0}{1+\bar{z}_0 z} \right), \qquad z \in K, \quad s \in U.$$

From the definition of the class $Q(\alpha,k,1,1)$ it follows that

$$\sum_{s \in U} \left| \frac{z+z_0}{1+\bar{z}_0 z} \right|^{k_s \alpha_s} |p_s^{(2)}(z)|^{\alpha_s} \le 1, \qquad z \in K. \tag{5}$$

Since $|p_s^{(2)}(z)| \le 1$ for $z \in K$, $s \in U$, there exist radial

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

limits $p_s^{(2)}(e^{it}) := \lim_{r\to 1^-} p_s^{(2)}(re^{it})$ for $s \in U$ and for almost all $t \in \langle 0,2\pi)$. Putting $z = re^{it}$ in (5) and passing to the limit with $r \to 1^-$, we obtain

$$\sum_{s\in U} \left|p_s^{(2)}(e^{it})\right|^{\alpha_s} \le 1 \quad \text{for} \quad t \in \langle 0,2\pi) \quad \text{a.e.} \tag{6}$$

It is obvious that $p_s^{(2)} \not\equiv 0$ for $s \in U$. Put

$$p_s^{(3)}(z) = \frac{p_s^{(2)}(z)}{A_s(z)}, \quad s \in U, \quad z \in K,$$

where A_s stands for the Blaschke product with zeros at zeros of the function $p_s^{(2)}$. (If any of the functions $p_s^{(2)}$ is different from 0 in the entire disc K, the respective Blaschke product is defined as a function identically equal to 1).

It is known (cf. [1]) that $|p_s^{(3)}(z)| \le 1$ for $z \in K$, $s \in U$, and that $|A_s(e^{it})| = 1$ for $s \in U$ and $t \in \langle 0,2\pi)$ a.e. The functions $p_s^{(3)}$, $s \in U$, are holomorphic and different from zero in K. Put $p_s^{(4)}(z) = \left[p_s^{(3)}(z)\right]^{\alpha_s/2} = c_0^{(s)} + c_1^{(s)}z + \ldots$, $z \in K$, $s \in U$, where, for any $s \in U$, $\left(p_s^{(3)}(z)\right)^{\alpha_s/2}$ denotes any fixed branch of the function $p_s^{(3)}$ with exponent $\alpha_s/2$. From the last information and from (6) it follows that

$$\sum_{s\in U} \left|p_s^{(4)}(e^{it})\right|^2 = \sum_{s\in U} \left|p_s^{(3)}(e^{it})\right|^{\alpha_s} = \sum_{s\in U} \left|p_s^{(2)}(e^{it})\right|^{\alpha_s} \le 1$$

for $t \in \langle 0,2\pi)$ a.e. Whereas from the fact that $|p_s^{(4)}(z)| \le 1$ for $s \in U$, $z \in K$, and from the Lebesgue theorem it follows that

$$\int_0^{2\pi} \sum_{s\in U} \left|p_s^{(4)}(e^{it})\right|^2 dt = \lim_{r\to 1^-} \int_0^{2\pi} \sum_{s\in U} \left|p_s^{(4)}(re^{it})\right|^2 dt =$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= 2\pi \sum_{n=0}^{\infty} \sum_{s\in U} \left|c_n^{(s)}\right|^2 .$$

Consequently,

$$\sum_{n=0}^{\infty} \sum_{s\in U} \left|c_n^{(s)}\right|^2 \le 1. \tag{7}$$

Taking into account the equality

$$\left|c_0^{(s)}\right|^2 = \left|p_s^{(4)}(0)\right|^2 = \left|\frac{p_s^{(1)}(z_0)}{A_s(0)}\right|^{\alpha_s}, \quad s \in U,$$

and of (7), we get

$$\sum_{s\in U} \left|\frac{p_s^{(1)}(z_0)}{A_s(0)}\right|^{\alpha_s} = \sum_{s\in U} \left|c_0^{(s)}\right|^2 \le 1. \tag{8}$$

Hence and from the definition of the set U as well as from the inequality $|A_s(0)| \le 1$ for $s \in U$, it follows that

$$\sum_{s=1}^{n} \left|p_s^{(1)}(z_0)\right|^{\alpha_s} = \sum_{s\in U} \left|p_s^{(1)}(z_0)\right|^{\alpha_s} \le 1,$$

which ends the proof of (3).

Suppose now that, at some point $z_0 \in K$, the equality $\sum_{s=1}^{n} \left|p_s^{(1)}(z_0)\right|^{\alpha_s} = 1$ holds. Then $U \neq \emptyset$. Besides, (8) implies $A_s(0) = 1$ for $s \in U$. So, $A_s(z) = 1$ for $z \in K$. From (8) and (7) it also follows that $c_n^{(s)} = 0$ for $n \ge 1$, $s \in U$. Consequently,

$$p_s^{(1)}\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) = p_s^{(2)}(z) = p_s^{(3)}(z) = \left[p_s^{(4)}(z)\right]^{2/\alpha_s} =$$

$$= \left[c_0^{(s)}\right]^{2/\alpha_s}, \quad z \in K, \quad s \in U.$$

Hence, it appears that the functions $p_s^{(1)}$, $s \in U$, are constant, which completes the proof of Theorem 2.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Remark. Denote by $\tilde{Q}(\alpha,k,R,M)$ the class of all functions $f = (f_1,\dots,f_n) : K_R \to C^n$ of the form (1) which are holomorphic in $\overline{K}_R$ and satisfy inequality (2) for $|z| = R$. Since, for $f \in \tilde{Q}(\alpha,k,R,M)$, the left-hand side of (2) is a plurisubharmonic function in $\overline{K}_R$, the maximum principle for plurisubharmonic functions implies that $\tilde{Q}(\alpha,k,R,M) \subset Q(\alpha,k,R,M)$. Thus, each function $f \in \tilde{Q}(\alpha,k,R,M)$ satisfies the assertion of Theorem 1. This statement can also be proved without referring to the maximum principle for plurisubharmonic functions. To this end, the reader will easily notice that, for a function $f \in \tilde{Q}(\alpha,k,R,M)$, the assertion of Theorem 1 can be shown similarly. This will even be simpler, inasmuch as the inequalities occurring in the proof of Theorem 1 are satisfied for $|z| = 1$.

Let $n \in N$, $\alpha = (\alpha_1,\dots,\alpha_n) \in R_+^n$ and $B_\alpha = \{(z_1,\dots,z_n) \in C^n : |z_1|^{\alpha_1} + \dots + |z_n|^{\alpha_n} < 1\}$. It is evident that each polynomial of the form $\left(z_1^{k_1},\dots,z_n^{k_n}\right)$, where $k_1,\dots,k_n \in N$, transforms B_α onto B_γ, where $\gamma = (\alpha_1/k_1,\dots,\alpha_n/k_n)$. Since B_α are full Reinhardt domains with centre at zero, each function holomorphic in B_α can be expanded in B_α into a Taylor series with centre at zero.

Theorem 3. Let $n \in N$, $\alpha = (\alpha_1,\dots,\alpha_n) \in R_+^n$, $k_1,\dots,k_n \in N$. Assume that a function $f = (f_1,\dots,f_n)$ is holomorphic in B_α and its expansion into Taylor series with centre at zero has the form

$$f_s(z_1,\dots,z_n) = z_s^{k_s} + \sum_{m=k_s+1}^{\infty} P_m^{(s)}(z_1,\dots,z_n), \qquad s=1,\dots,n,$$

where $P_m^{(s)}$, $s=1,\dots,n$, $m \geq k_s+1$, are homogeneous polynomials of degree m. Moreover, if $f : B_\alpha \to B_\gamma$, where $\gamma = (\alpha_1/k_1,\dots,\alpha_n/k_n)$, then $f(z_1,\dots,z_n) \equiv \left(z_1^{k_1},\dots,z_n^{k_n}\right)$.

Proof. Denote by r_0 the only root of the equation $r^{\alpha_1} + \dots$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\ldots + r^{\alpha_n} = 1$ from the interval $(0,1>$ and take any $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in T_n$. It is evident that $\varepsilon z \in B_\alpha$ for $|z| < r_0$. Put $h_\varepsilon(z) = f(\varepsilon z)$ for $|z| < r_0$. Obviously, $h_\varepsilon = (h_\varepsilon^{(1)},\ldots,h_\varepsilon^{(n)})$, where $h_\varepsilon^{(s)} = \varepsilon_s z^{k_s} + \sum_{m=k_s+1}^{\infty} P_m^{(s)}(\varepsilon_1,\ldots,\varepsilon_n) z^m$ for $|z| < r_0$, $s=1,\ldots,n$, and $h_\varepsilon \in Q(\gamma,k,r_0,1)$, where $k = (k_1,\ldots,k_n)$. Moreover, in inequality (4), corresponding to the function h_ε and $(\alpha,k,R,M)=(\gamma,k,r_0,1)$, the equality holds. So, in virtue of Theorem 2, we have $P_m^{(s)}(\varepsilon_1,\ldots,\varepsilon_n) = 0$, for $s = 1,\ldots,n$, $m > k_s$. Hence and from the arbitrariness of $\varepsilon \in T_n$, it follows that $P_m^{(s)} = 0$ for $s = 1,\ldots,n$ and $m > k_s$, which completes the proof.

In the case of $k_1 = k_2 = \ldots = k_n = 1$, Theorem 3 follows from the well-known uniqueness theorem of Cartan.

3. The question arises whether in Theorem 2 the assumption $P_m^{(s)} = 0$, for $s=1,\ldots,n$, $m=0,1,\ldots,k_s-1$, can be omitted. The author does not know the answer to it. We shall show, however, that the answer is affirmative under the additional assumption that f is holomorphic also on the boundary of B_α. Namely, we have the following result.

Theorem 4. Let $n \in N$, $\alpha = (\alpha_1,\ldots,\alpha_n) \in R_+^n$, $k_1,\ldots,k_n \in N$, $f : B_\alpha \to B_\gamma$, where $\gamma = (\alpha_1/k_1,\ldots,\alpha_n/k_n)$. Assume that $f = (f_1,\ldots,f_n)$ is holomorphic in $\bar{B}_\alpha$, and that

$$f_s(z) = \sum_{m=0}^{\infty} P_m^{(s)}(z), \quad z = (z_1,\ldots,z_n) \in \bar{B}_\alpha, \quad s=1,\ldots,n,$$

where $P_m^{(s)}$, $m \in N \cup \{0\}$, $s=1,\ldots,n$, are homogeneous polynomials of degree m, and $P_m^{(s)}(z_1,\ldots,z_n) \equiv z_s^{k_s}$, $s=1,\ldots,n$.

Then $f(z_1,\ldots,z_n) \equiv (z_1^{k_1},\ldots,z_n^{k_n})$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. On account of Theorem 3, it suffices to demonstrate that $P_m^{(s)} = 0$ for $1 \le s \le n$, $0 \le m < k_s$. The proof will be carried out inductively with respect to n. For n=1, the theorem is true. Assume that n>1 and that the theorem is valid for all positive integers less than n. Besides, assume that, for n, the assumptions of this theorem are satisfied. Since $P_m^{(s)}$, $m \in N \cup \{0\}$, s=1,...,n, are homogeneous polynomials of degree m, they can be represented in the form

$$P_m^{(s)}(z_1,\dots,z_n) = \sum_{\substack{\xi_1,\dots,\xi_n \ge 0 \\ \xi_1+\dots+\xi_n=m}} a^{(s)}_{\xi_1,\dots,\xi_n} z_1^{\xi_1}\dots z_n^{\xi_n}, \quad (z_1,\dots,z_n) \in C^n.$$

Note that $|f_1(z)|^{\alpha_1/k_1} + \dots + |f_n(z)|^{\alpha_n/k_n} \le 1$ for $z = (z_1,\dots,z_{n-1},0) \in \bar{B}_\alpha$. Hence, it appears that the function g defined by the formula

$$(9)\quad g(z_1,\dots,z_{n-1}) = (f_1(z_1,\dots,z_{n-1},0),\dots,f_{n-1}(z_1,\dots,z_{n-1},0))$$

maps B_{α^*} into B_{γ^*}, where $\alpha^* = (\alpha_1,\dots,\alpha_{n-1})$, $\gamma^* = (\alpha_1/k_1,\dots,\alpha_{n-1}/k_{n-1})$. By the induction assumption, we find

$$(10)\quad g(z_1,\dots,z_{n-1}) \equiv (z_1^{k_1},\dots,z_{n-1}^{k_{n-1}}).$$

Consequently, $a^{(s)}_{\xi_1,\dots,\xi_{n-1},0} = 0$ for $s=1,\dots,n-1$; $\xi_1,\dots,\xi_{n-1}, m \in N\cup\{0\}$, $m \ne k_s$; $\xi_1 + \dots + \xi_{n-1} = m$. By (9) and (10), we get

$$f_n(z_1,\dots,z_{n-1},0) = 0 \quad \text{for} \quad |z_1|^{\alpha_1} + \dots + |z_{n-1}|^{\alpha_{n-1}} = 0.$$

Hence, $a^{(n)}_{\xi_1,\dots,\xi_{n-1},0} = 0$ for $\xi_1,\dots,\xi_{n-1}, m \in N\cup\{0\}$. We have demonstrated that $a_{\xi_1,\dots,\xi_{n-1},0} = 0$ for $s=1,\dots,n$; $\xi_1,\dots,\xi_{n-1}, m \in N\cup\{0\}$, $m \ne k_s$; $\xi_1+\dots+\xi_{n-1}=m$. In an analogous way one can show a more general result, namely, that $a^{(s)}_{\xi_1,\dots,\xi_n} = 0$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $s=1,\dots,n$; $\xi_1,\dots,\xi_n, m\in N\cup\{0\}$; $m\neq k_s$; $\xi_1+\dots+\xi=m$, if at least one element of the sequence $(\xi_1,\dots,\xi_n)$ is equal to zero. So,

$$P_m^{(s)}(z_1,\dots,z_n) = \sum_{\substack{\xi_1,\dots,\xi_n\geq 1\\ \xi_1+\dots+\xi_n=m}} a^{(s)}_{\xi_1,\dots,\xi_n} z_1^{\xi_1}\dots z_n^{\xi_n}, \quad (z_1,\dots,z_n)\in C^n. \tag{11}$$

for $s=1,\dots,n$; $m\in N\cup\{0\}$; $m\neq k_s$. In order to show the assertion of Theorem 4, it is enough to prove, by Theorem 2, that $P_m^{(s)}=0$ for $1\leq s\leq n$, $0\leq m<k_s$. Suppose that it is not so. Thus, there exist $1\leq s_0\leq n$ and $0\leq m_0<k_{s_0}$ such that $P_{m_0}^{(s_0)}\neq 0$. With no loss of generality we may assume that $s_0=1$. Put $A'=\{s\in\{1,\dots,n\}: f_s\not\equiv z_s^{k_s}\}$. Obviously, $1\in A'$. From (11) it follows that, for any $s\in A'$, there exist $\xi_1,\dots,\xi_n\in N$ such that $a^{(s)}_{\xi_1,\dots,\xi_n}\neq 0$. Put $k_0=\max\limits_{1\leq s\leq n-1} k_s$ and

$$h_s=\min\{m\in N: \bigvee_{\xi_1,\dots,\xi_n\in N} a^{(s)}_{\xi_1,\dots,\xi_n}\neq 0 \wedge k_0\xi_1+\sum_{k=2}^{n-1}\xi_k=m\},$$

$s\in A'$. It is evident that $h_1<k_0k_1$ and $h_s>k_s$ for $s\in A'-\{1\}$. For any $r_n\in(0,1)$, denote by r_n' the only root of the equation

$$r^{k_0\alpha_1}+\sum_{k=2}^{n-1} r^{\alpha_k}=1-r_n^{\alpha_n}$$

from the interval $(0,1)$. Let $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{n-1})\in T_{n-1}$, $z_n\in C$, $|z_n|\in(0,1)$, $r_n=|z_n|$, $1\leq s\leq n$. Put

$$f_{\varepsilon,z_n}(z)=f(\varepsilon_1z^{k_0},\varepsilon_2z,\varepsilon_3z,\dots,\varepsilon_{n-1}z,z_n), \quad |z|\leq r_n',$$

$$f^{(s)}_{\varepsilon,z_n}(z)=f_s(\varepsilon_1z^{k_0},\varepsilon_2z,\varepsilon_3z,\dots,\varepsilon_{n-1}z,z_n), \quad |z|\leq r_n'.$$

It is easy to verify that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$f^{(1)}_{\varepsilon,z_n}(z) = \sum_{m=h_1}^{\infty} b^{(1)}_{m,z_n}(\varepsilon) z^m, \qquad |z| \le r'_n,$$

$$f^{(s)}_{\varepsilon,z_n}(z) = \varepsilon_s^{k_s} z^{k_s} + \sum_{m=h_s}^{\infty} b^{(s)}_{m,z_n}(\varepsilon) z^m, \quad |z| \le r'_n, \quad s \in A' - \{1\},$$

$$f^{(n)}_{\varepsilon,z_n}(z) = \begin{cases} z_n^{k_n} + \sum_{m=h_n}^{\infty} b^{(n)}_{m,z_n}(\varepsilon) z^m, & |z| \le r'_n, \text{ when } n \in A', \\ z_n^{k_n}, & |z| \le r'_n, \text{ when } n \notin A', \end{cases}$$

where

$$b^{(s)}_{m,z_n}(\varepsilon) = \sum_{k=1}^{\infty} c^{(s)}_{k,m}(\varepsilon) z_n^k,$$

$$c^{(s)}_{k,m}(\varepsilon) = \sum_{\substack{\xi_1,\ldots,\xi_n \ge 1 \\ k_0\xi_1+\xi_2+\ldots+\xi_{n-1}=m}} a^{(s)}_{\xi_1,\ldots,\xi_{n-1},k}\, \varepsilon_1^{k_0\xi_1} \varepsilon_2^{\xi_2} \ldots \varepsilon_{n-1}^{\xi_{n-1}}.$$

It is evident that all $c^{(s)}_{k,m}$ are polynomials of variables $\varepsilon_1,\ldots,\varepsilon_{n-1}$. The definition of h_1 implies that there exist $\xi_1,\ldots,\xi_{n-1},w \in N$ such that

$$a^{(1)}_{\xi_1,\ldots,\xi_{n-1},w} \neq 0 \wedge k_0\xi_1 + \sum_{k=2}^{n-1} \xi_k = h_1.$$

Hence, it appears that the polynomial $c^{(1)}_{w,h_1}$ is not identically zero. Consequently, there exists $\varepsilon' = (\varepsilon'_1,\ldots,\varepsilon'_{n-1}) \in T_{n-1}$ such that $c^{(1)}_{w,h_1}(\varepsilon') \neq 0$.

Lemma 1. There exist $r_0, |z_n^*| \in (0,1)$ such that

$$\left(r_0^{k_0\alpha_1} + \sum_{k=2}^{n-1} r_0^{\alpha_k} + |z_n^*|^{\alpha_n} = 1\right) \wedge \left(\bigwedge_{|z|=r_0} |f^{(1)}_{\varepsilon,z_n^*}(z)| > |z|^{k_0k_1} \right).$$

Proof. Let us put

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$A = \left\{(z,z_n) \in C^2 : |z|^{k_0\alpha_1} + \sum_{k=2}^{n-1} |z|^{\alpha_k} + |z_n|^{\alpha_n} \le 1\right\},$$

and, for $(z,z_n) \in A$,

$$g_{\varepsilon'}(z,z_n) = z^{-h_1} f_{\varepsilon',z_n}(z) = \sum_{m=h_1}^{\infty} b^{(1)}_{m,z_n}(\varepsilon') z^{m-h_1},$$

$$h_{\varepsilon'}(z,z_n) = g_{\varepsilon'}(z,z_n) - b^{(1)}_{h_1,z_n}(\varepsilon) = \sum_{m=h_1+1}^{\infty} b^{(1)}_{m,z_n}(\varepsilon') z^{m-h_1}.$$

The function $h_{\varepsilon'}$ is holomorphic in A. Since A is compact and $h_{\varepsilon'}(0,z_n) = 0$ for $|z_n| \le 1$, it follows from the continuity of $h_{\varepsilon'}$ on A that

(12) $$\bigwedge_{\eta>0} \bigvee_{\delta>0} \bigwedge_{(z,z_n)\in A} (|z| < \delta) \Longrightarrow \left(|h_{\varepsilon'}(z,z_n)| < \eta\right).$$

From the definition of $b^{(1)}_{h_1,z_n}(\varepsilon')$ we have

$$b^{(1)}_{h_1,z_n}(\varepsilon') = \sum_{k=1}^{\infty} c^{(1)}_{k,h_1}(\varepsilon') z_n^k, \qquad |z_n| \le 1.$$

Since $c^{(1)}_{w,h_1}(\varepsilon') \neq 0$, the above function of the variable z_n is not identically zero. Consequently, the function

$$p(r) = \max_{|z_n|=r} |b^{(1)}_{h_1,z_n}(\varepsilon')|, \qquad 0 \le r \le 1,$$

is strictly increasing. Putting $\eta = \frac{1}{2}p(\frac{1}{2})$ and making use of (12), we ascertain that there exists $\delta > 0$ such that

$$\bigwedge_{(z,z_n)\in A} (|z| < \delta_1) \Longrightarrow \left(|h_{\varepsilon',z_n}(z)| < \tfrac{1}{2}\, p(\tfrac{1}{2})\right).$$

Moreover, it is clear that there exists $\delta_2 \in (0,1)$ such that, for each $r \in (0,\delta_2>$, the root $r_n(r)$ of the equation

$$r^{k_0\alpha_1} + \sum_{k=2}^{n-1} r^{\alpha_k} + (r_n(r))^{\alpha_n} = 1$$

belongs to the interval $(\frac{1}{2},1)$. The inequality $h_1 < k_0k_1$ implies

(13) $$\bigvee_{\delta_3>0} \bigwedge_{|z|\le\delta_3} |z|^{k_0k_1-h_1} < \tfrac{1}{2}\, p(\tfrac{1}{2}).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Put $r_0 = \min\{\delta_1, \delta_2, \delta_3\}$. Then $r_n(r_0) \in (\frac{1}{2}, 1)$. Besides, there exists z_n^* such that $|z_n^*| = r_n(r_0)$ and $|b^{(1)}_{h_1, z_n^*(\varepsilon')}| = p(r_n(r_0)) > p(\frac{1}{2})$. Of course,

$$r_0^{k_0\alpha_1} + \sum_{k=2}^{n-1} r_0^{\alpha_k} + |z_n^*|^{\alpha_n} = 1.$$

Moreover, for any $|z| = r_0$, the inequality

$$|g_{\varepsilon'}(z, z_n^*)| \geq |b^{(1)}_{h_1, z_n^*(\varepsilon')}| - |h_{\varepsilon'}(z, z_n^*)| \geq$$

$$\geq p(\tfrac{1}{2}) - \tfrac{1}{2} p(\tfrac{1}{2}) = \tfrac{1}{2} p(\tfrac{1}{2})$$

holds. Consequently, in virtue of (13), we get

$$|f^{(1)}_{\varepsilon', z_n^*}(z)| = |z|^{h_1} |g_{\varepsilon'}(z, z_n^*)| > |z|^{h_1} \tfrac{1}{2} p(\tfrac{1}{2}) > |z|^{k_0 k_1}$$

for $|z| = r_0$, which ends the proof of Lemma 1.

We come back to the proof of Theorem 4. Let r_0 and z_n^* be the same as in Lemma 1. From the inequality $|f^{(1)}_{\varepsilon', z_n^*}(z)| > |z|^{k_0 k_1}$ for $|z| = r_0$ it follows that

$$(14) \qquad \sum_{s=2}^{n} |f^{(s)}_{\varepsilon', z_n^*}(z)|^{\alpha_s/k_s} < 1 - |z|^{k_0\alpha_1} = 1 - r_0^{k_0\alpha_1}$$

for $|z| = r_0$. Consequently, $\left(f^{(2)}_{\varepsilon', z_n^*}, \ldots, f^{(n)}_{\varepsilon', z_n^*}\right) \in \tilde{Q}(\alpha', k', r_0, 1 - r_0^{k_0\alpha_1})$, where $\alpha' = (\alpha_2/k_2, \ldots, \alpha_n/k_n)$, $k' = (k_2, \ldots, k_{n-1}, 0)$. Moreover, from the Lemma 1 it follows that

$$\sum_{s=2}^{n-1} \left| r_0^{k_s} \varepsilon_s' \right|^{\alpha_s/k_s} + \left| r_0^0 z_n^{*k_n} \right|^{\alpha_n/k_n} = 1 - r_0^{k_0\alpha_1}.$$

So, we see that in inequality (4) (corresponding to the parameters $\alpha', k', r_0, 1 - r_0^{k_0\alpha_1}$) the equality holds. Hence and from our remark, we find

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$f^{(s)}_{\varepsilon',z_n^*}(z) = \varepsilon_s' z^{k_s} z^{k_s}, \quad |z| \le r_0, \quad s = 2,\dots,n-1,$$

$$f^{(n)}_{\varepsilon',z_n^*}(z) = z_n^{*k_s}, \quad |z| \le r_0,$$

which contradicts (14) and, thereby, completes the proof of theorem 4.

Let $n,k \in N$, $\alpha = (\alpha_1,\dots,\alpha_n) \in R_+^n$ and $A = [a_{js}]_{1\le j,s\le n}$ be any non-singular square matrix of degree n with complex terms. Put $f_{A,k}(z_1,\dots,z_n) = (a_{11}z_1^k+\dots+a_{1n}z_n^k,\dots,a_{n1}z_1^k+\dots+a_{nn}z_n^k)$ for $(z_1,\dots,z_n) \in C^n$, and

$$D_{A,\alpha,k} = \left\{(w_1,\dots,w_n) \in C^n: \sum_{m=1}^{n}\left|\frac{\det A_m(w_1,\dots,w_n)}{\det A}\right|^{\alpha_m/k} < 1\right\},$$

where $A_m(w_1,\dots,w_n)$ denotes the matrix formed out of the matrix A by replacing the terms of its m-th column with the successive numbers $w_1,\dots,w_n$. From the Cramer theorem it easily follows that $f_{A,k} : B_\alpha \xrightarrow{\text{onto}} D_{A,\alpha,k}$.

Theorem 5. Let $\alpha = (\alpha_1,\dots,\alpha_n) \in R_+^n$, $A = [a_{j,s}]_{1\le j,s\le n}$, $a_{j,s} \in C$ for $1 \le j,s \le n$ and $\det A \ne 0$. Assume that $f : B_\alpha \to D_{A,\alpha,k}$ is holomorphic and of the form $f = \sum_{m=0}^{\infty} P_m$, where $P_m : C^n \to C^n$ are homogeneous polynomials of degree m and $P_k = f_{A,k}$. Then

(a) if $P_m = 0$ for $0 \le m \le k-1$, then $f = f_{A,k}$,

(b) if f is holomorphic in $\bar{B}_\alpha$, then $f = f_{A,k}$.

Proof. Let $f = (f_1,\dots,f_n)$ and $g = (g_1,\dots,g_n)$, where

$$g_m = \frac{\det A_m(f_1,\dots,f_n)}{\det A}, \quad m = 1,\dots,n.$$

Since $f : B_\alpha \to D_{A,\alpha,k}$, so $g : B_\alpha \to B_\gamma$, with $\gamma = (\alpha_1/k,\dots,\alpha_n/k)$. It is easy to check that in the expansion of g into

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

a Taylor series with centre at zero the k-th term is identically equal to the function $(z_1^k,\dots,z_n^k)$. So, if $P_m = 0$ for $0 \le m \le k-1$ or if f is holomorphic in $\bar{B}_\alpha$, then the function g satisfies the assumption of Theorem 3 or of Theorem 4. Thus, $g(z_1,\dots,z_n) \equiv (z_1^k,\dots,z_n^k)$ and, in consequence, $f = f_{A,k}$, which ends the proof of Theorem 5.

Author announced the results of this paper at the conference on analytic functions at Błażejewko in 1982 (cf. [2]).

REFERENCES

[1] **E.P. Collingwood, A.J. Lohwater:** The theory of cluster sets, Cambridge, 1966 (cf. also the edition in Russian language, Moscow, 1971, p. 47-52).

[2] **E. Janiec:** A uniqueness theorem concerning bounded maps in the unit ball, Conference On Analytic Functions, Błażejewko, August 19-27, 1982, Univ. of Łódź, 1982, p. 23.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ,
90-238 ŁÓDŹ, POLAND

Received February 20, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Władysław Sosulski

ON NEUTRAL PARTIAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS OF HYPERBOLIC TYPE

1. Introduction

We shall consider here, as basic space for solutions of partial differential inclusions of hyperbolic type, the space $AC(P,R^n)$ of all absolutely continuous functions defined on any rectangle $P = [0,a]\times[0,b]$ with values in n-dimensional space $(R^n,|\cdot|)$. This space will be considered as a normed space with a norm $\|\cdot\|_{AC}$ defined by

$$\|z\|_{AC} = |z(0,0)| + \int_0^a |z_x(s,0)|ds + \int_0^b |z_y(0,t)|dt +$$

$$+ \iint_P |z_{xy}(s,t)|dsdt.$$

Furthermore, we use the space $C_x(P,R^n)$ of all functions $u:P \to R^n$ such that $u(\cdot,y):[0,a] \to R^n$ is measurable for fixed $y\in[0,b]$, $u(x,\cdot):[0,b] \to R^n$ is continuous for fixed $x\in[0,a]$ and such that

$$\|u\|_x = \int_0^a \max_{y\in[0,b]} |u(x,y)|dx < \infty.$$

We will consider $C_x(P,R^n)$ together with the norm defined above. Similarly, we can define the space $C_y(P,R^n)$ with the norm $\|\cdot\|_y$. It was proved in [4] that $(C_x(P,R^n), \|\cdot\|_x)$ and $(C_y(P,R^n), \|\cdot\|_y)$ are Banach spaces. Furthermore, we use the space $C(P,R^n)$ of all continuous functions $u:P \to R^n$ with the norm $\|\cdot\|_C$ defined by $\|z\|_C = \sup_{(x,y)\in P} |z(x,y)|$ and the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

space $L(P,\mathbb{R}^n)$ of all L-integrable functions $u:P \to \mathbb{R}^n$ with the norm $\|u\|_L = \iint_P |u(x,y)|dxdy$.

The values of right-hand sides of considered inclusions will be taken from the metric space $(\mathrm{Comp}(\mathbb{R}^n),h)$ of all nonempty compact subsets of $\mathbb{R}^n$ with the Hausdorff metric h defined as usual by setting $h(A,B) = \max\{\bar{h}(A,B), \bar{h}(B,A)\}$ for $A,B \in \mathrm{Comp}(\mathbb{R}^n)$, where $\bar{h}(B,A) = \max_{b\in B} \mathrm{dist}(b,A)$ with $\mathrm{dist}(b,A) = \inf_{a\in A} |b-a|$.

Let $G:P \to P(\mathbb{R}^n)$ be set-valued function, $P(\mathbb{R}^n)$ being the space of all nonempty subsets of $\mathbb{R}^n$. Denote by $\mathcal{F}(G)$ the set of all L-integrable functions $w:P \to \mathbb{R}^n$ such that $w(x,y) \in G(x,y)$ a.e. in P. The set $\mathcal{F}(G)$ is said to be subtrajectory integrals of the set-valued functions G.

Let $A(P,\mathbb{R}^n)$ denote the metric space of all set-valued functions $G:P \to \mathrm{Comp}(\mathbb{R}^n)$ that are L-measurable and integrable bounded with a metric d defined by

$$d(F,G) = \iint_P h[F(x,y), G(x,y)]dxdy \quad \text{for} \quad F,G \in A(P,\mathbb{R}^n).$$

A family $\{G_\lambda\}_{\lambda\in\Lambda} \subset A(P,\mathbb{R}^n)$ is called uniformly integrable, if for every $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that $\|\iint_E G_\lambda(x,y)\| \le \varepsilon$ holds for every $\lambda\in\Lambda$ and all measurable sets $E\subset P$ with $\mu(E)<\delta$, where $\iint_E G_\lambda(x,y)dxdy$ denotes the Aumann integral of G_λ on E and

$$\|\iint_E G_\lambda(x,y)dxdy\| = h\Big(\iint_E G_\lambda(x,y)dxdy, \{0\}\Big).$$

We call a sequence (G_k) of $A(P,\mathbb{R}^n)$ weakly covergent to $G\in A(P,\mathbb{R}^n)$, if for every measurable set $U\subset P$ a sequence $\Big\{\iint_U G_k(x,y)dxdy\Big\}$ of $\mathrm{Comp}(\mathbb{R}^n)$ converges in the Hausdorff metric h to $\iint_U G(x,y)dxdy$. The weak convergence in $A(P,\mathbb{R}^n)$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

can be also defined by other conditions (see [2]).

We shall need the following well-known mappings

$$(0.1)\qquad (\mathcal{T}_0 u)(x,y) := \int_0^x \int_0^y u(s,t)\,ds\,dt \quad \text{for } u \in L(P,\mathbb{R}^n),$$

$$(0.2)\qquad (\mathcal{T}_1 u)(x,y) := \left[\int_0^y u(x,t)\,dt, \int_0^x u(s,y)\,ds \right] \quad \text{for } u \in L(P,\mathbb{R}^n),$$

$$(0.3)\qquad (\mathcal{T} u)(x,y) := [\mathcal{T}_0 u)(x,y),\ (\mathcal{T}_1 u)(x,y)] \quad \text{for } u \in L(P,\mathbb{R}^n),$$

$$(0.4)\qquad (\mathcal{D} z)(x,y) := [z_x(x,y),\ z_y(x,y)] \quad \text{for } z \in AC(P,\mathbb{R}^n).$$

It is easy to see that $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}$, $\mathcal{D}$ are linear mappings

$$\mathcal{T}_0 : L(P,\mathbb{R}^n) \longrightarrow AC^0(P,\mathbb{R}^n),\quad \mathcal{T}_1 : L(P,\mathbb{R}^n) \longrightarrow C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n),$$

$$\mathcal{T} : L(P,\mathbb{R}^n) \longrightarrow AC^0(P,\mathbb{R}^n) \times C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n),$$

$$\mathcal{D} : AC(P,\mathbb{R}^n) \longrightarrow C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n),$$

where $AC^0(P,\mathbb{R}^n)$ is the space of all functions $z \in AC(P,\mathbb{R}^n)$ such that $z(x,0) = z(0,y) = 0$ for all $(x,y) \in P$.

Given a set $A \in L(P,\mathbb{R}^n)$, we will denote by $\overline{[A]}^{w}_{L}$ its closure in the weak topology of $L(P,\mathbb{R}^n)$. If $(p,q) \in C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n)$, then $\|(p,q)\| = \max(\|p\|_x, \|q\|_y)$. Let K_λ, M_λ and S_λ denote closed balls of $C(P,\mathbb{R}^n)$, $C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n)$ and $L(P,\mathbb{R}^n)$, respectively, with centers at the origins and a radius $\lambda > 0$. We shall consider a family of all set-valued functions $G : P \times K_\lambda \times M_\lambda \times S_\lambda \longrightarrow \mathrm{Comp}(\mathbb{R}^n)$ such that

$G(\cdot,\cdot,z,p,q,r) \in A(P,\mathbb{R}^n)$ for each fixed $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$.

We will deal with equivalence classes of members of the above family, where equivalence is defined by the equality a.e. in P for each fixed $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$. The space of all such equivalence classes is denoted by $H_\lambda(P,\mathbb{R}^n)$. Consider also a set $\mathcal{H}_\lambda(P,\mathbb{R}^n) \subset H_\lambda(P,\mathbb{R}^n)$ of all $G \in H_\lambda(P,\mathbb{R}^n)$ such that a family $\{G(\cdot,\cdot,z,p,q,r)\}_{(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda} \subset A(P,\mathbb{R}^n)$ is uniformly integrable. It is easy to see that $\mathcal{H}_\lambda(P,\mathbb{R}^n)$ is a complete metric space with a metric δ defined by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(0.5) $\delta(F,G) = \sup\{ d[F(\cdot,\cdot,z,p,q,r), G(\cdot,\cdot,z,p,q,r)]: (z,p,q,r)\in K_\lambda\times M_\lambda\times S_\lambda \}$ for $F,G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$.

The Hausdorff metric in the space $Cl(L(P,\mathbb{R}^n))$ of all nonempty closed and bounded subsets of $L(P,\mathbb{R}^n)$ induced by the norm of $L(P,\mathbb{R}^n)$ will be denoted by H_L.

Recall, we say that $\mathcal{F}(G)$ is Lipschitz continuous on $K_\lambda\times M_\lambda\times S_\lambda$, if there is a number $\alpha>0$ such that

(0.6) $H_L[\mathcal{F}(G)(z,p,q,r), \mathcal{F}(G)(\bar z,\bar p,\bar q,\bar r)] \le$
$\le \alpha \max(\|z-\bar z\|_C , \|p-\bar p\|_X , \|q-\bar q\|_Y , \|r-\bar r\|_L)$

for each $(z,p,q,r),(\bar z,\bar p,\bar q,\bar r)\in K_\lambda\times M_\lambda\times S_\lambda$. Similarly, we say that $\mathcal{F}(G)$ is Lipschitz continuous on $M_\lambda\times S_\lambda$ uniformly with respect to $z\in K_\lambda$, if there is a number $\alpha>0$ such that

(0.7) $H_L[\mathcal{F}(G)(z,p,q,r), \mathcal{F}(G)(z,\bar p,\bar q,\bar r)] \le$
$\le \alpha \max(\|p-\bar p\|_X , \|q-\bar q\|_Y , \|r-\bar r\|_L)$

for each $z\in K_\lambda$ and $(p,q,r),(\bar p,\bar q,\bar u)\in M_\lambda\times S_\lambda$.

We say that $\mathcal{F}(G)$ is H-upper [H-lower] semicontinuous on $K_\lambda\times M_\lambda\times S_\lambda$, if for every sequence $\{(z_n,p_n,q_n,r_n)\}$ of $K_\lambda\times M_\lambda\times S_\lambda$ such that $\|z-z_n\|_C \to 0$, $\|p-p_n\|_X \to 0$, $\|q-q_n\|_Y \to 0$, $\|r-r_n\|_L \to 0$, as $n\to\infty$, one has

$$\lim_{n\to\infty} \overline{H}_L[\mathcal{F}(G)(z_n,p_n,q_n,r_n), \mathcal{F}(G)(z,p,q,r)] = 0$$

$$[\lim_{n\to\infty} \overline{H}_L[\mathcal{F}(G)(z,p,q,r), \mathcal{F}(G)(z_n,p_n,q_n,r_n)] = 0].$$

We say that $\mathcal{F}(G)$ is weakly-weakly sequentially upper [lower] semicontinuous on $K_\lambda\times M_\lambda\times S_\lambda$, if for every weakly closed set $M\subset L(P,\mathbb{R}^n)$ the set

$$\mathcal{F}(G)^{-1}(M) = \{(z,p,q,r)\in K_\lambda\times M_\lambda\times S_\lambda : \mathcal{F}(G)(z,p,q,r)\cap M \neq \emptyset\}$$

$$[\mathcal{F}(G)^{-}(M) = \{(z,p,q,r)\in K_\lambda\times M_\lambda\times S_\lambda : \mathcal{F}(G)(z,p,q,r)\subset M\}]$$

is sequentially weakly closed in the following sense: for every sequence $\{(z_n,p_n,q_n,r_n)\}$ of $\mathcal{F}(G)^{-1}(M)$ $[\mathcal{F}(G)^{-}(M)]$ such that $\|z-z_n\|_C \to 0$, $p_n \rightharpoonup p$, $q_n \rightharpoonup q$, $r_n \rightharpoonup r$, as $n\to\infty$ one has $(z,p,q,r)\in\mathcal{F}(G)^{-1}(M)$ $[\mathcal{F}(G)^{-}(M)]$, where "$\rightharpoonup$" denotes a

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

weak convergence. Similarly, we say that $\mathcal{F}(G)$ is strongly-weakly sequentially upper [lower] s emicontinuous on $K_\lambda \times M_\lambda \times S_\lambda$, if for every closed set $M \subset L(P,\mathbb{R}^n)$ the set $\mathcal{F}(G)^{-1}(M)$ $[\overline{\mathcal{F}(G)}^{-}(M)]$ is sequentially weakly closed in the above sense. The weak-strong sequential upper and lower semicontinuity we define similarly as above.

It will be convenient to denote the weak-weak, strong-weak and weak-strong upper [lower] sequential semicontinuities by (w-w.s.u.s.c), (s-w.s.u.s.c) and (w-s.s.u.s.c) [(w-w.s.l.s.c.), (s-w.s.l.s.c) and (w-s.s.l.s.c)], respectively. Instead of (s-s.s.u.s.c) and (s-s.s.l.s.c) we write (u.s.c) and (l.s.c), respectively.

We say that G is Lipschitz continuous with respect to its last four variables, if there is a Lebesgue integrable function $k:P \to \mathbb{R}$ such that for every (z,p,q,r), $(\bar{z},\bar{p},\bar{q},\bar{r}) \in K_\lambda \times M_\lambda \times S_\lambda$ and a. e. $(x,y) \in P$ one has

$$H[G(x,y,z,p,q,r), G(x,y,\bar{z},\bar{p},\bar{q},\bar{r})] \leq \tag{0.8}$$
$$\leq k(x,y) \max (\|z-\bar{z}\|_C, \|p-\bar{p}\|_X, \|q-\bar{q}\|_Y, \|r-\bar{r}\|_L).$$

A set-valued function $G \in H_\lambda(P,\mathbb{R}^n)$ is said to be A-Lipschitz continuous with respect to z,p,q,r, if there is a continuous function $K:P \to \mathbb{R}$ with $K(x,0) = K(0,y) = 0$ for $x \in [0,a]$, $y \in [0,b]$ such that

$$\int_0^x \int_0^y h[G(s,t,z,p,q,r), G(s,t,\bar{z},\bar{p},\bar{q},\bar{r})]dsdt \leq \tag{0.9}$$
$$\leq K(x,y) \max (\|z-\bar{z}\|_C, \|p-\bar{p}\|_X, \|q-\bar{q}\|_Y, \|r-\bar{r}\|_L)$$

for $(x,y) \in P$ and $(z,p,q,r),(\bar{z},\bar{p},\bar{q},\bar{r}) \in K_\lambda \times M_\lambda \times S_\lambda$.

We say that $G \in H_\lambda(P,\mathbb{R}^n)$ is weakly-strongly sequentially upper [lower] semicontinuous with respect to z,p,q,r, if for every $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$ and every sequence $\{(z_n,p_n,q_n,r_n)\}$ of $K_\lambda \times M_\lambda \times S_\lambda$ such that $\|z-z_n\|_C \to 0$, $\|p-p_n\|_X \to 0$, $\|q-q_n\|_Y \to 0$, $\|r-r_n\|_L \to 0$, as $n \to \infty$, we have

$$\lim_{n\to\infty} \bar{h}\left[\iint_U G(x,y,z_n,p_n,q_n,r_n)dxdy, \iint_U G(x,y,z,p,q,r)dxdy\right] = 0$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\Big[\lim_{n\to\infty} \bar{h}\Big[\iint_U G(x,y,z,p,q,r)\,dxdy,\ \iint_U G(x,y,z_n,p_n,q_n,r_n)\,dxdy\Big] = 0\Big]$$

for each measurable set $U \subset P$.

We say that G is strongly-weakly sequentially upper [lower] semicontinuous with respect to z,p,q,r, if for every $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$ and every sequence $\{(z_n,p_n,q_n,r_n)\}$ of $K_\lambda \times M_\lambda \times S_\lambda$ such that $\|z-z_n\|_C \to 0$, $p_n \to p$, $q_n \to q$ and $r_n \to r$, as $n\to\infty$, one has

$$\lim_{n\to\infty} \iint_P \bar{h}[G(x,y,z_n,p_n,q_n,r_n),\ G(x,y,z,p,q,r)]\,dxdy = 0 \tag{0.10}$$

$$\Big[\lim_{n\to\infty} \iint_P \bar{h}[G(x,y,z,p,q,r),\ G(x,y,z_n,p_n,q_n,r_n)]\,dxdy = 0\Big].$$

We say that G is weakly-weakly sequentially upper [lower] semicontinuous with respect to z,p,q,r, if for every $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$ and every sequence $\{(z_n,p_n,q_n,r_n)\}$ from $K_\lambda \times M_\lambda \times S_\lambda$ such that $\|z-z_n\|_C \to 0$, $p_n \to p$, $q_n \to q$ and $r_n \to r$, as $n\to\infty$, (0.10) holds for every measurable set $U \subset P$. Similarly, we define strong-strong sequentially upper [lower] semicontinuity of $G \in H_\lambda(P,\mathbb{R}^n)$ with respect to its last four variables and write (w-w.s.u.s.c), (w-s.s.u.s.c), (s-w.s.u.s.c), [(w-w.s.l.s.c), (w-s.s.l.s.c), (s-w.s.l.s.c), (s-s.u.s.c) and (s-s.l.s.c)], respectively, for the types of semicontinuity of $G \in H_\lambda(P,\mathbb{R}^n)$ with respect to z, p, q, r. We say that $G \in H_\lambda(P,\mathbb{R}^n)$ is upper [lower] semicontinuous (upper [lower] semicontinuous weakly) with respect to z,p,q,r, if for every $(z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$ and every sequence $\{(z_n,p_n,q_n,r_n)\}$ of $K_\lambda \times M_\lambda \times S_\lambda$ such that $\|z-z_n\|_C \to 0$, $\|p_n-p\|_X \to 0$, $\|q_n-q\|_Y \to 0$ and $\|r_n-r\|_L \to 0$, $[\|z-z_n\|_C \to 0$, $p_n \to p$, $q_n \to q$, $r_n \to r]$, as $n\to\infty$, one has

$$\lim_{n\to\infty} \bar{h}[G(x,y,z_n,p_n,q_n,r_n),\ G(x,y,z,p,q,r)] = 0$$

$$\Big[\lim_{n\to\infty} \bar{h}[G(x,y,z,p,q,r),\ G(x,y,z_n,p_n,q_n,r_n)] = 0\Big]$$

for a.e. $(x,y) \in P$. In the above cases we will write (u.s.c),

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$[(l.s.c)]$ $((u.s.c.w), [(l.s.c.w)])$.

Finally, we say that $G \in H_\lambda(P,\mathbb{R}^n)$ is A-continuous with respect to z,p,q,r, if the function $K_\lambda \times M_\lambda \times S_\lambda \ni (z,p,q,r) \longrightarrow G(\cdot,\cdot,z,p,q,r) \in A(P,\mathbb{R}^n)$ is continuous on $K_\lambda \times M_\lambda \times S_\lambda$. It is clear that $G \in (P,\mathbb{R}^n)$ is A-continuous with respect to z,p,q,r if and only if it is (s-s.l.s.c) and (s-s.u.s.c) with respect to z, p, q, r. We say that

$$G : P \times C(P,\mathbb{R}^n) \times C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n) \times L(P,\mathbb{R}^n) \longrightarrow Comp(\mathbb{R}^n)$$

has the Volterra property, if $G(x,y,p,q,r) = G(x,y,\bar{z},\bar{p},\bar{q},\bar{r})$ for every $(x,y) \in P$ and

$$(z,p,q,r),(\bar{z},\bar{p},\bar{q},\bar{r}) \in L(P,\mathbb{R}^n) \times C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n) \times L(P,\mathbb{R}^n)$$

such that $z|_\Pi = \bar{z}|_\Pi$, $p|_\Pi = \bar{p}|_\Pi$, $q|_\Pi = \bar{q}|_\Pi$, $r|_\Pi = \bar{r}|_\Pi$, where $w|_\Pi$ denotes restriction of $w : P \longrightarrow \mathbb{R}^n$ to the rectangle $\Pi := [0,x] \times [0,y] \subset P$.

Similar results for ordinary differential inclusions has been obtained in [9].

1. Existence theorems

We shall present here some sufficient conditions for the existence of solution of an initial value problem

$$(1.1) \quad \begin{cases} z_{xy}(x,y) \in G(x,y,z,z_x,z_y,z_{xy}) & \text{for a.e. } (x,y) \in P, \\ z(x,0) = 0 & \text{for } x \in [0,a], \\ z(0,y) = 0 & \text{for } y \in [0,b], \end{cases}$$

with $G \in H_\lambda(P,\mathbb{R}^n)$. Let $G \in \mathcal{H}_\lambda(P,\mathbb{R}^n)$ be such that

$$(1.2) \quad \iint_P \|G(x,y,z,p,q)\| dxdy \le \lambda \quad \text{for every} \quad (z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda$$

and let $\Omega_\lambda = \bigcup \{\mathcal{F}(G)(z,p,q,u,r) : (z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda\}$. It is easy to see that Ω_λ is relatively sequentially weakly compact, since the family $\{G(\cdot,\cdot,z,p,q,r) : (z,p,q,r) \in K_\lambda \times M_\lambda \times S_\lambda\}$ is uniformly integrable and for every $r \in \Omega_\lambda$ we have $\|r\|_L \le \lambda$. Therefore, $\Omega_\lambda \subset S_\lambda$. Since S_λ is convex and closed, it is also weakly closed in $L(P,\mathbb{R}^n)$. Thus $\overline{[\Omega_\lambda]}^W_L \subset S_\lambda$. Put $\Lambda_\lambda = \overline{co[\Omega_\lambda]}^W_L$. It is clear, by Krein-Smulian theorem (see [5]), that Λ_λ is a

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

convex weakly compact subset of $L(P,\mathbb{R})$. Furthermore, $\Lambda_\lambda \subset S_\lambda$ and of course, $\Lambda_\lambda \neq \emptyset$. Hence, in particular, it follows that the set $\mathcal{K}_\lambda = \mathcal{T}_0(\Lambda_\lambda)$ is convex and compact subset of K_λ.

Now, given above $G \in \mathcal{H}_\lambda(P,\mathbb{R}^n)$ satisfying (1.2), we can define on $P \times \mathcal{K}_\lambda$ a set-valued function $G \cdot \mathcal{D}$ by setting $(G \cdot \mathcal{D})(x,y,z) = G(x,y,z,\mathcal{D}z)$ for $(x,y,z) \in P \times \mathcal{K}_\lambda$, where $\mathcal{D}$ is a mapping defined by (0.4). Similarly, we define on $P \times \Lambda_\lambda$ a mapping $G \cdot \mathcal{T}$ by taking $(G \cdot \mathcal{T})(x,y,r) = G(x,z,\mathcal{T}r,r)$ for $(x,y,r) \in P \times \Lambda_\lambda$ with $\mathcal{T}$ defined by (0.3). Latter on, we shall denote by $\mathcal{F}(G \cdot \mathcal{D})(z)$ and $\mathcal{F}(G \cdot \mathcal{T})(r)$ subtrajectory integrals of $(G \cdot \mathcal{D})(\cdot,\cdot,z)$ and $(G \cdot \mathcal{T})(\cdot,\cdot,r)$ for fixed $z \in \mathcal{K}_\lambda$ and $u \in \Lambda_\lambda$, respectively. Let us observe that, similarly as in the proof of Lemma 3.6 in [12], it can be verified that for every $G \in \mathcal{H}_\lambda(P,\mathbb{R}^n)$ satisfying (1.2) and (s-w.s.l.s.c) with respect to z,p,q,r, a set-valued function $\mathcal{F}(G \cdot \mathcal{D})$ is (l.s.c) on $\mathcal{K}_\lambda$. Hence, immediately from Michael's continuous selection theorem (see [10]) the existence of continuous selections of $\mathcal{F}(G \cdot \mathcal{D})$ follows, whenever G has convex values. Using the procedure presented in [7], the convexity in the above assertion can be removed. In this situation we obtain the following continuous selection theorem.

Theorem 1.1. If $G \in \mathcal{H}_\lambda(P,\mathbb{R}^n)$ is (s-w.s.l.s.c) with respect to (z,p,q,r) and such that (1.2) holds, then there exists a continuous function $g: \mathcal{K}_\lambda \longrightarrow L(P,\mathbb{R}^n)$ such that $g(z) \in \mathcal{F}(G \cdot \mathcal{D})(z)$ for $z \in \mathcal{K}_\lambda$.

We shall need the following fixed point theorems.

Theorem 1.2. (Covitz-Nadler). If (X,ρ) is a complete metric space and $F: X \longrightarrow Cl(X)$ a set-valued contraction mapping i.e., such that $h[F(x),F(y)] \leq k\rho(x,y)$ for every $(x,y) \in X$ with any $k \in [0,1]$, then there exists $x \in X$ such that $x \in F(x)$.

Theorem 1.3. (Kakutani-Ky Fan). Let X be locally convex topological Hausdorff space, S a nonempty compact convex subset of X and CCl(S) a family of all nonempty closed convex subset of S. If $F: S \longrightarrow CCl(S)$ is (u.s.c), then there exists $x \in S$ such that $x \in F(x)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now, as a corollary to Theorem 1.3, we obtain its following weak form (see also [1]).

Theorem 1.4. Let X be a Banach space and S a weakly compact convex subset of X. Suppose F is a set-valued mapping of S into the space of all nonempty weakly closed convex subsets of S. If F is (w-w.s.u.s.c) on S, then there is $x \in S$ such that $x \in F(x)$.

Finally, from the continuous selection theorem of Michael and the fixed point theorem of Schauder the following fixed point theorem follows (see [8], [11]).

Theorem 1.5. Let X and Y be Banach spaces, S nonempty bounded closed and convex subset of X and CCl(S) a family of all nonempty closed convex subsets of S. Assume $F: Y \times S \longrightarrow CCl(S)$ is (l.s.c) and H-(u.s.c) on $Y \times S$. If, furthermore, F is such that $F(y,\cdot)$ is a contraction uniformly with respect to $y \in Y$ and $\Gamma: S \longrightarrow Y$ is completely continuous, then there exists $x \in S$ such that $x \in F(\Gamma(x),x)$.

Now, we can prove the following existence theorems.

Theorem 1.6. If $G: P \times C(P,\mathbb{R}^n) \times C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n) \times L(P,\mathbb{R}^n) \longrightarrow Comp(\mathbb{R}^n)$ is such that

(i) $G(\cdot,\cdot,z,p,q,r) \in A(P,\mathbb{R}^n)$ for fixed (z,p,q,r),

(ii) G is Lipschitz continuous on $C(P,\mathbb{R}^n) \times C_x(P,\mathbb{R}^n) \times C_y(P,\mathbb{R}^n) \times L(P,\mathbb{R}^n)$ with respect to z, p, q, r with constant $k<1$,

then (1.1) has at least one solution.

Proof. Let $G \cdot \mathcal{T}$ be defined on $P \times L(P,\mathbb{R}^n)$ by setting

$$(G \cdot \mathcal{T})(x,y,r) = G(x,y,\mathcal{T}r,r) \quad \text{for} \quad (x,y,r) \in P \times L(P,\mathbb{R}^n)$$

and let $\mathcal{F}(G \cdot \mathcal{T})(r)$ be the subtrajectory integrals of $(G \cdot \mathcal{T})(\cdot,\cdot,r)$ for each fixed $r \in L(P,\mathbb{R}^n)$. We can verify that

$$H_L[\mathcal{F}(G \cdot \mathcal{T})(r), \mathcal{F}(G \cdot \mathcal{T})(\bar{r})] \leq K \|r-\bar{r}\|_L \quad \text{for every} \quad r,\bar{r} \in L(P,\mathbb{R}^n).$$

Take $K \in (0,1)$ and put $X = L(P,\mathbb{R}^n)$, $\rho(r,\bar{r}) = \|r-\bar{r}\|_L$ for every $r,\bar{r} \in X$. Obviously (X,ρ) is a complete metric space and $\mathcal{F}(G \cdot \mathcal{T})$ is a set-valued contractive mapping of X into Cl(X), since, by (i) of Lemma 1.4 in [12], $\mathcal{F}(G \cdot \mathcal{T})(r) \in Cl(L(P,\mathbb{R}^n))$ for every $r \in X$. Now, by Theorem 1.2, there is $r \in X$ such that $r \in \mathcal{F}(G \cdot \mathcal{T})(r)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Put $z=\mathcal{T}_0 r$. One obtains $z(x,0)=0$, $z(0,y)=0$ and $z_{xy}(x,y)\in$ $\in G(x,y,z,z_x,z_y,z_{xy})$ for a.e. $(x,y)\in P$. Then z is a solution of (1.1).

Theorem 1.7. If $G:P\times C(P,\mathbb{R}^n)\times C_x(P,\mathbb{R}^n)\times C_y(P,\mathbb{R}^n)\times L(P,\mathbb{R}^n) \to$ $\to \mathrm{Comp}(\mathbb{R}^n)$ is such that

(i) $G(\cdot,\cdot,z,p,q,r)\in A(P,\mathbb{R}^n)$ for fixed (z,p,q,r),

(ii) G is A-Lipschitz continuous with respect to z,p,q,r with Lipschitz function K,

(iii) $K(a,b)<1$,

then (1.1) has at least one solution.

Proof. Take $L=K(a,b)$ and put $X=L(P,\mathbb{R}^n)$, $\rho(r_1,r_2)=\|r_1-r_2\|_L$. Similarly as in the proof of Lemma 2.6 in [12], we obtain

$$H_L[\mathcal{F}(G)(z,p,q,r),\ \mathcal{F}(G)(\bar z,\bar p,\bar q,\bar r)]\le$$
$$\le K(a,b)\ \max(\|z-\bar z\|_C, \|p-\bar p\|_x, \|q-\bar q\|_y, \|r-\bar r\|_L)$$

for $(z,p,q,r),(\bar z,\bar p,\bar q,\bar r)\in C(P,\mathbb{R}^n)\times C_x(P,\mathbb{R}^n)\times C_y(P,\mathbb{R}^n)\times L(P,\mathbb{R}^n)$. Hence, in particular it follows $H_L[\mathcal{F}(G\cdot\mathcal{T})(r),\ \mathcal{F}(G\cdot\mathcal{T})(\bar r)]\le$ $\le L\|r-\bar r\|_L$. Now, similarly as in the proof of Theorem 1.6, it follows the existence of $r\in L(P,\mathbb{R}^n)$ such that $z=\mathcal{T}_0 r$ is a solution of (1.1).

If the condition (iii) of Theorem 1.7 is not satisfied, we can obtain the following local existence theorem.

Theorem 1.8. If $G:P\times C(P,\mathbb{R}^n)\times C_x(P,\mathbb{R}^n)\times C_y(P,\mathbb{R}^n)\times L(P,\mathbb{R}^n) \to$ $\to \mathrm{Comp}(\mathbb{R}^n)$ is such that

(i) $G(\cdot,\cdot,z,p,q,r)\in A(P,\mathbb{R}^n)$ for fixed (z,p,q,r),

(ii) G is A-Lipschitz continuous with respect to z,p,q,r,

(iii) G has the Volterra property,

then there are $\alpha\in[0,a]$, $\beta\in[0,b]$ and absolutely continuous function $z:P_{\alpha\beta}\to\mathbb{R}^n$ such that

$$\begin{cases} z_{xy}\in G(x,y,z_z,z_y,z_{xy}) & \text{for a.e. } (x,y)\in P_{\alpha\beta}=[0,\alpha]\times[0,\beta], \\ z(x,0)=0 & \text{for } x\in[0,\alpha], \\ z(0,y)=0 & \text{for } y\in[0,\beta]. \end{cases}$$

Proof. If $K(a,b)<1$, then $\alpha=a$, $\beta=b$. Suppose $K(a,b)>1$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Since $K(x,0)=0$, $K(0,y)=0$ and K is continuous on P, there exist $\alpha\in[0,a]$, $\beta\in[0,b]$ such that $K(\alpha,\beta)<1$. Let X be a subspace of $L(P,\mathbb{R}^n)$ containing all functions $r\in L(P,\mathbb{R}^n)$ such that $r(x,y)=0$ for a.e. $(x,y)\in P\setminus P_{\alpha\beta}$. It is clear that $(X,\|\cdot\|_L)$ is also a Banach space. For every $r,\bar{r}\in X$ one obtains

$$\iint_{P\setminus P_{\alpha\beta}} h[(G\cdot\mathcal{T})(x,y,r)(G\cdot\mathcal{T})(x,y,\bar{r})]dxdy=0.$$

Therefore, for every $r,\bar{r}\in X$ one has

$$\iint_{P} h[(G\cdot\mathcal{T})(x,y,r),((G\cdot\mathcal{T})(x,y,\bar{r})]dxdy\le K(\alpha,\beta)\|r-\bar{r}\|_L.$$

Hence, similarly as in the proof of Theorem 1.7, there exists $r\in X$ such that $r(x,y)\in G(x,y,\mathcal{T}r,r)$ for a.e. $(x,y)\in P_{\alpha\beta}$. Therefore $z=\mathcal{T}_0 r$ satisfies (1.1).

Theorem 1.9. Suppose $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ is such that (1.2) holds and G has the Volterra property. If G is Lipschitz continuous with respect to z,p,q,r, then (1.1) has at least one solution in K_λ.

Proof. Let Λ_λ be, as above, a closed subset of $L(P,\mathbb{R}^n)$. We have $\Lambda_\lambda\subset S_\lambda$ and $\mathcal{F}(G\cdot\mathcal{T})(r)\subset\Lambda_\lambda$ for each $r\in\Lambda_\lambda$. Now, let $L\in(0,1)$ and $\alpha\in[0,a]$, $\beta\in[0,b]$ be such that $K(\alpha,\beta)<1$. Then

$$H[\mathcal{F}(G\cdot\mathcal{T})(r),\mathcal{F}(G\cdot\mathcal{T})(\bar{r})]\le L\|r-\bar{r}\|_L \quad \text{for } r,\bar{r}\in\Lambda_\lambda.$$

Put $X=\Lambda_\lambda$ and $\rho(r_1,r_2)=\|r_1-r_2\|_L$. Obviously (X,ρ) is a complete metric space and $\mathcal{F}(G\cdot\mathcal{T})$ is a contractive mapping of X into $(Cl(X),H_L)$. Therefore, by Theorem 1.2, there exists $r\in\Lambda_\lambda$ such that $r\in(G\cdot\mathcal{T})(r)$. Hence it follows that $z=\mathcal{T}_0 r$ is a solution of (1.1) on $P_{\alpha\beta}$.

In a similar way we obtain the result as follows.

Theorem 1.10. If $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ satisfies (1.2) and is A-Lipschitz continuous with respect to z,p,q,r with the Lipschitz function K such that $K(a,b)<1$, then (1.1) has at least one solution.

Now, using Theorem 1.1, we obtain the following existence theorem.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1.11. If $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ is (s-w.l.s.c) with respect to z, p, q, r and such that (1.2) holds, then (1.1) has at least one solution.

Proof. By Theorem 1.1 there exists a continuous function $g:K_\lambda\longrightarrow L(P,\mathbb{R}^n)$ such that $g(z)\in\mathcal{F}(G\cdot\mathcal{D})(z)$ for each $z\in K_\lambda$. Let $\mathcal{T}_g$ be a mapping defined on K_λ by taking $\mathcal{T}_g(z)=\mathcal{T}_0 g(z)$ for each $z\in K_\lambda$. For every $z\in K_\lambda$ we have $g(z)\in\Lambda_\lambda$ and therefore $\mathcal{T}_g(z)\in K_\lambda$, since $K_\lambda=\mathcal{T}_0\Lambda_\lambda$. Clearly, $\mathcal{T}_g$ is continuous on K_λ and K_λ is compact convex subset of the Banach space $C(P,\mathbb{R}^n)$. Then, by Schauder's fixed point theorem, there is $z\in K_\lambda$ such that $z=\mathcal{T}_g(z)=\mathcal{T}_0 z$. Hence $z_{xy}=g(z)\in\mathcal{F}(G\cdot\mathcal{D})(z)$, i.e. z is a solution of (1.1).

We shall now consider an initial value problem (1.1) with $\mathcal{H}_\lambda(P,\mathbb{R}^n)$ having convex values and prove the following existence theorems.

Theorem 1.12. If $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ has convex values, is (w-w.s.l.s.c) with respect to z,p,q,r and such that (1.2) holds, then (1.1) has at least one solution.

Proof. By Lemma 2.4 in [12], a set-valued function $\mathcal{T}_0\mathcal{F}(G)$ is (s-w.s.l.s.c) on $K_\lambda\times M_\lambda\times S_\lambda$, so $\mathcal{T}_0\mathcal{F}(G\cdot\mathcal{D})$ is (l.s.c) on K_λ. Furthermore, by Lemmas 1.4-1.6 in [12], it has compact convex values. Thus, by Michael's selection theorem, there is a continuous function $f:K_\lambda\longrightarrow C(P,\mathbb{R}^n)$ such that $f(z)\in\mathcal{T}_0\mathcal{F}(G\cdot\mathcal{D})(z)$ for $z\in K_\lambda$. Then, f is continuous mapping on compact convex subset of $C(P,\mathbb{R}^n)$ such that $f(K_\lambda)\subset K_\lambda$. Therefore, by Schauder's fixed point theorem, there is $z\in K_\lambda$ such that $z=f(z)$. Since $f(z)\in\mathcal{T}_0\mathcal{F}(G\cdot\mathcal{D})(z)$, we have $z\in\mathcal{T}_0\mathcal{F}(G\cdot\mathcal{D})(z)$.

By definitions of $\mathcal{T}_0$ and $\mathcal{F}(G\cdot\mathcal{D})$, one has $z(x,0)=0$, $z(0,y)=0$ and $z_{xy}(x,y)\in G(x,y,z,z_x,z_y,z_{xy})$ for a.e. $(x,y)\in P$.

Theorem 1.13. If $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ is (w-w.s.u.s.c) with respect to z, p, q, r, has convex values and satisfies (1.2), then (1.1) has at least one solution.

Proof. By virtue of Lemma 2.3 in [12], $\mathcal{F}(G)$ is (w-w.s.u.s.c) on $K_\lambda\times M_\lambda\times S_\lambda$. Hence, by Proposition 2.1 in [9], it fol-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

lows that $\mathcal{F}(G\cdot\mathcal{T})$ is (w-w.s.u.s.c) on Λ_λ. Furthermore, it is easy to see that $\mathcal{F}(G\cdot\mathcal{T})(r)$ is a convex weakly compact subset of Λ_λ for each $r\in\Lambda_\lambda$. Therefore, by Theorem 1.4, there is $r\in\Lambda_\lambda$ such that $r\in\mathcal{F}(G\cdot\mathcal{T})(z)$. Thus, $z=\mathcal{T}_0 r$ is a solution of (1.1).

Theorem 1.14. If $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ has convex values, is continuous with respect to z, Lipschitz continuous with respect to p, q, r, uniformly with respect to z, has the Volterra property and satisfies (1.2), then (1.1) has at least one solution on $P_{\alpha\beta}$ for some $\alpha>0$, $\beta>0$.

Proof. Let us observe that G is continuous with respect to z, p, q, r. Then $\mathcal{F}(G)$ is (l.s.c) and H-(u.s.c) on $K_\lambda\times M_\lambda\times S_\lambda$. Furthermore, $\mathcal{F}(G)$ is Lipschitz continuous with respect to p, q, r, uniformly with respect to $z\in K_\lambda$ and, by eventual changing of the rectangle P on $P_{\alpha\beta}$, its Lipschitz constant L can be taken from the interval (0.1). By (1.2), then there is a nonempty closed bounded convex subset $S:=\Lambda_\lambda\subset L(P,\mathbb{R}^n)$ such that, for every $(z,p,q,r)\in K_\lambda\times M_\lambda\times S_\lambda$ with $K_\lambda=\mathcal{T}\Lambda_\lambda$, $\mathcal{F}(G)(z,p,q,r)$ is a closed convex subset of S. Therefore, the restriction of $\mathcal{F}(G)$ to $K_\lambda\times\Lambda_\lambda$ satisfies the assumptions of Theorem 1.5. Then there is $r\in\Lambda_\lambda\subset S_\lambda$ such that $r\in\mathcal{F}(G\cdot\mathcal{T})(z)$. Thus, $z=\mathcal{T}_0 r$ is a solution of (1.1) on the rectangle $P_{\alpha\beta}$.

Theorem 1.15. Suppose $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ has convex values, satisfies (1.2), is A-continuous with respect to z and is A-Lipschitz continuous with respect to p,q,r, uniformly with respect to z. If a,b>0 are such that K(a,b)<1, then (1.1) has at least one solution.

Proof. Similarly as in proof of Theorem 1.14, we can verify that the restriction $\mathcal{F}(G)$ to $K_\lambda\times\Lambda_\lambda$ satisfies the assumptions of Theorem 1.5. Therefore, there is $r\in\Lambda_\lambda$ such that $r\in\mathcal{F}(G\cdot\mathcal{T})(r)$. Then $z=\mathcal{T}_0 r$ is a solution of (1.1).

2. Properties of the set of solution

Let $G\in\mathcal{H}_\lambda(P,\mathbb{R}^n)$ satisfy (1.2). Denote by S(G) the set of all fixed points of $\mathcal{F}(G\cdot\mathcal{T})$. It is easy to see that $\mathcal{E}(G):=\mathcal{T}_0 S(G)$ is a set of all solutions of (1.1). It is clear that all properties of $\mathcal{E}(G)$ follow immediately from those of

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$S(G)$ and of the mapping $\mathcal{T}_0$. We deal with properties of sets $S(G)$ and mapping $H^C_\lambda(P,\mathbb{R}^n) \ni G \longrightarrow S(G) \subset \Lambda_\lambda$, where $H^C_\lambda(P,\mathbb{R}^n)$ is a subset of $\mathcal{H}_\lambda(P,\mathbb{R}^n)$ containing all $G \in \mathcal{H}_\lambda(P,\mathbb{R}^n)$ with convex values that are (w-w.s.u.s.c.) with respect to z,p,q,r and such that (1.2) holds.

We shall consider $H^C_\lambda(P,\mathbb{R}^n)$ together with the metric δ defined by (0.5). It is not difficult to verify that $(H^C_\lambda(P,\mathbb{R}^n),\delta)$ is a complete metric space. We can also define in $H^C_\lambda(P,\mathbb{R}^n)$ a weak convergence of sequence of $H^C_\lambda(p,\mathbb{R}^n)$ in such a way that $H^C_\lambda(p,\mathbb{R}^n)$ becomes sequentially weakly complete. Namely, we say that a sequence (G_n) of $H^C_\lambda(P,\mathbb{R}^n)$ is weakly converging to $G \in H^C_\lambda(P,\mathbb{R}^n)$, if for every measurable set $U \subset P$ one has

$$\lim_{n\to\infty} \sup_{(z,p,q,r)\in K_\lambda \times M_\lambda \times S_\lambda} h\Big[\iint_U G_n(x,y,z,p,q,r)\,dxdy, \iint_U G(x,y,z,p,q,r)\,dxdy\Big] = 0.$$

We shall prove the following theorems.

Theorem 2.1. For every $G \in H^C_\lambda(P,\mathbb{R}^n)$ the set of all fixed points of $\mathcal{F}(G\cdot\mathcal{T})$ is a nonempty compact subset of A_λ.

Proof. By Theorem 1.13, we have $S(G) \neq \emptyset$. It is clear that $S(G) \subset \Lambda_\lambda$. Since Λ_λ is weakly compact in $L(P,\mathbb{R}^n)$, so $S(G)$ is relatively weakly compact in $L(P,\mathbb{R}^n)$, so $S(G)$ is relatively weakly sequentially compact in $L(P,\mathbb{R}^n)$. Then, by Eberlein-Smulian theorem [7], $\overline{[S(G)]}^W_L$ is weakly compact. We shall show that $S(G)$ is weakly closed. Indeed, let $r \in \overline{[S(G)]}^W_L$. By the Smulian theorem (see [7] Th. 8.12.4), there exists a sequence (r_n) of $S(G)$ weakly converging to r, as $n\to\infty$. Then for every measurable set $U \subset P$ one has

$$\Big| \iint_U r_n(x,y)\,dxdy - \iint_U r(x,y)\,dxdy \Big| \longrightarrow 0, \quad \text{as } n\to\infty.$$

Furthermore, $r_n \in \mathcal{F}(G\cdot\mathcal{T})(r_n)$, since $r_n \in S(G)$ for $n=1,2,\ldots$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then, in particular, for each measurable set $U \subset P$

$$\iint_U r_n(x,y)\,dxdy \in \iint_U G(x,y,\mathcal{T}r_n,r_n)\,dxdy, \quad n=1,2,\ldots$$

Therefore, for each measurable set $U \subset P$ and $n=1,2,\ldots$ one has

$$\operatorname{dist}\left(\iint_U r(x,y)\,dxdy, \iint_U G(x,y,\mathcal{T}r,r)\,dxdy\right) \leq$$

$$\leq \left|\iint_U r(x,y)\,dxdy - \iint_U r_n(x,y)\,dxdy\right| +$$

$$+ \operatorname{dist}\left[\iint_U r_n(x,y)\,dxdy, \iint_U G(x,y,\mathcal{T}r_n,r_n)\,dxdy\right] +$$

$$+ h\left[\iint_U G(x,y,\mathcal{T}r_n,r_n)\,dxdy, \iint_U G(x,y,\mathcal{T}r,r)\,dxdy\right].$$

Then $\iint_U r(x,y)\,dxdy \in \iint_U G(x,y,\mathcal{T}r,r)\,dxdy$ for each measurable set $U \subset P$. Hence, it follows that $r \in S(G)$. Thus $\overline{[S(G)]}^{W}_{L} \subset S(G)$ and, therefore, $S(G)$ is weakly closed in $L(P,\mathbb{R}^n)$.

We shall now consider a set-valued mapping $S(\cdot): H^C_\lambda(P,\mathbb{R}^n) \ni G \to S(G) \in Cl(L(P,\mathbb{R}^n))$. We say that $S(\cdot)$ is (w-w.s.u.s.c) on $2^{H^C_\lambda(P,\mathbb{R}^n)}$, if for every $G \in H^C_\lambda(P,\mathbb{R}^n)$, for each sequence (G_n) of $2^{H^C_\lambda(P,\mathbb{R}^n)}$ weakly converging in $H^C_\lambda(P,\mathbb{R}^n)$ to G and for each sequence (r_n) of $L(P,\mathbb{R}^n)$ such that $r_n \in S(G_n)$, $n=1,2,\ldots$, there exists a subsequence of (r_n) weakly converging to $r \in S(G)$. Similarly, we say that $S(\cdot)$ is (w-s.s.u.s.c) on $2^{H^C_\lambda(P,\mathbb{R}^n)}$, if for every $G \in H^C_\lambda(P,\mathbb{R}^n)$ and each sequence (G_n) of $2^{H^C_\lambda(P,\mathbb{R}^n)}$ converging in the δ-metric topology to G every sequence (r_n) of $L(P,\mathbb{R}^n)$ such that $r_n \in S(G_n)$, $n=1,2,\ldots$, has a subsequence weakly converging to any $r \in S(G)$.

We can also consider a set-valued mapping $\mathcal{C}(\cdot): H^C_\lambda(P,\mathbb{R}^n) \ni G \to \mathcal{C}(G) \in Comp(C(P,\mathbb{R}^n))$. Similarly as above, we can define strong-weak and strong-strong sequentially upper semicontinuity [(s-w.s.u.s.c) and (s-s.s.u.s.c)] of $\mathcal{C}(\cdot)$ on $H^C_\lambda(P,\mathbb{R}^n)$. We simply write (u.s.c) instead of (s-s.s.u.s.c).

Theorem 2.2. The mapping $S(\cdot)$ is (w-w.s.u.s.c) on $2^{H^C_\lambda(P,\mathbb{R}^n)}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let G be an arbitrary fixed element of $H_\lambda^C(P,\mathbb{R}^n)$ and (G_n) a sequence of $H_\lambda^C(P,\mathbb{R}^n)$ to G. Let (r_n) be a sequence of $L(P,\mathbb{R}^n)$ such that $r_n \in S(G_n)$ for each $n=1,2,\ldots$. Then $r_n(x,y) \in G_n(x,y,\mathcal{T}r_n,r_n)$ $n=1,2,\ldots$, for a.e. $(x,y)\in P$. Since $S(G_n)\subset\Lambda_\lambda$, $n=1,2,\ldots$, and Λ_λ is weakly compact, there is a subsequence, say (r_k), of (r_n) weakly converging in $L(P,\mathbb{R}^n)$ to some $r\in\Lambda_\lambda$. For each measurable set $U\subseteq P$ and $k=1,2,\ldots$ one has

$$\operatorname{dist}\Big[\iint_U r(x,y)\,dxdy, \iint_U G(x,y,\mathcal{T}r,r)\,dxdy\Big] \le$$

$$\le \Big|\iint_U [r(x,y)-r_k(x,y)]\,dxdy\Big| +$$

$$+ \operatorname{dist}\Big[\iint_U r_k(x,y)\,dxdy, \iint_U G_k(x,y,\mathcal{T}r_k,r_k)\,dxdy\Big] +$$

$$+ \sup_{(z,p,q,r)} h\Big[\iint_U G_k(x,y,z,p,q,r)\,dxdy, \iint_U G(x,y,z,p,q,r)\,dxdy\Big] +$$

$$+ h\Big[\iint_U G(x,y,\mathcal{T}r_k,r_k)\,dxdy, \iint_U G(x,y,z,\mathcal{T}r,r)\,dxdy\Big].$$

Hence it follows that $r\in S(G)$.

Corollary 2.1. The mapping $S(\cdot)$ is (w-s.s.u.s.c) on $H_\lambda^C(P,\mathbb{R}^n)$.

Indeed, it follows from Theorem 2.2, since δ-converging sequence of $H_\lambda^C(P,\mathbb{R}^n)$ is also weakly converging in $H_\lambda^C(P,\mathbb{R}^n)$.

Corollary 2.2. The mapping $\mathcal{T}(\cdot)$ is (s-w.s.u.s.c.) on $H_\lambda^C(P,\mathbb{R}^n)$, as a mapping of $H_\lambda^C(P,\mathbb{R}^n)$ into $\operatorname{Comp}(C^0(P,\mathbb{R}^n))$, and is (w-w.s.u.s.c) as a mapping of $H_\lambda^C(P,\mathbb{R}^n)$ into subsets $AC^0(P,\mathbb{R}^n)$. In particular, $\mathcal{T}(\cdot)$ is (u.s.c) as a mapping of $H_\lambda^C(P,\mathbb{R}^n)$, into $\operatorname{Comp}(C^0(P,\mathbb{R}^n))$.

REFERENCES

[1] **O. Arino, S. Fautier, I.P. Penot: A fixed point theorem for sequentially continuous mappings with applications to ordinary differential equations, Funkc. Ekv. 27 (1984) 237-279.**

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[2] Z. Arstein: Weak convergence of set-valued functions control, SIAM.J.Control 13 (1985) 865-878.

[3] R.J. Aumann: Integrals of set-valued functions, J. Math. Anal. 12 (1965) 1-12.

[4] L. Deimling: A Carathéodory theory for systems of integral equations, Ann. di Math. (1969) 217-260.

[5] N. Dunford, I.T. Schwartz: Linear operators, Int. Publ. INC, New York 1967.

[6] R.E. Edwards: Functional analysis theory and applications, Holt. Rinehard abd Windson, New York 1965.

[7] A. Fryszkowski: Continuous selection for a class of non-convex multivalued maps, Studia Math. 76 (1983) 163-174.

[8] M. Kisielewicz: Generalized functional-differential equation on neutral type, Ann. Polon Math. 42 (1983) 140-148.

[9] M. Kisielewicz: Subtrajectory integrals of set-valued functions and neutral functional differential inclusions, Funkc. Ekv. (in press).

[10] E. Michael: Continuous selections. I, Ann. Math. (1956) 361-382.

[11] L. Rybiński: Multivalued contraction with parameter, Ann. Polon. Math. 45 (1985) 275-282.

[12] W. Sosulski: Subtrajectory of set-valued functions depending on parameters, Disscusiones Math. (in press).

INSTITUTE OF MATHEMATICS, HIGHER SCHOOL OF ENGINEERING
65-246 ZIELONA GÓRA, POLAND

Received March 3, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Anatolij Dvurečenskij, František Kôpka

ON REPRESENTATION THEOREMS FOR OBSERVABLES IN WEAKLY COMPLEMENTED POSETS

In this note, we present a generalization of a representation lemma for σ-homomorphisms on quantum logics ([1]) to σ-homomorphisms on weakly orthocomplemented σ-posets. As a special case we obtain also a representation lemma in F-quantum spaces ([2],[3]). This result enables us to prove a variant of the ergodic theorem in F-quantum spaces ([4],[5]) and probably some other limit theorems, too ([7]).

We shall work with a special kind of partially ordered sets.

Definition 1. A partially ordered set M with a mapping $a \to a'$ is said to be a weakly orthocomplemented σ-poset, if

(i) $(a')' \geq a$ for every $a \in M$;

(ii) $a, b \in M$, $a \leq b$ implies $b' \leq a'$;

(iii) if $(a_i)_i \subset M$, $a_i \leq a_j'$ $(i \neq j)$, then there exists $\bigvee_i a_i$ in M;

(iv) $a \neq a'$ for every $a \in M$.

These posets were studied, e.g. in [13].

We note that for every $a \in M$ we have $a' = a'''$. Indeed, since $a \leq a''$, then $a' \geq a'''$ and $(a')' \leq (a')''$. Analogically, we may show that $\{a \in M : a = a''\} = \{b' : b \in M\}$.

Two elements a,b from M are orthogonal and we write $a \perp b$ if $a \leq b'$.

Examples. 1. Every σ-logic L ([11],[12]) satisfies all the above conditions even of L is only a partially ordered set, not necessarily a lattice.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

2. A set F of functions $f : X \to \langle 0,1\rangle$ is an F-quantum space if the following conditions are satisfied:

a) F contains the constant function 0 and does not contain the constant function 1/2;

b) if $f \in F$ then also $f' = 1-f \in F$;

c) if $f_n \in F$ $(n=1,2,\dots)$, then $\sup_n f_n \in F$.

Evidently every F-quantum space satisfies the above assumptions.

3. Motivated by some physical reasons, J. Pykacz [10] suggested to substitute the property c), in Example 2., by a weaker one:

c1) if $f_n \in F$ $(n=1,2,\dots)$ and $f_n \le f_m' = 1-f_m$ $(n \neq m)$, then $\sup_n f_n \in F$.

Evidently also the weaker form of an F-quantum space satisfies the above assumptions. It is simple to show that it is not true, in general, that $f \vee f' = 1$.

4. A q-σ-algebra ([14]) is a family of subsets of a given set X satisfying the following conditions:

1) $\emptyset \in Q$;

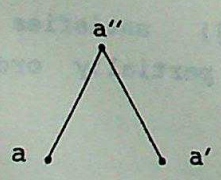
2) if $A \in Q$ then also $X \setminus A \in Q$;

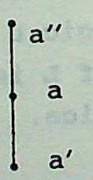
3) if $A_n \in Q$ $(n=1,2,\dots)$ and A_n are pairwise disjoint, then $\bigcup_n A_n \in Q$.

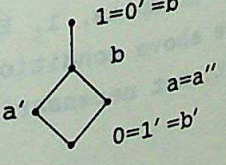
Put now $F = \{\chi_A : A \in Q\}$. Evidently F satisfies the assumptions of Examples 1-3 and hence, the assumptions of Definition 1, too.

5. Let H be a pre-Hilbert space, P be the set of all closed subspaces of H. Then P satisfies the assumptions in Definition 1 (with $A' = \{x \in H : \forall a \in A \ (x,a)=0\}$), but P need not be a logic. There are examples of subspaces of H such that $A'' \neq A$, and $A \vee A' = H$.

6. Below we present some simple examples of weakly orthocomplemented σ-posets:







CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

According to quantum logic theory, we say that two elements a and b from a weakly orthocomplemented σ-poset are orthogonal, and we write $a \perp b$, if $a \le b'$.

Definition 2. Let $\mathcal{B}$ and M be two weakly orthocomplemented σ-posets. A mapping $x : \mathcal{B} \to M$ is called a σ-homomorphism if

1) $x(E^c) = (x(E))'$ for every $E \in \mathcal{B}$;

2) $x(E) \perp x(F)$ if $E \cap F = \emptyset$, $E, F \in \mathcal{B}$;

3) if $E_n \in \mathcal{B}$ $(n=1,2,\ldots)$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$x(\bigcup_n E_n) = \bigvee_n x(E_n).$$

In particular, if $\mathcal{B}=\mathcal{B}(R)$ ($\mathcal{B}(R)$ is the set of all Borel subsets in R), then a σ-homomorphism x is called an observable.

It is not difficult to see that every σ-homomorphism satisfies the following conditions: If A_n $(n=1,2,\ldots)$ are subsets from a Boolean σ-algebra $\mathcal{B}$ and x is an observable, then there exist $\bigvee_n x(A_n)$ and $\bigwedge_n x(A_n)$ in M, and

$$x(\bigcup_n A_n) = \bigvee_n x(A_n), \quad x(\bigcap_n A_n) = \bigwedge_n x(A_n).$$

If $A_1 \subset A_2$, then $x(A_1) \le x(A_2)$.

We say that a nonempty subset $A \subseteq M$ is a Boolean σ-algebra of M if

(i) there are $0_A, 1_A \in M$ such that $0_A \le a \le 1_A$ and $a \vee a' = 1_A$, $a \wedge a' = 0_A$, for any $a \in A$;

(ii) A with respect to $\le, \vee, \wedge, ', 0_A, 1_A$ is a Boolean σ-algebra in the sense of Sikorski [9].

In particular, if $x : \mathcal{B} \to M$ is a σ-homomorphism, where $\mathcal{B}$ is a Boolean σ-algebra, then the range of x, that is $x(\mathcal{B}) = \{x(E) : E \in \mathcal{B}\}$, is a Boolean σ-algebra.

If A_1 and A_2 are two Boolean σ-algebras of M with $A_1 \cap A_2 \neq \emptyset$, then $1_{A_1} = 1_{A_2}$, and $0_{A_1} = 0_{A_2}$. Indeed, let $a \in A_1 \cap A_2$, then $1_{A_1} = a \vee a' = 1_{A_2}$. In [8], the following theorem is proved:

Theorem 1. Let M be a weakly orthocomplemented σ-poset. Let $y, z : \mathcal{B}(R) \to M$ be two observables and $z(\mathcal{B}(R)) \le y(\mathcal{B}(R))$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Then there is a Borel measurable mapping $T : R \to R$, such that $z(E) = y(T^{-1}(E))$ for every $E \in \mathcal{B}(R)$.

Corollary 1. If $\mathcal{B} = \mathcal{B}(Y)$ is a Borel σ-algebra of a complete separable metric space Y, then the statement of Theorem 1 is valid for all observables $z, y : \mathcal{B} \to M$ such that $z(\mathcal{B}) \subset y(\mathcal{B})$.

Proof. Due to a classical theorem of the separable descriptive theory ([6, 33, Th. 2]), we have that $\mathcal{B}(Y)$ is σ-isomorphic to $\mathcal{B}(R)$.

Now we shall present a theorem which is in a certain sense a generalization of Theorem 1. It holds, in particular, in much more general topological spaces, of course, the observables y and z are assumed to satisfy some further conditions.

Let us recall some facts from Boolean algebra theory.

Recall that a (σ-) filter of a Boolean σ-algebra $\mathcal{B}$ of subsets of a set Y is a non-void subset $\mathcal{A} \subseteq \mathcal{B}$ such that

(i) $G_n \in \mathcal{A}$, $n=1,2,$ $(n \geq 1)$ implies $\bigcap_n G_n \in \mathcal{A}$;

(ii) if $G \subseteq H$, $G \in \mathcal{A}$; $H \in \mathcal{B}$, then $H \in \mathcal{A}$.

A filter $\mathcal{A}$ is maximal iff it is a not proper of any proper filter of $\mathcal{B}$, or, equivalently, $A \notin \mathcal{A}$ implies $A^c \in \mathcal{A}$.

A σ-algebra $\mathcal{B}$ is said to be σ-perfect, if any maximal σ-filter $\mathcal{A}$ of $\mathcal{B}$ is determined by some point $t \in Y$, i.e. $\mathcal{A} = \{E \in \mathcal{B} : t \in E\}$, for some $t \in Y$.

Theorem 2. Let $\mathcal{B}$ be a σ-algebra of subsets of a set $Y \neq \emptyset$. Let M be a weakly orthocomplemented σ-poset. Then $\mathcal{B}$ is σ-perfect iff, for any two σ-homomorphisms $y, z : \mathcal{B} \to M$ such that y is injective and $z(\mathcal{B}) \subseteq y(\mathcal{B})$, there exists a $\mathcal{B}$-measurable mapping $T : Y \to Y$, such that $z = y \circ T^{-1}$.

Proof. Let $\mathcal{B}$ be σ-perfect. For any $t \in Y$ we define $F_t = \{y(E) : t \in E \in \mathcal{B}\}$ and $G_t = \{G \in \mathcal{B} : z(G) \in F_t\}$. Due to the assumptions on y, G_t is a maximal σ-filter of $\mathcal{B}$, i.e., there exists some point $T(t) \in Y$, say, such that G_t is determined by $T(t)$. We claim to show that the mapping $T : Y \to Y$ defined

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

via $t \to T(t)$, $t \in Y$, is measurable and $y(T^{-1}(G)) = z(G)$ for any $G \in \mathcal{B}$. Due to our assumptions, y is injective. Hence, for any $G \in \mathcal{B}$, there exists a unique $E \in \mathcal{B}$ such that $z(G) = y(E)$. We assert that $T^{-1}(G) = E$. Let $t \in T^{-1}(G)$, then $T(t) \in G$, and if $t \notin E$ then $t \in E^c$, and $z(G^c) = (z(G))' = (y(E))' = y(E^c)$, i.e. $E^c \in F_t$ and $G^c \in G_t$, $T(t) \in G^c$, which yields the contradiction. If now $t \in E$ then $y(E) \in F_t$. Because $y(E) = z(G)$, we have $G \in G_t$, $T(t) \in G$, i.e. $t \in T^{-1}(G)$. We have proved that $T^{-1}(G) = E$ and $z(G) = y(E) = y(T^{-1}(G))$.

Now we show that if for all σ-homomorphisms z and injective y with $z(\mathcal{B}) \leq y(\mathcal{B})$, there exists a $\mathcal{B}$-measurable transformation $T : T \to Y$ such that $z(B) = y(T^{-1}(B))$, $B \in \mathcal{B}$, then $\mathcal{B}$ is σ-perfect. Suppose the converse, i.e. let $\mathcal{B}$ be not σ-perfect. Denote by X the set of all maximal σ-filters of $\mathcal{B}$. Define, for any $A \in \mathcal{B}$, the set h(A) of all maximal σ-filters of $\mathcal{B}$ which contain A. Put $\mathcal{A} = \{h(A) : A \in \mathcal{B}\}$. Then it is easy to see, that $\mathcal{A}$ is a σ-algebra of subsets of the set $X \neq \emptyset$, and the mapping $A \to h(A)$, $A \in \mathcal{B}$, is a σ-isomorphism ([9]) of $\mathcal{B}$ onto $\mathcal{A}$.

Define a mapping $\varphi : Y \to X$ as follows

$$\varphi(x) = \nabla_x := \{A \in \mathcal{B} : x \in A\}.$$

Then $\varphi^{-1}(A) = h^{-1}(A)$, $A \in \mathcal{A}$, and $\varphi(x) \in \mathcal{A}$ iff $\mathcal{B}$ is σ-perfect. Fix a point $x_0 \in Y$, and let ∇ be a maximal σ-filter of $\mathcal{B}$ which is not determined by any point of Y. Then there is a set $A_0 \in \mathcal{B}$ such that $x_0 \in A_0$ and $A_0 \notin \nabla$, and put $B_0 = h(A_0)$.

Define a mapping $T_x : X \to X$ as follows

$$T_x(t) = \begin{cases} \nabla_{x_0} & \text{if } t \in B_0, \\ \nabla & \text{if } t \notin B_0. \end{cases}$$

It is simple to prove that T_x is an $\mathcal{A}$-measurable transformation. Let y_0 and z_0 be two homomorphisms from $\mathcal{A}$ into M defined via

$$y_0 = y \circ h^{-1}, \quad z_0 = y_0 \circ T_x^{-1}.$$

Then $z := z_0 \circ h$ is a σ-homomorphism from $\mathcal{B}$ into M with $z(\mathcal{B}) \leq y(\mathcal{B})$. We assert that for these z and y there is no $\mathcal{B}$-measurable transformation T with $z = y \circ T^{-1}$.

Indeed, let x be a point from Y that does not belong to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

A_0. Then there are two possible cases:
(i) $T\,x \in A_0$;
(ii) $T\,x \notin A_0$.

In the first case, $z(A_0) = y \circ h^{-1} \circ T_x^{-1} \circ h(A_0) = y(A_0)$ and $y(T^{-1}(A_0)) \neq y(A_0)$ since $T^{-1}(A_0)$ contains the point x. In the second one, there exists a set $A_1 \in \mathcal{B}$ such that $T\,x \in A_1$, and $A_1 \notin \nabla$. Let $A_2 = A_1 \setminus A_0 \in \mathcal{B}$, then $T\,x \in A_2$ and $A_2 \notin \nabla$. Therefore $z(A_2) = y(\emptyset)$ and $y(T^{-1}(A_2)) \neq y(\emptyset)$, which shows that our assumption that $\mathcal{B}$ is not σ-perfect was false. Q.E.D.

Theorem 3. Let $\mathcal{B}$ be a σ-algebra of subsets of a set $Y \neq \emptyset$ containing a countable generator of $\mathcal{B}$. Let M be a weakly orthocomplemented σ-poset. Let $y, z : \mathcal{B} \to M$ be arbitrary two σ-homomorphisms such that $y(E) = y(\emptyset)$ iff $E = \emptyset$ and $z(\mathcal{B}) \subseteq y(\mathcal{B})$. Then there is a $\mathcal{B}$-measurable mapping $T : Y \to Y$ such that $z = y \circ T^{-1}$.

Proof. We claim to show, that $\mathcal{B}$ is σ-perfect. Let $\mathcal{F}$ be a maximal σ-filter of $\mathcal{B}$ and let $(F_i)_{i=0}^{\infty}$ be a countable generator of $\mathcal{B}$. Then either $F_i \in \mathcal{F}$ or $F_i^C \in \mathcal{F}$ $(i = 1,2,\ldots)$. Without loss of generality we may assume that $F_i \in \mathcal{F}$ for every $i = 1,2,\ldots$. Then $C = \bigcap_i F_i \in \mathcal{F}$ and $C \neq \emptyset$.

Let us put $X = \{A \in \mathcal{B} : A \cap C \neq \emptyset$ or $C \subset A\}$. The system X is a σ-algebra containing all F_i $(i = 1,2,\ldots)$. Hence $X = \mathcal{B}$ and consequently, $\mathcal{F}$ is determined by any point $t \in C$.

For the rest of the proof, see Theorem 2. Q.E.D.

Example 7. There exists a σ-perfect σ-algebra of subsets, which is not countably generated.

Indeed, let X be a set such that $\bar{\bar{X}} > \aleph_0$. Denote by

$$\mathcal{B} = \{A \subset X : \bar{\bar{A}} \leq \aleph_0 \text{ or } \bar{\bar{A}}^C \leq \aleph_0\}.$$

The family $\mathcal{B}$ is a σ-algebra. The σ-algebra $\mathcal{B}$ is not countably generated, because the set $\mathcal{F} = \{A \in \mathcal{B} : \bar{\bar{A}}^C \leq \aleph_0\}$ is a maximal σ-filter and is not generated by any point from X. On the other hand, if, similarly as in the second part of the proof of Theorem 2, Y is the set of all maximal σ-filters of $\mathcal{B}$, then $\mathcal{B}$ is σ-isomorphic to the σ-perfect σ-algebra $\mathcal{A} = \{h(A) : A \in \mathcal{B}\}$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

with $h(A) = \{\nabla\in Y : A\in\nabla\}$. This σ-algebra is that of question.

Example 8. Let $\mathcal{B}$ be a σ-algebra of subsets of a set from Example 7. Let $\omega\notin X$ and put $X' = X\cup\{\omega\}$. Let M be the power set of the set X'. Define an injective σ-homomorphism $y : \mathcal{B} \to M$ via

$$y(A) = \begin{cases} A & \text{if } \bar{\bar{A}}\le\aleph_0 \\ A\cup\{\omega\} & \text{if } \bar{\bar{A}}>\aleph_0, \end{cases}$$

fix a point $x_0\in X$ and define a σ-homomorphism $z : \mathcal{B} \to M$ as follows:

$$z(A) = \begin{cases} \{x_0\} & \text{if } x_0\in A \text{ and A is countable} \\ \emptyset & \text{if } x_0\notin A \text{ and A is countable} \\ X'-\{x_0\} & \text{if } x_0\notin A \text{ and A is uncountable} \\ X' & \text{if } x_0\in A \text{ and A is uncountable.} \end{cases}$$

Then $z(\mathcal{B})\le y(\mathcal{B})$ and there is no $\mathcal{B}$-measurable transformation $T : X \to X$ with $z = y\circ T^{-1}$.

Corollary 2. Let $\mathcal{B}$ be a σ-algebra generated by open sets of a Hausdorff topological space Y (in particular, let Y be a metric space) with the second axiom of countability. Then the assertion of Theorem 3 holds.

Theorem 4. Let $\mathcal{B}$ in Theorem 3 be a σ-algebra of subsets of a set Y containing a countable generator $\{F_i : i\in I\}$, such that

(i) $\bigcup_{i\in I} F_i = Y$,

(ii) for all $s,t\in Y$, $s\neq t$, there are disjoint subsets F_j, F_k $(F_j, F_k\in\{F_i : i\in I\})$ such that $s\in F_j$, $t\in F_k$,

(iii) for any disjoint $F_j, F_k\in\{F_i : i\in I\}$ there is $J\subset I$ such that $\bigcup_{i\in J} F_i = (F_j\cup F_k)^c$.

Then the assertion of Theorem 3 holds.

Proof. Define, for any $t\in Y$, a maximal σ-filter

$$G_t = \{F\in\mathcal{B} : t\in F\}.$$

Due to (i), we see that there is at least one $F_i\in G_t$. If not, then $F_i^c\in G_t$ for any $i\in I$, which yields $\emptyset = \bigcap_{i\in I} F_i^c \in G_t$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Therefore, $G_t = \mathcal{B}$.

Put $C = \bigcap \{F_i : F_i \in G_t\}$. Then $C \in G_t$ and the maximality of G_t entails $C \neq \emptyset$. Inasmuch as $\{F_i : i \in I\}$ has the properties (ii) and (iii), C contains exactly one point t. Indeed, let there be two different points $s,t \in C$. By (ii) there are two disjoint sets F_j, F_k with $s \in F_j, t \in F_k$. If, for example, $F_j \in G_t$, then $s,t \in C$, $C \subset F_j$. On the other hand, if $F_j, F_k \notin G_t$, then $F_j^c, F_k^c \in G_t$, consequently, $(F_j \cup F_k)^c \in G_t$. The property (iii) entails that there is a subsystem $\{F_i : i \in J \subset I\}$ such that $\bigcup_{i \in J} F_i = (F_j \cup F_k)^c$. It is clear that $F_i \notin G_t$ for any $i \in J$. Hence, $G_t \not\ni F_j \cup F_k = \left(\bigcup_{i \in J} F_i\right)^c = \bigcap_{i \in J} F_i^c \in G_t$, which contradicts $s \neq t$. Consequently, any singleton $\{t\}$ belongs to $\mathcal{B}$. Q.E.D.

Corollary 3. Let $\mathcal{B}$ in Theorem 3 be a σ-algebra generated by open sets of locally compact Hausdorff topological space Y satisfying the second axiom of countability. Then the assertion of Theorem 3 holds.

Proof. The topological properties of Y imply that there is a countable basis consisting of σ-compact open sets. In other words, there is a system of compact sets $\{F_i : i \in I\}$ satisfying the conditions (i)-(iii) of Theorem 4.

Finally, we note that, in the quantum logic theory, an important role plays the notion of compatible observables, that is such a system of observables whose ranges may be covered by a common σ-algebra. For these observables a so-called calculus is built up, which enables us to construct various operations with them. In particular, it is proved in [12], that if $(y_n)_{n=1}^{\infty}$ is a system of compatible observables, then there is an observable u and a sequence of Borel measurable real valued functions $(f_n)_{n=1}^{\infty}$ such that

(1) $\quad y_n = u \circ f_n^{-1}$, $n \geq 1$.

This result holds for any separable, complete metric space, and in [15] an example is shown of a logic with two

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

compatible σ-homomorphism for which (1) does not hold. We recall that in that example $\mathcal{B}$ is a Borel σ-algebra of a Banach space X of nonmeasurable cardinality whose topological character (i.e. the infimum of the cardinalities of the bases for open sets of X) is bigger than $2^{\aleph_0}$.

Below we present a new simple example of two compatible σ-homomorphisms z and y for which there is no σ-homomorphism u and no $\mathcal{B}$-measurable mappings $f,g : X \longrightarrow X$ such that

$$(2) \quad \begin{cases} z = u \circ f^{-1}, \\ y = u \circ g^{-1}. \end{cases}$$

Example 9. Take into account Example 8. Then for those z and y there is no $f,g : X \longrightarrow X$ for which (2) holds for any σ-homomorphism u.

Proof. Suppose that $y = u \circ g^{-1}$. We assert that g is a one-to-one mapping from X onto X. Indeed let $x \in X$, then $g_t^x = \{t \in X : g(t) = x\}$ is at most countable. If not, then there are at most countably many points $x_1, x_2, \ldots \in X$ such that the range of g, $g(X) \in \mathcal{B}$, consists of $x, x_1, x_2, \ldots$. Calculate $y(g(X)) = \{x, x_1, x_2, \ldots\}$ and $u(g^{-1}(g(X))) = u(X) \neq y(g(X))$. Hence $g_t^x = \{t_1, t_2, \ldots\}$ and $\{x\} = y(\{x\}) = u(g_t^x) = u(\{t_1\}) \cup u(\{t_2\}) \cup \ldots$ which yields that g_t^x is a singleton.

Therefore, for any $x \in X$ there is a unique $t \in X$ with $\{x\} = u(\{t\})$, which says that g is a bijection, and consequently, u is an injective σ-homomorphism.

Suppose now (2) holds. In same way as above, we may show that for the point $x_0 \in X$ there is a unique point $t_0 \in X$ such that $\{x_0\} = z(\{x_0\}) = u(f^{-1}\{x_0\}) = u(\{t_0\})$.

We prove that, due to (2), f is defined only in t_0. In fact, let $x \neq x_0$, then $\emptyset = z(\{x\}) = u(\{t \in X : g(t) = x\})$. The injectivity of u implies $\{t \in X : g(t) = x\} = \emptyset$. Consequently, for our z and y (2) does not hold. Q.E.D.

Finally, we note that the above quoted example of Pták [15] gives a σ-perfect σ-algebra in which (2) does not hold,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

consequently, the σ-perfectness of a σ-algebra is not the sufficient condition in order (2) holds.

REFERENCES

[1] **A. Dvurečenskij, B. Riečan:** On the individual ergodic theorem on a logic, CMUC 21 (1980) 358-391.

[2] **B. Riečan:** A new approach to some notions of statistical quantum mechanics, BUSEFAL 35 (1988), 4-6.

[3] **A. Dvurečenskij, B. Riečan:** On joint observables for F-quantum spaces, BUSEFAL 35 (1988), 10-14.

[4] **B. Harman, B. Riečan:** On the individual ergodic theorem in F-quantum spaces, Submitted to Proc. Poznań 1987.

[5] **S. Pulmannová:** Individual ergodic theorem on a logic, Math. Slovaca 32 (1982), 413-416.

[6] **K. Kuratowski:** Topology I. Academic Press 1966.

[7] **B. Harman, B. Riečan:** On the martingale convergence theorem in quantum theory, In: Trans. Ninth Prague Conf. Infor. Theory. Academia, Prague 1983, 275-280.

[8] **F. Kôpka, B. Riečan:** On representation of observables by Borel measurable functions. Proc. 1-st. Winter School on Meas. Th., Lipt. Ján, Jan. 10-15, 1988, (1988) 68-71.

[9] **R. Sikorski:** Boolean algebras. Springer Verlag 1964.

[10] **J. Pykacz:** Quantum logics and soft fuzzy probability spaces, BUSEFAL, 32 (1987), 150-157.

[11] **V.S. Varadarajan:** Probability in physics and a theorem on simultaneous observability, Comm. Pure Appl. Math. 15 (1962), 189-217 (Errata, 18 1965).

[12] **V.S. Varadarajan:** Geometry of quantum theory. Van Nostrand, Princeton 1968.

[13] **G. Cattaneo, A. Mania:** Abstract orthogonality and orthocomplementation, Proc. Camb. Phil. Soc. 76 (1974), 115-132.

[14] **P. Suppes:** The probability argument for a non-classical logic of quantum mechanics, Phil. Sci. 33 (1966) 14-21.

[15] **P. Pták:** Spaces of observables, Czech. Math. J. 34 (109) (1984) 552-561.

MATHEMATICAL INSTITUTE, SLOVAK ACADEMY OF SCIENCES,
CS - 814 73 BRATISLAVA, CZECHOSLOVAKIA;

TECHNICAL UNIVERSITY CS - 031 04 LIPTOVSKY MIKULÁŠ,
CZECHOSLOVAKIA.

Received March 17, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

EMONSTRATIO MATHEMATICA
Tol. XXIII No 4 1990

Zbigniew Dudek

ON MULTIPLICATIVE OPERATORS IN COMMUTATIVE ALGEBRAS

The concept of a Quasi-Leibniz right invertible operator n a commutative algebra was introduced by Przeworska-Rolewicz 5], [6]. It was observed [2] that there is a canonical 1-1 orrespondence between the set of all multiplicative operators n a algebra X and the set of all Quasi-Leibniz operators in modulo certain natural equivalnce relation.

In this paper we consider the structural properties of ultiplicative operators is a commutative algebra X with the rther goal of utilizing these to obtain a more comprehensive escription of the set of Quasi-Leibniz operators. The main esult of this paper demonstrates that every multiplicative erator induces a decomposition of the algebra into a direct m of its two principal ideals, which are canonically linked the operator. It is also shown that such a decomposition is ossible if and only if there are nontrivial idempotent lements in X.

1. Introduction

Let X denote a commutative algebra, that is a commutative near ring, over the field of scalars K, with the identity ement denoted by e. The field of scalars is assumed to be ther the field of real numbers R, or the field of complex mbers C. The product of two elements $x, y \in X$ shall be noted by xy. The set of all linear operators with domains d ranges in X shall be denoted by L(X). Given operator A in

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$L(X)$, its domain shall be denoted by $Dom(A)$ and its range by $Ran(A)$. The identity map in X shall be denoted by I. Given $z \in X$, the principal ideal generated by z shall be denoted by $\langle z \rangle$ [3]. The direct algebraic sum of two ideals I_1 and I_2 in X shall be denoted by $I_1 \oplus I_2$ [8].

An operator $D \in L(X)$ satisfies the Quasi-Leibniz condition, and itself is called a QL operator [2], if there is a constant $d \in K$ such that

(1.1) $D(xy) = (Dx)y + x(Dy) + d(Dx)(Dy)$ for all $x,y \in Dom(D)$;

The constant d in (1.1) solely depends on the operator D[1]. It is known [1] that if D is a QL operator and d_D denotes the corresponding constant that appears in (1.1), then

(1.2) $$d_{\alpha D} = \frac{d_D}{\alpha} \quad \text{for all } \alpha \in K,\ \alpha \neq 0.$$

An operator $A \in L(X)$ is called multiplicative [2], or M operator, if A preserves the algebraic operations of X, without necessarily $A(e) = e$. We shall not assume here that the difference of an M operator and the identity operator is right invertible as we did in [2]. In all of the follows X will have the above defined meaning with e being its iedntity element. We shall assume that multiplication in X is not trivial, and that the domains of all M operators and QL operators considered in the paper are closed with respect to the underlying algebraic operations from the algebra X.

2. The conncection between M operators and QL operators

Denote by QL(X) the set of all QL operators and by M(X) the set of all M operators in the algebra X. Consider the following two subset of L(X):

(2.1) $$QL^*(X) = \{D \in QL(X): d_D \neq 0 \text{ and } D \neq 0\}$$

and

(2.2) $$M^*(X) = \{A \in M(X): A \neq I\}.$$

Let us introduce the relation $\approx$ in the set $QL^*(X)$ by setting

(2.3) $D_1 \approx D_2$ if $Dom(D_1) = Dom(D_2)$ and $d_{D_1} D_1 = d_{D_2} D_2$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

It is easy to check that the relation $\approx$ is an equivalence relation in the set $QL^*(X)$ and that the equivalence class determined by a $D \in QL^*(X)$ is

$$[D] = \{\alpha D: \alpha \in K \text{ and } \alpha \neq 0\}. \tag{2.4}$$

Moreover, in virtue of (2.1), $d_D = \alpha d_{\alpha D}$ for all $\alpha \in K$, $\alpha \neq 0$. The following theorem relates the sets $M^*(X)$ and $QL^*(X)/\approx$.

Theorem 2.1. (see [2]). There is a 1-1 correspondence between the sets $M^*(X)$ and $QL^*(X)/\approx$ given by the formulas

$$\Delta(D) = (I + d_D D) \in M^*(X) \quad \text{for } D \in QL^*(X), \tag{2.5}$$

and

$$d_D D(\Delta) = (\Delta - I) \in QL^* \quad \text{for } \Delta \in M^*(X). \tag{2.6}$$

The above theorem implies that the knowledge of all M operators in the algebra X may be utilized in describing of all nontrivial QL opreators in X via formula (2.6).

3. Structural properties of M operators in algebra X

It is clear that every inner homomorphism of an algebra X is an M operator. But there are M operators not being inner homomorphisms. This is illustrated by the following two examples.

Example 3.1. Let $X = C(R)$ be the algebra of continuous real valued functions on the real line with the standard vector operations and with the usual pointwise multiplication: $(xy)(t) = x(t)y(t)$ for $t \in R$, $x, y \in C(R)$. Given nonzero number h, define $S_h: X \longrightarrow X$ as the shift operator

$$(S_h x)(t) = x(t + h) \quad \text{for } t \in R, \ x \in X. \tag{3.1}$$

It is clear that S_h is an M operator in X such that $(S_h e)(t) = 1 = e(t)$ for all $t \in R$, i.e. S_h is an inner homomorphism of X.

Example 3.2. Given $\delta \in (0,1)$, let $\Omega_1 = (-\infty, -\delta]$, $\Omega_2 = [\delta, +\infty)$ and $\Omega = \Omega_1 \cup \Omega_2$. Let $X = C(\Omega)$ with the standard vector operations and define $\Delta: X \longrightarrow X$ by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.2)\qquad (\Delta x)(t) = \begin{cases} x(t) & \text{for } t\in\Omega_1 \\ & \text{, for } x\in X. \\ 0 & \text{for } t\in\Omega_2 \end{cases}$$

It is easy to see that Δ is an M operator in X that is not an innear homomorphism of X.

The following propositions describe basic properties of M operators.

Proposition 3.1. Let A be an M operator in X with $e\in Dom(A)$. Then Ran(A) is an algebra with the unit element $\varepsilon=A(e)$.

Proof. If A(e) = e or A(e) = 0, the assertion is obvious. Thus, assume that $\varepsilon=A(e)$ = and $0=A(e)$. First, we will show that Ran(A) is closed with respect to the algebraic operations inherited from X. Let $u, v \in Ran(A)$ and $\alpha, \beta \in K$. Then there are $x, y \in Dom(A)$ such that $u=Ax$, $v=Ay$, and $\alpha u + \beta v = \alpha\, Ax + \beta\, Ay = A(\alpha x + \beta y) \in Ran(A)$.

Furthemore, $uv = Ax\, Ay = A(xy) \in Ran(A)$. Moreover, for every $u\in Ran(A)$ $u\varepsilon = u\varepsilon = Ax\, Ae = A(xe) = Ax = u$, ie. ε is the unit element in Ran(A).

Proposition 3.2. Let A_1, A_2 be M operators in X with domains $Dom(A_1)$ and $Dom(A_2)$. Then, the product

$$(3.3)\qquad A = A_2\, A_1 \text{ with } Dom(A) = A_1^{-1}[Dom(A_2) \cap Ran(A_1)]$$

is also an M operator.

Proof. Linearity and multiplicativity of A is obvious. It remains to show that its domain is a subalgebra closed with respect to algebraic operations of X. Assume that $x, y \in Dom(A)$ and $\alpha, \beta \in K$. Then

$$(3.4)\qquad A_1x, A_1y \in Dom(A_2) \cap Ran(A_1), \text{ so}$$

$$(3.5)\qquad A_1(\alpha x + \beta y) = \alpha\, A_1x + \beta\, A_1y \in Dom(A_2) \cap Ran(A_1), \text{ and}$$

$$(3.6)\qquad A_1(xy) = A_1x\, A_1y \in Dom(A_2) \cap Ran(A_1).$$

Thus $\alpha x + \beta y \in Dom(A)$ and $xy \in Dom(A)$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

The connection between M operators and homomorphisms of X is given by the following proposition.

Proposition 3.3. Let A be an M operator in X with $e \in Dom(A)$ and $Ran(A) = X$. Then A is a homomorphism of Dom(A) onto X.

Proof. It suffices to show that $Ae = e$. Let $\varepsilon = Ae$. Then, for an arbitrary $y \in X$ there is an $x \in Dom(A)$ such that $y = Ax$, and

$$\varepsilon y = y\varepsilon \quad Ax \quad Ae = A(xe) = Ax = y, \tag{3.7}$$

so ε is the unit element in X, and therefore $e = \varepsilon = Ae$. Thus A is a homomorphism of Dom(A) onto X.

Given M operator A in X, it is clear that ker(A) is an ideal in Dom(A). The following theorem explains the "split" present in Example 3.2.

Theorem 3.1. Let A be an M operator in X with $e \in Dom(A)$. Then

(i) $\langle Ae \rangle$ and $\langle e - Ae \rangle$ are commutative algebras with the identity elements Ae and (e - Ae), respectively;

(ii) $X = \langle Ae \rangle \oplus \langle e - Ae \rangle$;

(iii) $Ran(A) \subset \langle Ae \rangle$.

Proof. The case when $Ae = e$ or $Ae = 0$ is trivial. Thus, assume that Ae is different from e and 0.

By definition, $\langle Ae \rangle$ and $\langle e-Ae \rangle$ are principal ideals in X.

To prove (i) it suffices to show that the elements Ae and (e - Ae) are idempotent, since this would imply that for arbitrary elements $x \in \langle Ae \rangle$ and $y \in \langle e-Ae \rangle$, we would have

$$x\,Ae = Ae\,x = x, \tag{3.8}$$

and

$$y(e - Ae) = (e - Ae)y = y. \tag{3.9}$$

But Ae is an idempotent by Proposition 3.1, and for (e - Ae) we have that $(e - Ae)^2 = e - 2Ae + (Ae)^2 = e - Ae$, which proves (i).

To prove (ii), let us assume that $x \in X$, and define

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3.10) $x_1 = x\,Ae \in \langle Ae \rangle$,

and

(3.11) $x_2 = x(e - Ae) \in \langle e - Ae \rangle$.

Then

(3.12) $x = xe = x_1 + x_2$,

therefore $X \subset \langle Ae \rangle + \langle e - Ae \rangle$. Let $u \in \langle Ae \rangle \cap \langle e - Ae \rangle$. Then

(3.13) $u = u\,Ae = u(e - Ae) = 0$,

therefore

(3.14) $\langle Ae \rangle \cap \langle e - Ae \rangle = \{0\}$,

and

(3.15) $X = \langle Ae \rangle \oplus \langle e - Ae \rangle$,

which proves (ii).

To prove (iii), let us notice that for $y \in Ran(A)$ we get from Proposition 3.1 that

(3.14) $y = y\varepsilon = y\,Ae \in \langle Ae \rangle$, that is $Ran(A) \subset \langle Ae \rangle$.

The decomposition described in Theorem 3.1 is possible in algebras with nontrivial elements only, as indicated by the next theorem.

Theorem 3.2. Let z be a nontrivial idempotent element in an algebra X, i.e. $z \neq 0$, $z \neq e$ and $z^2 = z$. Then

(i) $X = \langle z \rangle \oplus \langle e-z \rangle$,

(ii) $\langle z \rangle$ and $\langle e-z \rangle$ are commutative algebras with the identity elements z and $(e-z)$ respectively.

Proof. Define a map $A_z: X \longrightarrow X$ by $A_z(x) = zx$ for $x \in X$. Then A_z is obviously linear and multiplicative, i.e. A_z is an M operator such that $A_z(e) = z$. The remaining part of the proof follows from Theorem 3.1.

4. Conclusion

The results obtained can be dually stated in terms involving QL operators via formulas (2.5) and (2.6). However, in order to use M operators to investigate QL operators that are right invertible, one needs a characterization of those multiplicative operators M for which the difference M-I is

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

right invertible. Disregard of this interdependence may lead to erroneous assertions about QL operators.

Przeworska-Rolewicz asserts in [6] that in a commutative algebra, a right invertible operator D with a nonzero kernel, is a QL operator iff the operator $\hat{D} = (I + d_D D)$ is right invertible and multiplicative ([6], Proposition 6.1.1, p. 360). The following counterexample refutes this assertion.

Example 4.1. Let X be as in Example 3.2, and define D by

$$(Dx)(t) = \begin{cases} x(t+1) - x(t), & t \in \Omega_1 \\ -x(t), & t \in \Omega_2 \end{cases}, \quad x \in C(\Omega),$$

that is, D is a direct sum of a difference operator on $C(\Omega_1)$ and of -I on $C(\Omega_2)$. One may prove that D is right invertible with a nonzero kernel [4]. It is easy to verify that it satisfies the QL condition with the corresponding constant $d_D = 1$. However, the operator $\hat{D} = I + d_D D$ given by

$$(\hat{D}x)(t) = \begin{cases} x(t+1), & t \in \Omega_1 \\ 0, & t \in \Omega_2 \end{cases}, \quad x \in C(\Omega),$$

is not right invertible, though multiplicative.

REFERENCES

[1] Z. Dudek: Some properties of Wronskian in D-R spaces of the type QL, II, Demonstratio Math. 13 (1980), 987-993.

[2] Z. Dudek: On decompositions of Quasi-Leibniz D-R algebras, Demonstratio Math. 14 (1981), 745-757.

[3] R.E. Johnson: University Algebra. Englewood Cliffs N.Y., Prentice Hall Inc. 1966.

[4] V. Lakshmikantham, D. Trigiante: Theory of difference equations, numerical methods and aplications: Academic Press, Inc., Boston, San Diego, New York, Berkeley, London, Sydney, Tokyo, Toronto, 1988.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[5] **D. Przeworska-Rolewicz:** On trigonometric identity for right invertible operators, Comment. Math. 21 (1978), 267-278.

[6] **D. Przeworska-Rolewicz:** Algebraic Analysis. PWN-Polish Scientific Publishers Warsaw, D. Reidel Publishing Company Boston-Dordrecht(Lanaster) Tokyo, 1988.

[7] **H. von Trotha:** Structure properties of D-R vector spaces, Dissertations Math. 184, Warsaw, 1978.

[8] **O. Zariski, P.Samuel:** Commutative Algebra, D. van Nostrand Company, Inc., Princeton, New Jersey, Toronto and London, 1967.

DEPARTMENT OF MATHEMATICS, TRENT UNIVERSITY, PETERBOROUGH, ONT. CANADA

Received March 28, 1989; revised version March 20, 1990.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Roger Yue Chi Ming

INJECTIVITY AND ANNIHILATORS

Introduction

A non-trivial generalization of injectivity, noted m-injectivity, is introduced. Rings satisfying certain annihilator conditions are studied. LIS rings and m-injective modules are defined below. This note contains the following results: (1) A is biregular iff A is a semi-prime ring such that for any finite number of elements $a_1,\ldots,a_n \in A$, $\sum_{i=1}^{n} Aa_iA$ is the complement left annihilator of an element of A; (2) If A is semi-prime LIS, then (a) the left annihilator of any left ideal is generated by a central idempotent; (b) any complement left ideal which is two-sided is generated by a central idempotent; (3) Any left and right non-singular LIS ring is a Baer ring; (4) Any m-injective left ideal of A is a direct summand of ${}_AA$; (5) If A is left m-injective, then A/J is von Neumann regular, where the Jacobson radical J of A coincides with the left singular ideal of A; (6) A is quasi-Frobeniusean iff A is left Artinian left and right m-injective; (7) If A is left continuous left m-injective; then $A = F \oplus D \oplus C$, where F, D are isomorphic left ideals of A and C is a left right continuous strongly regular ring with identity.

Throughout, A denotes an associative ring with identity and A-modules are unital. J,Z stand respectively for the Jacobson radical and the left singular ideal of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal. A is called left non-singular iff Z=0. Recall

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

that (a) A is von Neumann regular iff $b \in bAb$ for each $b \in A$; A is a left V-ring iff every simple left A-module is injective; (c) A is biregular iff for each $a \in A$, AaA is generated by a central idempotent. It is well-known that von Neumann regular rings need not be V-rings and the converse is not true either. However, a theorem of I. Kaplansky assert that a commutative ring is regular iff it is a V-ring. Some time ago, we introduces p-injective modules to study regular rings, V-rings and associated rings (cf. for example, [10]). As before, a left A-module M is called p-injective iff for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. It is well-known that A is von Neumann regular iff every left (right) A-module is flat. Another characterization is "every left (right) A-module is p-injective". Flatness and p-injectivity are distinct concepts. Note that if I is a p-injective left ideal of A, then A/I is a flat left A-module. A is called left p-injective iff ${}_AA$ is p-injective. A theorem of M. Ikeda - T. Nakamaya [4] guarantees that A is left p-injective iff every principal right ideal of A is a right annihilator. As an analogy to left V-rings, A is called a left p-V-ring if every simple left A-module is p-injective. P-V-rings are studied in [1], [8], [9].

An important generalization of strongly regular rings is the class of biregular rings. We start with a characterization of biregular rings.

Theorem 1. The following conditions are equivalent:

(1) A is biregular;

(2) A is semi-prime such that for any finite number of elements $a_1,\ldots,a_n \in A$, $\sum_{i=1}^{n} Aa_iA$ is the complement left annihilator of an element of A;

(3) A is semi-prime such that for any finite number of elements $a_1,\ldots,a_n \in A$, $\sum_{i=1}^{n} Aa_iA$ is the complement right annihilator of an element of A.

Proof. If A is biregular, for any elements

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$a_1,\dots,a_n \in A$, $\sum_{i=1}^{n} Aa_iA$ is generated by a central idempotent [11, Remark 1].

Therefore, (1) implies (2) and (3).

Assume (2). If $z \in Z$, let $AzA = 1(w)$ for some $w \in A$. Then $AzA = 1(AwA)$. Since A is semi-prime, $AwA \cap AzA = AwA \cap 1(AwA) = 0$. Now $AwA + AzA$ is a complement left ideal by hypothesis. In as much as $AwA \oplus AzA = AwA \oplus 1(AwA)$ is an essential left ideal of A, we have $AwA \oplus AzA = A$, which yields $z = 0$ (since Z contains no non-zero idempotent), whence $Z=0$. For any $b \in A$, let $AbA = 1(c)$ for some $c \in A$. Then $AbA = 1(AcA)$ and $r(AbA + AcA) = r(AbA) \cap r(AcA)$ implies that $r(AbA + AcA) = 0$ (because $r(AcA) = 1(AcA) = AbA$). Since $Z = 0$ and $AcA \oplus AbA = AcA \oplus 1(AcA)$ is an essential left ideal which is the left annihilator of an element, the $AcA \oplus AbA = A$ which proves that AbA is generated by a central idempotent. Thus (2) implies (1).

Similarly, (3) implies (1).

Rings satisfying the maximum condition on right annihilators constitute a well-known generalization of right Noetherian rings (for example, in the theory of right Goldie rings). We now give a condition for such rings to be biregular. If A is semi-prime, any ideal of A which is a complement left annihilator.

Then, with slight modifications, the proof of Theorem 1 yields

Proposition 2. Let A be a semi-prime ring with maximum condition on right annihilators. If, for any finite number of elements $a_1,\dots,a_n \in A$, $\sum_{i=1}^{n} Aa_iA$ is the complement left ideal of A, then A is biregular.

(Note that the rings in Proposition 2 need not be von Neumann regular.)

We now consider the following generalization of right self-injective rings (cf. [4, Theorem 1]).

Notation. Write "A is a LIS ring" if A has the following property: for any right ideals R_1, R_2 of A, $1(R_1 \cap R_2) =$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$= 1(R_1) + 1(R_2)$.

Remark 1. If every one-sided ideal of A is an annihilator, then A is LIS.

Proposition 3. Let A be a semi-prime LIS ring. Then any left ideal which is the left annihilator of a left ideal is generated by a central idempotent.

Proof. Let $I = 1(S)$ for some left ideal S of A. Then $I = 1(T)$, where $T = SA$ is an ideal of A. Now $I \cap T = 1(T) \cap T = 0$ (because A is semi-prime) which implies that $1(I) + 1(T) = 1(I \cap T) = A$. Since $1(I) \cap 1(T) = 1(I) \cap I = 0$, then $A = 1(I) \oplus I$. In as much as A is semi-prime and I is an ideal of A, then I is generated by a central idempotent.

Corollary 3.1. If A is semi-prime LIS, then the right annihilator of any right ideal of A is generated by a central idempotent.

Corollary 3.2. Let A be a semi-prime right self-injective ring. Then any finitely generated left ideal of A which is an ideal is generated by a central idempotent.
(Apply [4, Theorem 1].)

Corollary 3.3. If A is semi-prime LIS, any one-sided complement ideal which is an ideal of A is generated by a central idempotent.

The next corollary is motivated by Theorem 1.

Corollary 3.4. The following conditions are equivalent for a LIS ring A: (a) A is biregular; (b) A is semi-prime such that AaA is a complement left ideal for each $a \in A$; (c) A is semi-prime such that AaA is a complement right ideal for each $a \in A$.

Remark 2. Corollary 3.2 implies the following: IF A is semi-prime right self-injective, for any non-zero element u in Z or J, Au cannot be an ideal of A.

Proposition 4. Let A be LIS ring whose left annihilators coincide with complement left ideals. Then any complement left of A is a direct summand of ${}_AA$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Let C be a complement left of A. Then there exists a complement left ideal K of A such that $L = C \oplus K$ is an essential left ideal of A. Since any left annihilator ideal is a complement left ideal, then $Z = 0$. Consequently, $r(L) = 0$ which implies that $0 = r(C) \cap r(K)$. Since A is LIS, then $A = l(r(C) \cap r(K)) = l(r(C)) + l(r(K)) = C + K$ (because C,K are left annihilators). Since $C \cap K = 0$, then $C \oplus K = A$.

It is well-known that Z=0 iff every left annihilator ideal in A is complement left ideal [2, Lemma 1].

Corollary 4.1. A left non-singular, left p-injective LIS ring whose complement left ideals are left annihilators is left continuous regular.

Corollary 4.2. The following conditions are equivalent: (1) A is left and right self-injective regular; (2) A is right self-injective, left non-singular left p-injective whose complement left ideals are left annihilators.

It is well-known that a left A-module M is injective iff for any left ideal I of A, any left A-homomorphism $g : I \longrightarrow M$, there exists an element $y \in M$ such that $g(b) = by$ for all $b \in I$.

We introduce the following generalization of injectivity.

Definition. A left A-module Q is called m-injective if, for any left ideal I of A, any left A-monomorphism $h : I \longrightarrow Q$, there exists $y \in Q$ such that $h(b) = by$ for all $b \in I$.

Remark 3. Any direct summand of a m-injective left A-module is m-injective.

Proposition 5. Let I be a m-injective left ideal of A. Then I is generated by an idempotent.

Proof. If $j : I \longrightarrow I$ is the identity map, ${}_AI$ is m-injective, there exists $y \in I$ such that $j(b) = by$ for all $b \in I$. Therefore $I = Iy \subseteq Ay \subseteq I$ yields $I = Ay$. Now $y = j(y) = y^2$ which proves the proposition.

Corollary 5.1. If every principal left ideal of A is m-injective, there A is von Neumann regular.

Corollary 5.2. If, for each $a \in A$, AaA is a m-injective left

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ideal, then A is biregular.

Proof. For any $a \in A$, $AaA = Ae$, $e = e^2 \in A$. Therefore A is semi-prime hence e is central in A.

Lemma 6. Any p-injective simple left A-module is m--injective.

Proof. Let V be a simple p-injective left A-module, I a left ideal of A, $g: I \longrightarrow V$ a non-zero left A-monomorphism. Then $g(I) = V$ which implies that g is an isomorphism. Therefore I is a principal left ideal and since ${}_AV$ is p-injective, there exists $v \in V$ such that $g(b) = bv$ for all $b \in I$. This proves that ${}_AV$ is m-injective.

Remark 4. Since left p-V-rings generalize effectively left V-rings, then m-injectivity non-trivially generalizes incjectivity.

Proposition 7. Suppose that every m-injective left A-module is injective. The following conditions are then equivalent:

(1) A is a left Noetherian, left V-ring;

(2) A is a left p-V-ring satisfying the maximum condition on left annihilators.

Proof. Obviously, (1) implies (2).

Assume (2). By Lemma 6, is a left V-ring. Let M be a completely reducible left A-module. If I is a left ideal of A; $g: I \longrightarrow M$ a non-zero left A-monomorphism, then $g(I) = N$ is a direct summand of ${}_AM$ which implies that ${}_AN$ is completely reducible. Now $g: I \longrightarrow N$ is an isomorphism which implies that I is a completely reducible left ideal of A. Therefore I is a direct sum of a family of minimal left ideals of A. Since any minimal left ideal is a direct summand of ${}_AA$ and since A satisfies the maximum condition on left annihilators, then I is a finite direct sum of minimal left ideals; whence ${}_AI$ is injective. Therefore g extends to a left A-homomorphism $h: A \longrightarrow M$, proving that ${}_AM$ is m-injective. By hypothesis, ${}_AM$ is injective and by [3, Corollary 20,3E], A is left Noetherian. Thus (2) implies (1).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

A is called left m-injective if ${}_AA$ is m-injective. An element b of A is called left regular if $l(b) = 0$ [3].

Proposition 8. Let A be a left m-injective ring. Then

(1) Any left regular element of A is right invertible;

(2) $J = Z$;

(3) A/J is von Neumann regular.

Proof. (1) Let $c \in A$ such that $l(c) = 0$. If $g: Ac \longrightarrow A$ is the left A-homomorphism defined by $g(ac) = a$ for all $a \in A$, then g is a well-defined left A-monomorphism. Therefore $g(c) = 1$ and since ${}_AA$ is m-injective, there exists $d \in A$ such that $g(c) = cd$ which implies that c is right invertible in A.

(2) Let $z \in Z$. Then for any $a \in A$, $l(1-za) = 0$ which implies that $(1-za)$ is right invertible in A. Therefore $z \in J$, showing $Z \subseteq J$. Suppose there exists $u \in J$ such that $u \notin Z$. Let K be a non-zero left ideal of A such that $l(u) \cap K = 0$. Then $Ku \neq 0$ and if $g: Ku \longrightarrow A$ is the left A-homomorphism defined by $g(ku) = k$ for all $k \in K$, g is indeed a monomorphism which implies that there exists $d \in A$ such that $g(ku) = kud$ all $k \in K$. Therefore $K = Kud$ which yields $K(1-ud) = 0$. Now $(1-ud)\, w = 1$ for some $w \in A$ (because $u \in J$) which implies that $K = K(1-ud)\, w = 0$, a contradiction ! This proves that $J \subseteq Z$ whence $J = Z$.

(3) Let $b \in A$, $b \notin J$. Then there exists a non-zero complement left ideal C of A such that $L = l(b) \oplus C$ is an essential left ideal of A. Define $f: Cb \longrightarrow A$ by $f(cb) = c$ for all $c \in C$. then f is a monomorphism which implies that there exists $s \in A$ such that $f(cb) = cbs$ for all $c \in C$. Therefore $C = Cbs$ which implies that $C \subseteq l(b-bsb)$, whence $L \subseteq l(b-bsb)$, yielding $b-bsb \in Z = J$. This proves that A/J is von Neumann regular.

If I is a reduced left ideal of A, E an essential extension of ${}_AI$ in ${}_AA$, then E is a reduced left ideal.

Corollary 8.1. If A is a prime left m-injective ring, then either A is a division ring or every non-zero ideal of A contains a non-zero nilpotent element.

Applying [14, Lemma 2] to Proposition 8(3), we get

Corollary 8.2. (1) If A is left m-injective with a reduced

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

maximal left ideal, then A is strongly regular;

(2) If A contains a p-injective, m-injective reduced maximal left ideal, then A is strongly regular with non-zero socle.

We now turn to LIS rings are m-injective.

Proposition 9. Let A be a left m-injective, LIS ring whose complement left annihilators. Then A is left continuous regular.

Proof. Apply Proposition 4 and Proposition 8.

A well-known result of Y. Utumi states that if A is left and right non-singular, then the maximal left and right quotient rings of A coincide iff every complement one-sided ideal of A is an annihilator. The next corollary then follows.

Corollary 9.1. Let A be left m-injective left non-singular LIS ring whose maximal left and right quotient rings coincide. Then A is left and right continuous regular.

Recall that A is a Baer ring if every left annihilator ideal (equivalently, every right annihilator ideal) is generated by an idempotent.

Proposition 10. Let A be a left and right non-singular LIS ring. Then A is Baer.

Proof. Let I be a non-zero proper left annihilator ideal. Then $I = l(S)$ for some non-zero subset S of A. Since $I = l(SA)$ is non-zero, then SA is not an essential right ideal of A. Let K be a non-zero complement right ideal of A such that $E = SA \oplus K$ is an essential right ideal of A. Since A is right non-singular, then $l(E) = 0$ which implies that $l(SA) \cap l(K) = 0$. In as much as A is LIS, $SA \cap K = 0$ implies that $A = l(SA \cap K) = l(SA) + l(K)$, whence $A = l(SA) \oplus (l(K)$. Thus I is a direct summand of ${}_AA$.

Corollary 10.1. The following conditions are equivalent for a LIS ring A:

(1) A is Baer regular;

(2) A is left p-injective left and right non-singular;

(3) A is right p-injective left and right non-singular.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Applying Proposition 8, we get

Corollary 10.2. If A is left m-injective left non-singular LIS, then A is Baer regular.

We now look at conditions which will ensure that minimal one-sided ideals are annihilators.

Proposition 11. Let A be a left m-injective left perfect ring. Then for any minimal left ideal $U = Au$, $u \in A$, uA is a minimal right ideal of A.

Proof. Since A is left perfect, uA contains a minimal right ideal $V=vA$, $v \in A$. Since $l(u) \subseteq l(v)$, if $g: Au \longrightarrow Av$ is the left A-homomorphism defined by $g(au) = av$ for all $a \in A$, then g is an isomorphism and if $j: Au \longrightarrow A$ is the natural injection, then $jg^{-1}: Av \longrightarrow A$ is a monomorphism which implies that there exists $y \in A$ such that $jg^{-1}(av) = avy$ for all $a \in A$. In particular, $jg^{-1}(v) = vy \in vA$ which implies that $u \in vA$, whence $uA \subseteq vA$. Since $vA \subseteq uA$, we have $uA = vA$ is a minimal right ideal of A.

Corollary 11.1. Let A be a left and right m-injective left perfect ring. Then any minimal ideal of A is a left annihilator.

Proof. If $U = Au$, $u \in A$, is a minimal left ideal of A, by Proposition 11, uA is a minimal right ideal of A. It is clear that $Au \subseteq l(r(Au))$. Let $0 \neq c \in l(r(Au))$. Then $r(u) = r(Au) = r(l(r(Au))) \subseteq r(c)$ and we may define a right A-homomorphism $f: uA \longrightarrow cA$ by $f(ua) = ca$ for all $a \in A$. Now $\ker f = 0$ (because $f(u) = c \neq 0$). If $j: cA \longrightarrow A$ is the natural injection, then $jf: uA \longrightarrow A$ is a right A-monomorphism. In as much as A_A is m-injective, $jf(u) = wu$ for some $w \in A$. Therefore $c = wu \in Au$ which proves that $l(r(Au)) \subseteq Au$, whence $U = Au = l(r(Au))$.

We are now in a position to give some characteristic properties of quasi-Frobeniusean rings.

Theorem 12. The following conditions are equivalent:

(1) A is quasi-Frobeniusean;

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2) A is a left m-injective; left Noetherian, left cogenerator;

(3) A is a left p-injective left FPF ring with maximum condition on left annihilators;

(4) A is left Artinian left and right m-injective.

Proof. Obviously, (1) implies (2) through (4).

Assume (2). Then A/J is semi-simple Artinian by Proposition 8. Since A is a left cogenerator, then A is left pseudo-Frobeniusean, which implies that A is left self-injective. Then (2) implies (1) by [3, Theorem 24.20].

Assume (3). Since every principal right ideal of A is a right annihilator, then A satisfies the descending chain condition on principal right ideals which means that A is left perfect. Since J=Z is nilpotent, then A is semi-primary. Now a right perfect left FPF ring is left pseudo-Frobeniusean. Then A is left self-injective with maximum condition on left annihilators and hence (3) implies (1).

Assume (4). Then A is left and right perfect, left and right m-injective. By Corollary 11.1, any minimal left or right ideal of A is an annihilator. Then (4) implies (1) by [6, Proposition 1].

Recall that a left A-module M is continuous iff any left submodule of M isomorphic to a complement left submodule of M is a direct summand of $_AM$. In particular, this will imply that, given B, C direct summands of $_AM$ such that $B \cap C = 0$, then $B \oplus C$ is a direct summand of $_AM$. A is left continuous iff $_AA$ is continuous (cf. [7]). A continuous m-injective left A-module needs not be injective (otherwise, left p-V-rings would always be left V-rings!). If A is left self-injective, then obviously A is a left continuous, left m-injective ring.

Carefull scruting of the proof of [12, Theorem 9] yields the following decomposition result:

Theorem 13. Let A be a left continuous, left m-injective ring. Then $A = B \oplus C$, where B and C are left ideals of A such that B is a direct sum of pair of isomorphic left ideals of A and C contains no direct sum of mutually isomorphic non-zero

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

left ideals of A.

A left A-module M is called semi-simple if J(M), the Jacobson radical of M, is zero [5]. If A is a left p-V-ring, then any left ideal of A is semi-simple (cf.[7, Theorem 9]). It follows that if I is a left ideal in a Neumann regular ring A, then I is reduced iff I contains no direct sum of two isomorphic non-zero left ideals of A. We may now have a nice decomposition result.

Theorem 14. Let A be a left continuous left m-injective regular ring. Then $A = F \oplus D \oplus C$, where F, D are isomorphic left ideals of A and C is a left and right continuous strongly regular ring with identity.

Proof. By Theorem 13, $A = B \oplus C$, where $B = F \oplus D$, F, D being isomorphic left ideals of A and C contains no direct sum of two isomorphic non-zero left ideals of A. Then C is a reduced left ideal of A. Let $B = Ae$, $e = e^2 \in A$, $C = Av$, where $v = 1-e$. Since C is reduced, $eA \subseteq l(v) = B$, which shows that e is central in A (B being an ideal of A). Consequently, C is a left and right continuous strongly regular ring with identity.

At this point, let us add a result on reduced rings.

Proposition 15. Let A be left non-singular containing no direct sum of mutually isomorphic non-zero left ideals. The A is reduced.

Proof. Suppose there exists $0 \neq a \in A$ such that $a^2 = 0$. Then there exists a non-zero left ideal K of A such that $l(a) \cap K = 0$. The map $f: Ka \longrightarrow K$ defined by $f(ka) = k$ for all $k \in K$ is a well-defined left A-homomorphism which is indeed an isomorphism. In as much as $Ka \subseteq l(a)$, A contains $Ka \oplus K$ which contradicts the hypothesis. This proves that A must be reduced.

Proposition 16. Let A be a semi-prime left m-injective ring whose complement left ideals are direct summands of ${}_AA$. Then $A = B \oplus C$, where C is the unique maximal reduced ideal of A which is a left m-injective left and right continuous strongly regular ring and B is the unique minimal direct summand of A

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

containing all the nilpotent elements of A.

Proof. Let C denote the sum of all reduced ideals of A. Then C is the unique maximal reduced ideal of A [13, Proposition 7] and C is a direct summand of ${}_AA$ [13, Remark 9]. Indeed, $A = C \oplus B$, where B is the minimal direct summand of ${}_AA$ containing all the nilpotent elements of A. Now C (and also B) is generated by a central idempotent. Since B is an ideal of A and A is left m-injective, then C is a left m-injective ring with identity. Since C is reduced, then C is strongly regular by Proposition 8(3). The proposition then follows.

Corollary 16.1. If A is semi-prime left self-injective, then $A = B \oplus C$, where C, the sum of all reduced ideals of A, is the unique maximal reduced ideal of A which is a left and right self-injective strongly regular ring and B is the unique minimal direct summand of ${}_AA$ containing all the nilpotent elements of A.

Proposition 17. Let A be a left m-injective left continuous regular ring containing a reduced complement left ideal K of A. If $A = B \oplus D$, where B, D are left ideals of A and B does not contain any left ideal isomorphic to K, then D contains a non-zero central idempotent.

Proof. We first note that K contains no direct sum of two isomorphic non-zero left ideals of A. Let S denote the set of all left A-monomorphism from some left submodule of K into B. Since S is inductive, it contains a maximal member g. If $g: M \longrightarrow B$, since ${}_AB$ is m-injective, if ${}_AN$ is an essential extension of ${}_AM$, in ${}_AK$, g extends to $h: N \longrightarrow B$ which is also a monomorphism. By the maximality of g, h=g which implies that N=M, showing that M is a complement left submodule of K, whence M is a complement left submodule of ${}_AA$. Since A is left continuous, M=Ae, $e = e^2 \in A$. Then $K = Ae \oplus At$. Since B does not contain any left ideal isomorphic to K, then $t \neq 0$. If $E = g(M)$, E=Au, $u = u^2 \in A$, and $B = E \oplus Av$, $v = v^2 \in A$. Suppose that $tAu \neq 0$. Then there exist non-zero idempotents t_1, u_1 in At, Au respectively such that ${}_AAt_1 \approx {}_AAu_1$. Therefore $g^{-1}(Au_1) \approx At_1$ and K contains $g^{-1}(Au_1) \oplus At_1$ which is a contradiction! Thus

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$tAu = 0$. Now suppose that $tAv \neq 0$. Then there exist non-zero idempotents t_2, v_2 in At, Av respectively such that ${}_AAt_2 \cong {}_AAv_2$ and hence g extends to an isomorphism of $M \oplus At_2$ onto $E \oplus Av_2$, contradicting the maximality of g. Therefore $tAv = 0$ which yields $tB = 0$, whence $l(B)$ is non-zero. Now $l(B) = Aw$, where w is a non-zero central idempotent. If $B = As$, $D = A(1-s)$, $s = s^2 \in A$, then $ws = 0$ implies that $w \in D$.

We note a last proposition by combining Proposition 5 with Proposition 8.

Proposition 18. The following conditions are equivalent:

(1) A is semi-simple Artinian;

(2) Every left A-module is m-injective;

(3) Every left ideal of A is m-injective;

(4) A is semi-prime left m-injective satisfying the maximum condition on left annihilators.

REFERENCES

[1] G. Baccella: Generalized V-rings and von Neumann regular rings, Rend.Sem.Mat. Univ. Padova 72 (1984), 117-133.

[2] K. Chiba, H. Tominaga: On strongly regular rings, II, Proc. Japan Acad. 50 (1974), 444-445.

[3] C. Faith: Algebra II: Ring Theory. Grundlehren 191, Springer-Verlag (1976).

[4] M. Ikeda, T. Nakayama: On some characteristic properties of quasi-Frobenius and regular rings, Proc. Amer. Math. Soc. 5(1954), 15-19.

[5] G.G. Michler, O.E. Villamayor: On rings whose simple modules are injective, J. Algebra 25 (1973), 185-201.

[6] H.H. Storrer: A note on quasi-Frobenius rings and ring epimorphisms, Canada. Math. Bull. 12 (1969), 287-292.

[7] Y. Utumi: On continuous rings and self-injective rings, Trans.Amer.Math.Soc. 118 (1965), 158-173.

[8] K. Varadarajan, K. Wehrhahn: P-injectivity of simple pre-torsion modules, Glasgow Math. J. 28 (1986), 223-225.

[9] M.F. Yousif: SI-modules, Math. J. Okayama Univ. 28

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(1986), 133-146.

[10] R. Yue Chi Ming: On von Neumann regular rings, II, Math. Scand. 39 (1976), 167-170.

[11] R. Yue Chi Ming: On regular rings and self-injective rings, II, Glasnik Mat. 18 (38) (1983), 221-229.

[12] R. Yue Chi Ming: On von Neumann regular rings, XIII, Ann. Univ. Ferrara Sez. VII, Sc. Mat. 31 (1985), 49-61.

[13] R. Yue Chi Ming: On semi-prime and reduced rings, Riv. Mat. Univ. Parma (4) 12 (1986), 167-175.

[14] R. Yue Chi Ming: Annihilators and strongly regular rings, Rend. Sem. Fac. Sc. Cagliari 57 (1987), 51-59.

UNIVERSITE PARIS VII, UFR DE MATHEMATIQUES; URA 212 CNRS, 75251 PARIS CEDEX 05, FRANCE
Received April 11, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Barbara Pilecka-Oborska

UNE MODIFICATION DE LA CONSTRUCTION DE LA FONCTION DE SORTIE

1. Préliminaire

Considérons un groupe $(G,\cdot)$ pour lequel e est l'élément neutre. Soit Γ un ensemble arbitraire et $F:\Gamma\times G \longrightarrow \Gamma$ une solution de l'équation de translation

$$F(F(\alpha,x_1),x_2) = F(\alpha,x_1x_2). \tag{1.1}$$

Par $(P,\cdot)$ nous entendons le semi-groupe avec l'élément neutre à droite y_0. Supposons momentanément que $(P,\cdot)$ possède la propriété de la réduction à gauche, à savoir

$$\bigwedge_{(y_1,y_2,y_3\in P)} (y_1y_2 = y_1y_3 \Rightarrow y_2 = y_3). \tag{1.2}$$

Soit $\lambda: \Gamma\times G \longrightarrow P$ une solution de l'équation

$$\lambda(\alpha,x_1x_2) = \lambda(\alpha,x_1)\cdot\lambda(F(\alpha,x_1),x_2), \tag{1.3}$$

où $F: \Gamma\times G \longrightarrow \Gamma$ est une solution fixée de l'équation (1.1).

Dans la théorie des automates la fonction λ satisfaisant à (1.3) est appelée fonction de sortie (voir par exemple [3]).

Théorème 1 (voir [3]). La solution générale de l'équation (1.1) est la famille des fonctions de la forme suivante

$$F(\alpha,x) = g_k^{-1}(g_k(f(\alpha))x) \quad \text{pour} \quad f(\alpha)\in\Gamma_k, \tag{1.4}$$

où

(a) f est une fonction arbitraire définie sur l'ensemble Γ telle que $f(f(\alpha)) = f(\alpha)$ pour $\alpha\in\Gamma$,

(b) la famille $\{\Gamma_k\}_{k\in K}$ des ensembles non vides, disjoints et

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

tels que $f(\Gamma) = \bigcup_{k\in K} \Gamma_k$ remplit la condition suivante:

(c) il existe pour chaque $k\in K$ un sous-groupe G_k de G tel que card Γ_k = card G/G_k, où G/G_k est la famille des classes d'équivalence à droite du groupe G par rapport à un sous-groupe G_k,

(d) la fonction g_k est une bijection de l'ensemble Γ_k sur la famille G/G_k.

Les sous-groupes G_k dans (c) nous appellerons les sous-groupes de stabilité de la solution F et les ensembles Γ_k les fibres transitives.

Soit $F: \Gamma\times G \longrightarrow \Gamma$ une solution fixée de l'équation (1.1).

Théorème 2 (voir [3]). La solution générale de l'équation (1.3) est la famille des fonctions de la forme suivante

(1.5) $\lambda(\alpha,x) = \lambda_k(\bar{x}_k(\alpha)x)/\lambda_k(\bar{x}_k(\alpha))$
pour $\alpha\in\Gamma_k$, $x\in G$,

(1.6) $\lambda(\alpha,x) = h(\alpha)\cdot\lambda(F(\alpha,e),x)$
pour $\alpha\in\Gamma\setminus\bigcup_{k\in K}\Gamma_k$, $x\in G$,

où

(1.7) la famille $\{\Gamma_k\}_{k\in K}$ est un ensemble des fibres transitives de la solution F; G_k est le sous-groupe de stabilité correspondant à Γ_k, l'élément $\alpha_k\in\Gamma_k$ est tel que $F(\alpha_k,G_k) = \{\alpha_k\}$;

(1.8) $\bar{\lambda}_k$, pour $k\in K$, est un homomorphisme du groupe G_k dans le semi-groupe $(P,\cdot)$ et $P_k := \bar{\lambda}_k(G_k)$;

(1.9) S_k est une sélection arbitraire de l'ensemble G/G_k et

$$s_k(x) := S_k(C) \quad \text{pour } x\in C\in G/G_k,\ k\in K;$$

(1.10) f_k, pour $k\in K$, est une fonction sur $s_k(G)$ des valeurs dans P telle que l'ensemble des valeurs Z_k de cette fonction satisfait à la condition suivante:

(1.11) tous les éléments de l'ensemble

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$P_kZ_k := \left\{ y : \bigvee_{(y_1 \in P_k)} \bigvee_{(y_2 \in Z_k)} y = y_1 y_2 \right\}$$

sont divisibles à droite, c'est-à-dire

$$\bigwedge_{(y_1, y_2 \in P_k Z_k)} \bigvee_{(y_3 \in P)} (y_1 = y_2 y_3)$$

[si $\overline{y}$, $\overline{\overline{y}} \in P_kZ_k$, alors par $\overline{y}/\overline{\overline{y}}$ on désigne le quotient à droite de l'élément $\overline{y}$ par l'élément $\overline{\overline{y}}$, c'est-à-dire $\overline{y} = \overline{\overline{y}}(\overline{y}/\overline{\overline{y}})$];

(1.12) $\lambda_k(x) := \overline{\lambda}_k(x\, s_k^{-1}(x)) \cdot f_k(s_k(x))$
pour $x \in G$ et $k \in K$;

(1.13) pour $\alpha \in \Gamma_k$, $\overline{x}_k(\alpha)$ est un élément tel que
$F(\alpha_k, \overline{x}_k(\alpha)) = \alpha$;

(1.14) h est une fonction arbitraire qui transforme l'ensemble $\Gamma \setminus \bigcup_{k \in K} \Gamma_k$ dans l'ensemble P.

La solution générale de (1.3) au cas de $\lambda : \Gamma \times B \longrightarrow C$, où $(B,\cdot)$, $(C,\cdot)$ sont les groupoïdes de Brandt, a été donné dans [2].

2. Modification de la construction de la fonction de sortie

Lemme 1. y_o est l'élément neutre du semi-groupe $(P,\cdot)$.

Démonstration. Supposons que nous avons $yy_0 = y$ pour chaque $y \in P$, donc $y_oy_o = y_o$. Il en résulte que $y_oy = (y_oy_o)y = y_o(y_oy)$, donc $y_oy = y_o(y_oy)$. Profitant de (1.2), nous obtenons $y = y_oy$. Cela signifie que y_o est aussi un élément neutre à gauche, ce qui termine la démonstration.

Lemme 2. Si $\lambda : \Gamma \times G \longrightarrow P$ est la solution de l'équation (1.3) et $\alpha \in \bigcup_{k \in K} \Gamma_k$, alors $\lambda(\alpha, e) = y_o$.

La démonstration de ce théorème, au cas de la fonction $F: \Gamma \times G \longrightarrow \Gamma$ satisfaisant à la condition d'identité, c'est-à-dire $\bigwedge_{\alpha \in \Gamma} F(\alpha, e) = \alpha$, se trouve dans [3]. Si $\alpha \in \bigcup_{k \in K} \Gamma_k$, il résulte du Théorème 1 que $F(\alpha, e) = \alpha$. Raisonnant comme dans

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3], nous obtenons la thèse du Lemme 2.

Lemme 3. Si pour certains $a,b \in P$ on a $ab = y_0$, on a aussi $ba = y_0$.

Démonstration. Nous avons de notre supposition que $y_0 a = (ab)a = a(ba)$, alors $ay_0 = a(ba)$. De là et d'aprés (1.2), nous avons la thèse du Lemmme 3.

Lemme 4. Si $A,B \subset P$, $y_0 \in A \cap B$,

$$A \cdot B := \left\{ y : \bigvee_{(y_1 \in A)} \bigvee_{(y_2 \in B)} (y = y_1 y_2) \right\}$$

et la condition

$$(2.1) \qquad \bigwedge_{(y_1, y_2 \in A \cdot B)} \bigvee_{(y_3 \in P)} (y_1 = y_2\ y_3)$$

est remplie, alors chaque élément de l'ensemble $A \cdot B$ a l'inverse dans P.

Démonstration est évidente, d'après la Lemme 3.

Nous obtenons du Lemme 4 les résultats suivants.

Corollaire 1. Si $y_0 \in Z_k$ et les relations (1.3), (1.11) ont lieu, alors chaque élément de l'ensemble $P_k Z_k$ a l'inverse dans P.

Corollaire 2. Si $\lambda : \Gamma \times G \longrightarrow P$ est une solution de l'équation (1.3) et $y_0 \in Z_k$, alors la condition (1.5) a la forme

$$(2.2) \qquad \lambda(\alpha, x) = \lambda_k(\overline{x}_k(\alpha))^{-1} \cdot (\lambda_k(\overline{x}_k(\alpha))\ x)$$

pour $x \in G$, $\alpha \in \bigcup_{k \in K} \Gamma_k$.

Corollaire 3. Si $y_0 \in Z_k$, la condition (1.11) est équivalente à la suivante: il existe un sous-groupe $\overline{P}$ du demi-groupe P tel que $Z_k \subset \overline{P}$.

Démonstration. Soit H l'ensemble de tous les éléments de P qui possèdent l'inverse dans P. L'ensemble H est un sous-groupe de $(P,\cdot)$ ([4] p. 47, exemple 7). Si la condition (1.11) est remplie, les éléments de l'ensemble Z_k, d'après le Corollaire 1, appartiennent à H. Alors, il suffit de prendre $\overline{P} := H$. Si $\overline{P}$ est un sous-groupe de P et $Z_k \subset \overline{P}$, chaque

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

élément de l'ensemble Z_k possède l'inverse dans $\overline{P}$. Il résulte des définitions des ensembles P_k et P_kZ_k que chaque élément de P_kZ_k a l'inverse dans P. Donc, si $y_1, y_2 \in P_kZ_k$, l'élément $y_3 = y_2^{-1}y_1$ remplit la condition $y_1 = y_2y_3$. Alors (1.11) a lieu.

Théorème 3. La solution générale de l'équation (1.3) est la famille des fonctions de la forme (2.2) et (1.6), où les paramètres $\overline{\lambda}_k$, f_k, s_k, h et $\overline{x}(\alpha)$ ont la même signification que dans le Théorème 2 et où on peut prendre $s_k(x) = e$, $f_k(s_k(x)) = y_o$ pour $x \in G_k$ et $k \in K$.

Démonstration. Soit $\lambda : \Gamma \times G \longrightarrow P$ une solution de l'équation (1.3). Notons par Γ_k, G_k, α_k la même chose que dans (1.7). Prenons la fonction S_k définie dans (1.9) et telle que $S_k(G_k) = e$. De la définition de la fonction s_k dans (1.9), nous avons alors $s_k(x) = e$ pour $x \in G_k$ et $k \in K$.

Posant $f_k(s_k(x)) := \lambda(\alpha_k, s_k(x))$ et $Z_k := f_k(s_k(G))$, on a $f_k(s_k(x)) = \lambda(\alpha_k, e)$ pour $x \in G_k$ et $k \in K$.

Le Lemme 2 nous donne $f_k(s_k(x)) = y_o$ pour $x \in G_k$ et $k \in K$, alors $y_o \in Z_k$ pour chaque $k \in K$. Posons

$$\overline{\lambda}_k(x) := \lambda(\alpha_k, x) \quad \text{pour } x \in G_k,\ k \in K. \tag{2.3}$$

De (1.3) et (1.7), pour $\alpha = \alpha_k$ et $x_1, x_2 \in G_k$, nous avons

$$\lambda(\alpha_k, x_1x_2) = \lambda(\alpha_k, x_1)\lambda(\alpha_k, x_2).$$

Il en résulte que $\overline{\lambda}_k$ est une homomorphie du groupe G_k dans le demi-groupe $(P, \cdot)$, alors $P_k := \overline{\lambda}_k(G)$ est un sous-groupe de P.

Soit λ_k une fonction définie dans (1.12). Nous allons démontrer que l'ensemble P_kZ_k est l'ensemble des valeurs de la fonction λ_k.

Pour $x \in G$ on a $x\, s_k^{-1}(x) \in G_k$. Si x parcourt une classe de la famille G/G_k, alors $s_k(x)$ est un élément fixe de cette classe, mais en même temps $xs_k^{-1}(x)$ parcourt le groupe G_k. Il en résulte, vu la définition de l'ensemble Z_k, que l'ensemble des valeurs de la fonction λ_k remplit l'ensemble P_kZ_k tout entier, si x parcourt le groupe G tout entier.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

D'après (1.3), (1.13), (1.12) et (2.3), nous avons

$$\begin{aligned}(2.4)\quad \lambda(\alpha_k,x) &= \lambda(\alpha_k, xs_k^{-1}(x)s_k(x)) = \\ &= \lambda(\alpha_k, xs_k^{-1}(x))\lambda(F(\alpha_k, xs_k^{-1}(x)), s_k(x)) = \\ &= \bar{\lambda}_k(xs_k^{-1}(x))\lambda(\alpha_k, s_k(x)) = \\ &= \bar{\lambda}_k(xs_k^{-1}(x))f_k(s_k(x)) = \lambda_k(x).\end{aligned}$$

Nous voyons encore, d'après (1.3) et (1.13), que

$$\begin{aligned}\lambda(\alpha_k, \bar{x}_k(\alpha)x) &= \lambda(\alpha_k, \bar{x}_k(\alpha))\lambda(F(\alpha_k, \bar{x}_k(\alpha)), x) = \\ &= \lambda(\alpha_k, \bar{x}_k(\alpha))\lambda(\alpha, x) \quad \text{pour } \alpha\in\Gamma_k.\end{aligned}$$

De là et de (2.4) nous obtenons $\lambda_k(\bar{x}_k(\alpha)x) = \lambda_k(\bar{x}_k(\alpha))\lambda(\alpha,x)$. Il en résulte qu'il existe le quotient

$$(2.5)\quad \lambda_k(\bar{x}_k(\alpha)x)/\lambda_k(\bar{x}_k(\alpha))$$

et (1.5) a lieu.

Il résulte de (1.13) que nous pouvons prendre pour $\bar{x}_k(\alpha)$ un élément quelconque du groupe, prenant $\alpha\in\Gamma_k$ espectivement. $\lambda_k(\bar{x}_k(\alpha))$ est donc un élément arbitraire de P_kZ_k. Si nous établissons $\bar{x}_k(\alpha)$, nous pouvons prendre l'élément $x\in G$ tel que $\lambda_k(\bar{x}_k(\alpha)x)$ sera l'élément quelconque de P_kZ_k. Il en résulte, d'après (2.5), que chaques deux éléments de P_kZ_k sont divisibles du côté droit. Alors (1.11) a lieu.

Puisque $y_o\in Z_k$, la condition (1.5) a donc la forme (2.2), conformément au Corollaire 1.

Pour démontrer (1.6) il suffit de prendre dans (1.3) $x_1=e$, $x_2=x$ et simultanément poser

$$h(\alpha) := \lambda(\alpha,e) \quad \text{pour } \alpha\notin\bigcup_{k\in K}\Gamma_k.$$

Le raisonnement que nous venons de faire est une petite modification de celui de [3] (Théorème 2). Si les paramètres $\bar{\lambda}_k$, f_k, s_k, h, $\bar{x}(\alpha)$ ont la même signification que dans le Théorème 2, $s_k(x) = e$ et $f_k(s_k(x)) = y_o$ pour $x\in G_k$ et $k\in K$, alors $y_o\in Z_k$. Vu Théorème 2 et Corollaire 1, la fonction définie par (2.2) et (1.6) est donc la solution de l'équation (1.3).

Il résulte du Théorème 3 qu'on ne rétrécit pas l'ensemble

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

des solutions de (1.3), pourvu qu'on prenne les paramètres s_k, f_k tels que $s_k(x) = e$ et $f_k(s_k(x)) = y_o$ pour $x \in G_k$.

Soit $H := \{x \in P : \bigvee_{x' \in P} xx' = y_o\}$. Il résulte du Théorème 3 et du Corollaire 3 la proposition suivante.

Corollaire 4. L'ensemble des solutions $\lambda : \Gamma \times G \longrightarrow P$ de l'équation (1.3) est identique à l'ensemble des solutions $\lambda : \Gamma \times G \longrightarrow H$ de cette équation.

Corollaire 5. Si dans [2] les groupoïdes B et C forment des groupes, alors la construction de la fonction $\lambda : \Gamma \times B \longrightarrow C$ satisfaisant à l'équation (1.3) est identique à la construction de a solution donnée par le Théorème 3.

TRAVAUX CITÉS

[1] **A. Krupińska:** Ogólne rozwiązanie równania translacji na grupoidzie Brandta, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, 41 (1970) 31-44.

[2] **A. Krupińska:** Structure générale de la fonction de sortie pour l'automate abstrait sur le grupoïde de Brandt, Demonstratio Math. 6 (1973) 707-724.

[3] **Z. Moszner:** Structure de l'automate plein, réduit et inversible, Aequationes Math. 9 (1973) 46-59.

[4] **S. Złonkiewicz:** Struktury algebraiczne, cz. I. Wyższa Szkoła Pedagogiczna w Rzeszowie, Rzeszów, 1971.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY,
30-084 KRAKÓW, POLAND

Received April 19, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

1.

In
with th
method
some si
converg
the ope
most i
polynom
Petrov
theorem

2.

Let

(1)

with ad
is an o
polynom

and f_j
Usi
the base
matrix

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

H.-G. Roos, E. Pfeifer

A CONVERGENCE RESULT FOR THE TAU METHOD

1. Introduction

In the last years many papers have been published dealing with the description and application of the so-called Tau method [4]. Looking at these papers it seems that except of some simpler cases there is no general result concerning the convergence of the constructed Tau approximants. Starting from the operational approach given in [1], we show that in the most interesting cases, using Chebyshev or Legendre polynomials the Tau method is in fact a method of Galerkin-Petrov type, thus convergence results follow from general theorems given in [2].

2. The operational approach of the Tau method

Let us consider the differential equation

(1) $(Dy)(x) = 0, \quad a \le x \le b,$

with additional conditions $(f_j, y) = s_j$, $j=1,2,\ldots,m$. Here D is an ordinary linear differential operator of order m with polynomial coefficients

$$D = \sum_{i=0}^{m} p_i(x) \frac{d^i}{dx^i}, \qquad p_i(x) = \sum_{j=0}^{\alpha_i} p_{ij} x^j = p_i \underline{x},$$

and f_j are some linear functionals acting on y.

Using a polynomial base $\underline{v} = \{v_0, v_1, v_2, \ldots\}$ related to the base $\{1, x, x^2, \ldots\}$ via a regular and lower triangular matrix V by $\underline{v} = V\underline{x}$ to construct an approximate solution

$$y_n(x) = \sum_{i=0}^{n+m} a_i^{(n)} v_i(x) = \underline{a}_n \underline{v}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

of (1) we start from an expansion of the exact solution y in the given base, namely

$$(2) \qquad y(x) = \sum_{i=0}^{\infty} a_i v_i(x) = \underline{\underline{a}}\,\underline{v}.$$

Setting (2) into (1) we get

$$(3) \qquad Dy = \underline{\underline{a}}\,\underline{\Pi}\,\underline{v}, \quad x \in R$$

and the additional conditions $(\underline{\underline{f}}, y) = \underline{\underline{a}}B = \underline{\underline{s}}$. The matrix Π arises by differentiation and multiplication of powers x^j, $\underline{\underline{f}}$ and $\underline{\underline{s}}$ are the vectors $(f_1,\ldots,f_m,0,\ldots)$ and so $(s_1,\ldots,s_m,0,\ldots)$, resp., and B is the matrix (b_{ij}) with $b_{ij} = (f_j, v_i)$. Because $\underline{v}$ is a base (3) turns out into

$$(4) \qquad \begin{cases} \underline{\underline{a}}\,\underline{\Pi}_0 = 0 & \underline{\underline{a}}\,\underline{B}_1 = s_1 \\ \underline{\underline{a}}\,\underline{\Pi}_1 = 0 \quad \text{and} & \\ \quad\vdots & \underline{\underline{a}}\,\underline{B}_m = s_m, \end{cases}$$

where $\underline{\Pi}_i$ and $\underline{B}_i$ are the i-th columns of Π and B. Instead of (4) we may write

$$(5) \qquad \underline{\underline{a}}\,G = \underline{\underline{s}}$$

with $G = (\underline{B}_1,\ldots,\underline{B}_m,\underline{\Pi}_0,\underline{\Pi}_1,\ldots)$.

To determine y_n we take from G only the first $n+m+1$ rows and columns, i.e., we solve the linear system

$$(6) \qquad \underline{\underline{a}}_n G_n = \underline{\underline{s}}.$$

The solution y_n $\underline{\underline{a}}_n\underline{v}$ is a s-called Tau approximant, because it generates the polynomial defect

$$(7) \qquad (Dy_n)(x) = \underline{\underline{a}}_n\,\underline{\Pi}\,\underline{v} = (\underline{\underline{a}}_n\underline{\Pi}_0)v_0 + \ldots + (\underline{\underline{a}}_n\underline{\Pi}_{n+m+h})v_{n+m+h'}$$

where $h = \max_i(\alpha_i - i)$ is the so-called height of the differential operator D. We notice that in view of (4) and (6) the components of the defect vanish with respect to the first $n+1$ elements of the base $\underline{v}$.

3. An abstract convergence theorem

Let be given some Banach spaces E, E_n, $n=1,2,\ldots$, connec-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ted by linear operators $P_n \in L(E,E_n)$, $n=1,2,\ldots$, satisfying

(8) $\|P_n y\|_{E_n} \longrightarrow \|y\|_E$ as $n\longrightarrow\infty$

for every $y\in E$.

We call a sequence $(y_n\in E_n : n\in N'\subset N)$ to be P-convergent to $y\in E$ and write $y_n \xrightarrow{P} y$ if

$$\|y_n - P_n y\|_{E_n} \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty,\ n\in N'.$$

Similar, a sequence $\{ y_n\in E_n : n\in N \}$ is said to be P-compact, if for every $N'\subset N$ there exists a set $N''\subset N'$ and an element $y\in E$ such that $y_n \xrightarrow{P} y$, $n\in N''$.

Let be given some further Banach spaces F, F_n, $n=1,2,\ldots$, connected by linear operators $Q_n\in L(F,F_n)$ for which a condition similar to (8) holds.

A sequence of linear operators $A_n\in L(E_n,F_n)$, $n=1,2,\ldots$, is said to be PQ-convergent to the linear operator $A\in L(E,F)$ if the conditions

(9) $\|A_n\|\leq c$, $n\in N$, c independent of n,

and

(10) $\|A_n P_n y - Q_n A y\|_{F_n} \longrightarrow 0$, $n\longrightarrow\infty$,

are true for at least a dense subset of E. This convergence is called regular if every bounded sequence $\{y_n\}$, $n\in N$, for which $\{A_n y_n\}$ is Q-compact performes a P-compact sequence. It holds the following theorem [2].

Theorem 1. We consider the equation

(11) $Ay=r$

which has to be solved by the discretizations

(12) $A_n y_n = r_n$, $n=1,2,\ldots$,

with $A\in L(E,F)$, $A_n\in L(E_n F_n)$, $r\in F$ and $r_n\in F_n$.

Let us assume that the following conditions are satisfied:

(13) $\ker A=\{0\}$,

(14) the operators A_n are of Fredholm type with index zero, $n\in N$,

(15) the sequence $\{A_n\}$ is regular PQ-convergent to the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

operator A,

(16) $r_n \xrightarrow{Q} r$.

Then, (11) admits a unique solution $y^* \in E$, the equations (12) possess unique solutions $y_n^* \in E_n$ for sufficiently large n and the error estimations

(17) $\|y_n^* - P_n y^*\|_{E_n} \leq \text{const} \|A_n P_n y^* - r_n\|_{F_n}$

are valid.

4. The convergence result for the Tau method

The applicability of the theorem above to the Tau method requires an additional assumption for the base $\underline{v}$ which is fulfilled in the practically most interesting cases. So let us assume that there exists a weight function w (with $w(x) \geq \text{const} > 0$) for which the base functions v_i are orthogonal with respect to the inner product

$$\int_b^a w(x) v_i(x) v_j(x) dx.$$

Now it is easily seen that the Tau method is equivalent to: Find some

$$y_n(x) = \sum_{i=0}^{n+m} a_i^{(n)} v_i(x)$$

satisfying

(18) $$\int_b^a w(x) (Dy_n)(x) v_j(x) dx = 0, \qquad j=0,1,\ldots,n$$

and the additional conditions

(19) $(f_j, y_n) = s_j, \quad j=1,\ldots,m.$

We define

- $E = W_2^m(a,b;w)$ with the norm

$$\|y\|_E = \left(\sum_{j=0}^{m} \int_b^a w(x) (y^{(j)}(x))^2 dx \right)^{\frac{1}{2}},$$

- $E_n = \text{span}\{1, x, \ldots, x^{n+m}\}$ with the norm $\|y_n\|_{E_n} = \|y_n\|_E$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$P_n y = \text{best}_{E_n} y$, i.e. the best approximation of y by elements from E_n in the norm of E,

$F^1 = W_2(a,b;w)$ with the norm

$$\|r^1\|_{F^1} = \left(\int_b^a w(x)(r^1(x))^2 dx \right)^{\frac{1}{2}},$$

$F^2 = R^m$ with the maximum norm of the components,

$F = F^1 \times F^2$ with the norm $\|r\|_F = \max \left\{ \|r^1\|_{F^1}, \|r^2\|_{F^2} \right\}$,

$F_n^1 = \text{span}\{ 1, x, \ldots, x^n \}$ with $\|r_n^1\|_{F_n^1} = \|r_n^1\|_{F^1}$,

$F_n^2 = F^2$,

$F_n = F_n^1 \times F_n^2$ with $\|r_n\|_{F_n} = \max \left\{ \|r_n^1\|_{F_n^1}, \|r_n^2\|_{F_n^2} \right\}$

and

$Q_n r = (q_n r^1, r^2)$, where $q_n r^1 = \text{best}_{F_n^1} r^1$ is the best approximation of r^1 by polynomials of degree not greater that n in the norm of F^1.

Of course, $\|P_n y\|_{E_n} \longrightarrow \|y\|_E$ and $\|Q_n r\|_{F_n} \longrightarrow \|r\|_F$, $n \to \infty$.

Theorem 2. Let us suppose that equation (1) with homogeneous conditions $(f_j, y) = 0$, $j = 1, \ldots, m$, has only the trivial solution $y \equiv 0$. Let the additional conditions satisfy

(20) $$\left\| \{(f_j, y)\}_{j=1,\ldots,m} \right\|_{R^m} \leq \text{const} \, \|y\|_E.$$

Let the leading coefficient $p_m(x)$ of D be identically equal to 1. Then, equations (18), (19) possess for $n \geq n_0$ unique solutions y_n^* in E_n converging to the unique solution of (1) in the norm of E. Further, the error estimate

(21) $$\|y_n^* - y^*\|_E \leq \text{const} \|P_n y^* - y^*\|_E$$

is valid.

Proof. Let us define

$$Ay = \begin{Bmatrix} Dy \\ (f_j, y),\ j = 1, \ldots, m \end{Bmatrix}, \quad r = \begin{Bmatrix} 0 \\ s_j,\ j = 1, \ldots, m \end{Bmatrix},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$A_n y_n = \left\{ \begin{array}{l} q_n Dy_n \\ (f_j, y_n),\ j=1,\dots,m \end{array} \right\} \text{ and } r_n = \left\{ \begin{array}{l} 0 \\ s_j,\ j=1,\dots,m \end{array} \right\}.$$

Then the equations (11) and (12) are the abstract formulations of (18) and (19).

As a direct consequence of

$$\|Dy\|_{F^1} \le \text{const } \|y\|_E$$

and assumptions (20) we get $A \in L(E,F)$. Now we demonstrate step by step validity of the conditions (13),(14),(15),(16):

(i) Our assumption about D results in condition (13).

(ii) Condition (14) follows from $\dim E_n = \dim F_n$.

(iii) We have

$$(20) \quad \left\{ \begin{array}{l} \|A_n y_n\|_{F_n} = \max\left\{ \|q_n Dy_n\|_{F_n^1},\ \|\{(f_j, y_n)\}_{j=1,\dots,m}\|_{R^m} \right\}, \\ \|q_n Dy_n\|_{F_n^1} \le \|Dy_n\|_{F^1} \le c\|y_n\|_{E_n}, \end{array} \right.$$

where c does not depend on n. Therefore it holds $A_n \in L(E_n, F_n)$ and (9). Further

$$(20) \quad \left\{ \begin{array}{l} \|A_n P_n y - Q_n Ay\|_{F_n} = \\ \quad = \max\{\|q_n DP_n y - q_n Dy\|_{F_n^1},\ \|\{(f_j, P_n y - y)\}_{j=1,\dots,m}\|_{R^m}\}, \\ \|q_n DP_n - q_n Dy\|_{F_n^1} \le \|DP_n y - Dy\|_{F^1} \le \text{const}\|P_n y - y\|_E \longrightarrow 0,\ n \to \infty, \end{array} \right.$$

prove the validity of (10).

Now, let $\|y_n\|_{E_n} \le \text{const}$ and $\{A_n y_n\}$ be Q-compact. In view of

$$|y_n^{(j)}(t) - y_n^{(j)}(x)| \le \int_x^t |y_n^{(j+1)}(s)|\,ds \le \text{const}\|y_n^{(j+1)}\|_{F^1}\sqrt{t-x}$$

and $|y_n^{(j)}| \le \text{const}$ for $j=0,\dots,m-1$, we can use Arzela's theorem and get compactness of every sequence $\{y_n^{(j)}\}$ $j=0,\dots,m-1$, in F^1. For the corresponding subsequences we have

$$y_n^{(j)} \longrightarrow y^{(j)},\ n \in N'',\ j=0,\dots,m-1,$$

where y depends on N''. The Q-compactness of $\{A_n y_n\}$ implies the existence of a function z for which

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\|q_n Dy_n - q_n z\|_{F_n^1} \longrightarrow 0, \quad n \in N''' \subset N''.$$

But,

$$q_n Dy_n = q_n\Big(\sum_{j=0}^{m} p_j y_n^{(j)}\Big) = y_n^{(m)} + q_n\Big(\sum_{j=0}^{m-1} p_j y_n^{(j)}\Big),$$

hence $\{y_n^{(m)}\}$ is compact in F^1 and $y_n^{(m)} \longrightarrow y^{(m)}$, $n \in N'''$.

Because of

$$\|y_n - Py_n\|_{E_n} \le \|y_n - y\|_E + \|y - p_n y\|_E \longrightarrow 0, \quad n \in N''',$$

we got the P-compactness of (y_n).

Thus, condition (15) is satisfied.

(iv) It is obvious that condition (16) is fulfilled. The existence of our solutions y^*, y_n^* follows now from Theorem 1 in section 3.

Finally, we prove the error estimate (21) starting from (17), (20) and

$$\|y_n^* - y^*\|_E \le \|y_n^* - p_n y^*\|_E + \|P_n y^* - y^*\|_E \le$$

$$\le \text{const} \max\Big\{ \big\|q_n DP_n y^*\big\|_{F^1}, \big\|\{(f_j, P_n y^*) - s_j\}_{j=1,\dots,m}\big\|_{R^m} \Big\} +$$

$$+ \|P_n y^* - y^*\|_E \le$$

$$\le \text{const} \max\Big\{ \big\|q_n D(P_n y^* - y^*)\big\|_{F^1}, \big\|\{(f_j, P_n y^* - y^*)\}_{j=1,\dots,m}\big\|_{R^m} \Big\} +$$

$$+ \|P_n y^* - y^*\|_E \le \text{const} \|P_n y^* - y^*\|_E .$$

This estimation completes the proof of Theorem 2.

Remark 1. For usual boundary conditions of the form

$$(f_j, y) = \sum_{i=0}^{m-1} \left(\alpha_{ij} y^{(j)}(a) + \beta_{ij} y^{(j)}(b)\right), \quad j = 1, \dots, m$$

condition (20) is satisfied.

Remark 2. There are no technical difficulties to discuss equation (1) with a nontrivial right hand side.

Remark 3. It is possible to derive from (21) error estimations in uniform norms using the well-known embedding result

$$\max_{a \le x \le b} |y_n^{*(j)}(x) - y^{*(j)}(x)| \le \text{const} \|P_n y^* - y^*\|_E$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

for $j=0,\dots,m-1$. Moreover, we have

$$\|P_n y^* - y^*\|_E \le \|z_{n+m} - y^*\|_E$$

for every polynomial from E_n and, if $y^{*(m)}$ is continuous,

$$\|z_{n+m} - y^*\|_E^2 \le \sum_{j=0}^{m} \int_b^a w(x) \left(z_{n+m}^{(j)}(x) - y^{*(j)}(x)\right)^2 dx \le$$

$$\le \text{const} \sum_{j=0}^{m} \max_{a\le x\le b} |z_{n+m}^{(j)}(x) - y^{*(j)}(x)|^2 \le$$

$$\le \text{const} \max_{a\le x\le b} |z_{n+m}^{(m)}(x) - y^{*(m)}(x)|^2$$

for every polynomial z_{n+m} from E_n with $z_{n+m}^{(j)}(a) = y^{*(j)}(a)$, $j=0,\dots,m-1$. Hence

$$\|P_n y^* - y^*\|_E \le \text{const} \max_{a\le x\le b} |(\text{best}_{C,n} y^{*(m)})(x) - y^{*(m)}(x)| ,$$

$$\max_{a\le x\le b} |y_n^{*(j)}(x) - y^{*(j)}(x)| \le \tag{22}$$

$$\le \text{const} \max_{a\le x\le b} |(\text{best}_{C,n} y^{*(m)})(x) - y^{*(m)}(x)| ,$$

$j=0,\dots,m-1$. Here we denoted by $\text{best}_{C,n} y^{*(m)}$ the best uniform approximation of $y^{*(m)}$ by polynomials of degree not greater than n. Estimation (22) is the same as for the collocation method with zeroes of the base elements v_n given in Theorem 15.5 in [3]. In [4] we derived an error estimation for the Tau method of the type (22) for systems of first order equations with the height zero using the fact that for this class equations the collocation method and the Tau method are equivalent. It seems that for more general problems this equivalence does not hold.

Remark 4. Because of $\|P_n y^* - y^*\|_E \le \|y_n^* - y^*\|_E$ estimation (21) is asymptotically the best possible one in the norm we chosed. Using a weight w for which

$$\left\|\text{best}_{F^1_n} y^* - y^*\right\|_{F^1} \le c_i n^{-i} \|y^{(i)}\|_{F^1} , \quad y\in F^1.$$

(for example $w(x)\equiv 1$ or $w(x) = \left((b-x)(x-a)\right)^{-1/2}$) and follow-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

ing the proof of Theorem 16.6 in [3] we get

$$\left\| y_n^{*(j)} - y^{*(j)} \right\|_{F^1} \le \text{const } n^{j-m} \left\| \text{best}_{F_n^1} y^{*(m)} - y^{*(m)} \right\|_{F^1},$$

$j=0,\dots,m$ and

$$\max_{a\le x\le b} |y_n^{*(j)}(x) - y^{*(j)}(x)| \le \text{const } n^{j-m+1/2} \left\| \text{best}_{F_n^1} y_n^{*(m)} - y_n^{*(m)} \right\|_{F^1},$$

$j=0,\dots,m-1$.

REFERENCES

[1] E.L. Ortiz, H. Samara: An operational approach to the Tau method for the numerical solution of non-linear differential equations, Computing 27, (1981) 15-25.

[2] G. Vainikko: Funktionalanalysis der Diskretisierungmethoden, Teuber Verlagsgesellschaft, Leipzig 1976.

[3] M.A. Krasnosiels'kii et al.: Approximate solution of operator equations. Wolters-Noordhof Publishing, Groningen 1972.

[4] H.-G. Roos, E. Pfeifer: Die Tau-Methode. Preprint 07-08-88 (1988) TU Dresden.

TU DRESDEN, SEKTION MATHEMATIK, DRESDEN, GERMANY 8027,
Received April 25, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Jerzy Zurawiecki

LOCALLY REDUCIBLE ITERATIVE SYSTEMS

1. Introduction

Many applications of discrete mathematics require recurrence equations of the form

$$(*) \qquad x_{k+i} = \varphi(x_i,\ldots,x_{i+k-1}),$$

where k is a positive integer - called the order of this equation - while φ is a k argument function, with the arguments and values from some finite set A. Infinite sequences of elements of A - satisfying (*) - are referred to as iterative sequences, generated by φ. The classical theory of such sequences - with its applications - may be found in [1,3,4]. A fundamental property of iterative sequences is their determinism in relation to the domain of φ generating these sequences; each $(a_1,\ldots,a_k)$ from the domain of φ corresponds to exactly one solution $s = s_1,s_2,\ldots$ such that $s_1 = a_1,\ldots,s_k = a_k$. Thereby we can define an iterative system of φ as an operator assigning to each element of the domain of φ the corresponding unique solution of (*).

Every iterative sequence satisfies an infinite set of equations of the form (*). Therefore it is important to find a minimal equation among them, that is an equation of least order satisfied by this sequence. The order of a minimal equation of some iterative sequence defines a complexity of this sequence (the order of iterative sequence). Sometimes it is convenient to consider a notion of a minimal equation for

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

an iterative sequence from some special class, for example, for sequences generated by linear functions, [1,4]. We can also search for a minimal equation for some finite set of iterative sequences. In this case, such an equation does not have to be minimal for each of these sequences, i.e. their orders can be smaller than the order of every equation that they jointly satisfy. The aim is to define some class of functions corresponding to such equations, and next, to elaborate general rules of construction and recognition of its elements. These functions will be referred to as locally reducible functions. They may be characterized as follows: each nonempty set of dependent iterative sequences generated by a locally reducible function can also be generated by a function with a smaller number of arguments, where sequences are called dependent if they become identical after having eventually deleted some finite number of initial elements. Locally reducible functions have never been a subject of separate research, except for the case of binary total locally reducible functions that may be replaced by two binary total functions of smaller order, only (elementary functions - [9]).

Many properties of the recurrence equations are inherited from equations of smaller order. Locally reducible functions "propagate" these attributes from a lower level onto a higher one. It lets us hope that a close study of such functions makes it possible to establish recurrence dependences - with respect to parameter k - between some classes of recurrence equations. Such a problem has been considered in [6] for functions generating - the well-known and important - de Bruijn sequences (see mainly [2], also [1,3,4]). In the case of recurrence equations with purely periodic solutions, the locally reducible functions generate sequences with minimal periods, that is, the cycle of each periodic solution of (*) is composed of a number of cycles of sequences generated by some locally reducible function, the same for all solutions of this equation. Such connections have been considered in [8]. But in the general case the problem of the utilization of them

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

is still open.

In this paper we would like to extend some results obtained in [9], for the binary elementary iterative systems, onto the class of all binary locally reducible iterative systems. But for this purpose we need to use a new method. Its idea lies in a "colouring" of iterative sequences with the one restriction that the dependent sequences have the same colour. This allows to identify each function considered here with some system of linear equations. If such a system of linear equations has a solution of some special form, then the corresponding function is locally reducible. In the case of elementary functions it leads to the criterion obtained in [9], but there exist binary total locally reducible functions that are not elementary. Then, in order to decide about the "local reducibility" of such functions we must solve the corresponding system of linear equations in a nonbinary finite field. This is one of the main results of the paper. Changing the interpretation of such a system, that is taking its coefficients as unknowns, and conversely, we can obtain some classes of locally reducible functions as solutions.

The results presented in this paper do not exhaust all problems for this class of recurrence equations. We restrict ourselves to the most general problems connected with a formalization of the notion of locally reducible function and problems of construction as well as recognition of such functions.

2. Iterative systems

In this paper we shall use standard mathematical notation. In particular: N denotes the set $\{1,2,\ldots\}$ of all natural numbers, $|S|$ - the cardinality of a set S and, for a given function f, by D_f and R_f we denote its domain and range, respectively. Moreover, for a constant function f, that is if $R_f = \{a\}$ for some a, we shall write $f \equiv a$.

Now we define a notion of the iterative system, equivalent to the well-known notion of feedback shift register (see for

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

instance [1,3,4]). For this purpose, let k denote an arbitrary fixed natural number while $A = \{0,1\}$. Then $\mathcal{F}^k$ denotes the set of all functions $\varphi: A^k \longrightarrow A$ such that:

(i) if $x_1,\ldots,x_k \in D_\varphi$, then $x_2,\ldots,x_k$, $\varphi(x_1,\ldots,x_k) \in D_\varphi$.

Elements of $\mathcal{F}^k$ are called feedback functions.

Let $\mathcal{T}^k$ denote the set of all total functions from A^k into A. The elements of $\mathcal{T}^k$ are examples of feedback functions, that is, $\mathcal{T}^k \subseteq \mathcal{F}^k$.

2.1. Definition. Let $\varphi \in \mathcal{F}^k$. The mapping $I_\varphi: D_\varphi \longrightarrow A^\infty$ such that for each $u \in D_\varphi$ we have: if $I_\varphi(u) = c_1,c_2,\ldots$, then:

$$c_1,\ldots,c_k = u$$

and

$$c_{i+k} = \varphi(c_i,\ldots,c_{i+k-1}), \text{ for } i \in N,$$

is said to be the iterative system of φ. The elements of the set $C_\varphi = \{I_\varphi(u): u \in D_\varphi\}$ are called computations of I_φ. If $\varphi \in \mathcal{T}^k$, then the iterative system I_φ is called total.

2.2. Definition. Let φ and ψ be arbitrary functions from $\mathcal{F}^k$. The iterative system I_ψ is called a subsystem of I_φ iff $D_\psi \subseteq D_\varphi$ and $\psi = \varphi$ on D_ψ.

Note that I_ψ is a subsystem of I_φ iff $C_\psi \subseteq C_\varphi$.

Now we establish a class of subsystems of an iterative system, representing - in a natural way - this system. For this purpose let us consider an arbitrary $\varphi \in \mathcal{F}^k$. We define a partition of D_φ in the following way. Elements u' and u'' of D_φ are called similar (in symbols: $u' \sim u''$) iff there exist natural numbers i and j such that:

$$\text{if } I_\varphi(u') = c'_1, c'_2,\ldots \text{ and } I_\varphi(u'') = c''_1, c''_2,\ldots,$$

$$\text{then } c'_i, c'_{i+1},\ldots = c''_j, c''_{j+1},\ldots .$$

We at once see that $\sim$ is an equivalence relation on D_φ. The family of all its cosets will be denoted by $D_\varphi/\sim$. Let us

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

observe that different feedback functions may correspond to the same relation of similarity. For a class of binary total feedback functions this problem has been studied by Łatko in [5].

2.3. Definition. Let $\varphi \in \mathcal{F}^k$. An iterative system I_α, for $\alpha \in \mathcal{F}^k$, is called an atom of I_φ iff the following conditions hold:

(a) $D_\alpha \in D_\varphi /\sim$,

(b) I_α is a subsystem of I_φ.

It is an immediate consequence of the above definition that if $\mathcal{A}$ is the set of all atoms of I_φ, then we have:

$$\text{if } I_\alpha \in \mathcal{A}, \text{ then } D_\alpha/\sim \; = \{D_\alpha\},$$

and

$$C_\varphi = \bigcup_{I_\alpha \in \mathcal{A}} C_\alpha .$$

2.4. Example. For $k = 4$ consider the function $\varphi:\{0,1\}^k \longrightarrow \{0,1\}$ defined as follows:

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in \{1010,1101\}, \\ 0 & \text{for } x \in \{0,1\}^4 - \{1010,1101\}. \end{cases}$$

It is easy to see that I_φ has the following computations: $0001(0)_\omega$, $001(0)_\omega$, $01(0)_\omega$, $1(0)_\omega$, $(0)_\omega$, $0011(0)_\omega$, $011(0)_\omega$, $11(0)_\omega$, $0111(0)_\omega$, $111(0)_\omega$, $11011(0)_\omega$, $1001(0)_\omega$, $1011(0)_\omega$, $1111(0)_\omega$, and $(01)_\omega$, $(10)_\omega$, where an equality $c = c_1, c_2, \ldots = c_1, \ldots c_p, (c_{p+1}, \ldots, c_{p+r})_\omega$ means that for some $p \geq 1$ and $r \geq 0$ we have $c_{p+i+nr} = c_{p+i}$, for all natural n and i.

In order to establish the atoms of I_φ note that the set C_φ corresponds to the directed graph presented in Figure 1. Then we see that I_φ consists of two atoms: $I_{\alpha'}$ and $I_{\alpha''}$, where

$$\alpha'(x) = \begin{cases} \varphi(x), & \text{for } x \in \{0101,1010\} \\ \text{undefined, otherwise,} \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\alpha''(x) = \begin{cases} \varphi(x), & \text{for } x \in \{0,1\}^4 - \{0101,1010\} \\ \text{undefined, otherwise.} \end{cases}$$

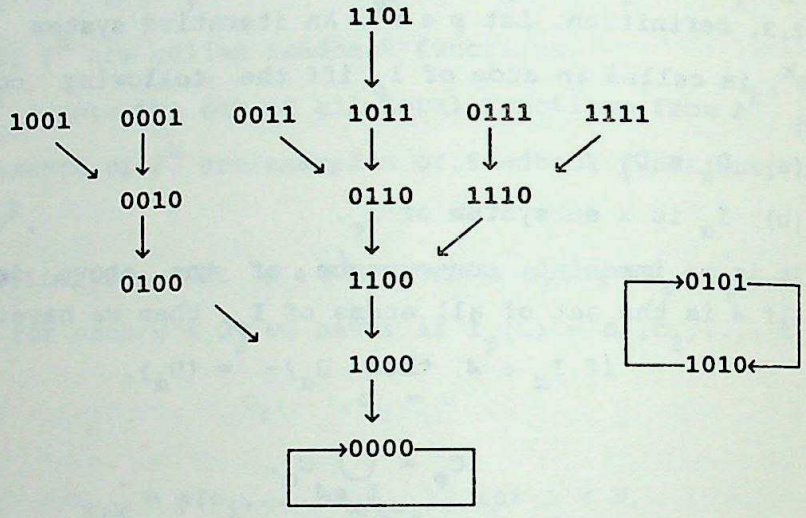


Figure 1.

Let us observe, additionally, that an atom of an iterative system need not be its minimal subsystem. For instance, $I_{\alpha''}$ contains the subsystem I_ψ, where $D_\psi = \{0000\}$ and $\psi(0000) = 0$, which is a minimal subsystem of I_φ.

2.5 Proposition. Let $\varphi \in \mathcal{F}^k$. A subsystem I_ψ of I_φ is its atom iff it is a minimal subsystem such that: either $\psi = \varphi$ or φ restricted to $D_\varphi - D_\psi$ is also a feedback function. □

The proof is obvious. It is easy to see that the decomposition of an iterative system into the family of all its atoms is unique.

3. Notion of locally reducible iterative system

Note that each function $\varphi \in \mathcal{F}^k$ determines the unique infinite sequence of the feedback functions $\psi_0, \psi_1, \ldots$ such that

$$\psi_0 \in \mathcal{F}^k, \quad \psi_1 \in \mathcal{F}^{k+1}, \quad \ldots$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$C_\varphi = C_{\psi_0} = C_{\psi_1} = \dots .$$

This sequence is defined as follows

(ii) $\psi_0 = \varphi$,

(iii) if $i > 0$ then: $D_{\psi_i} = \{(u, \psi_{i-1}(u)) : u \in D_{\psi_{i-1}}\}$ and

$$\psi_i(x_1, \dots, x_{k+i}) = \psi_{i-1}(x_2, \dots, x_{k+i}),$$

for $x_1, \dots, x_{k+1} \in D_{\psi_i}$.

If $k > 1$ then it may happen that there exists also $\xi \in \mathcal{F}^{k-1}$ such that $C_\xi = C_\varphi$. These observations lead to the following concept.

3.1. Definition. Let $\varphi \in \mathcal{F}^k (k>1)$. The iterative system I_φ is said to be reducible iff there exists $\xi \in \mathcal{F}^{k-1}$ such that $C_\varphi = C_\xi$.

It is obvious that the systems I_φ and I_ξ, in the above definition, are different and ξ is unique. This unique function ξ will be denoted by $\varphi^\cdot$ while φ will be called reducible. Examples of reducible iterative systems may be obtained by using the construction described at the beginning of this section.

Now we assume that if $x = x_1, \dots, x_{k-1}, x_k \in \{0,1\}^k$, then $x^\cdot$ denotes the element of $\{0,1\}^{k-1}$ such that $x^\cdot = x_1, \dots, x_{k-1}$. Similarly, for $G \subseteq \{0,1\}^k$ we define $G^\cdot = \{x^\cdot : x \in G\}$.

3.2. Proposition. If $\varphi \in \mathcal{F}^k$ is a reducible function, then $\varphi^\cdot$ is defined in the following way:

(a) $D_{\varphi^\cdot} = D_\varphi^\cdot$,

(b) if $x_1, \dots, x_k \in D_\varphi$ then $\varphi^\cdot(x_1, \dots, x_{k-1}) = x_k$.

Proof. This proof follows immediately from the definition of reducible iterative systems, if we note that for arbitrary elements x and y of D_φ, if $x^\cdot = y^\cdot$, the $x = y$. □

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3.3. Definition. An iterative system is said to be locally reducible iff each of its atoms is a reducible system.

By $\mathcal{LR}^k$ we denote the set of all elements of $\mathcal{I}^k$ the iterative systems of which are locally reducible. An example of $\varphi \in \mathcal{LR}^k$ is given in Example 3.8.

3.4. Definition. Let $\varphi \in \mathcal{LR}^k$. By a partition of I_φ we mean each family $\{I_{\varphi_0},\dots,I_{\varphi_m}\}$, $m \geq 0$, of its subsystems such that:

(a) each of these subsystems is reducible:

(b) if $i \neq j$, then $D_{\varphi_i} \cap D_{\varphi_j} = \emptyset$;

(c) $D_\varphi = \bigcup_{i=0}^{m} D_{\varphi_i}$.

The partition $\{I_{\varphi_0},\dots,I_{\varphi_m}\}$ will be denoted by $\mathcal{P}(\varphi_0,\dots,\varphi_m)$.

There exists at least one partition of a locally reducible iterative system; it is formed by the atoms of this system. Generally, we have the following dependence between the atoms of a locally reducible iterative system and arbitrary of its partitions.

3.5. Proposition. Let $\varphi \in \mathcal{LR}^k$. A family of reducible subsystems of I_φ forms one of its partitions iff each atom of I_φ is a subsystem of some system from this family.

Proof. The sufficiency is immediate. In order to prove the necessity, suppose that there exists an atom I_α of I_φ, that is not a subsystem of any system from a partition $\mathcal{P}(\varphi_0,\dots,\varphi_m)$, $m \geq 0$. This means that there exist computations c' and c'' of I_α such that $c' \in C_{\psi_1}$ and $c'' \in C_{\psi_2}$, for some different ψ_1 and ψ_2 from $\mathcal{P}(\varphi_0,\dots,\varphi_m)$. But on the other hand, because I_α is an atom of I_φ, there must exist natural numbers i and j for which $c'_i, c'_{i+1},\dots = c''_j, c''_{j+1},\dots$. Thereby $C_{\psi_1} \cap C_{\psi_2} \neq \emptyset$, which is inconsistent with the definition of a partition. □

3.6. Definition. Let $\varphi \in \mathcal{LR}^k$ and let $\mathcal{P}(\varphi_0,\dots,\varphi_m)$ be one of the partitions of I_φ. The family $\{I_{\varphi_0},\dots,I_{\varphi_m}\}$ is said to

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

be a resolution of I_φ. The elements of a resolution are said to be its components.

The resolution $\{I_{\xi_0},\dots,I_{\xi_m}\}$ of I_φ will be denoted by $\mathcal{R}(\xi_0,\dots,\xi_m)$.

3.7. **Definition.** Let $\varphi \in \mathcal{LR}^k$ and let $n \in N$. Then I_φ is said to be n-chromatic iff n is the smallest number for which there exists an n-element partition of I_φ. If I_φ is n-chromatic then n is said to be the chromatic number of I_φ while each of the n-element partitions (resolutions) of I_φ is said to be its chromatic partition (resolution).

The chromatic number of a locally reducible iterative system characterizes its complexity. (See Section 5, especially Theorem 5.3.). The unique 1-chromatic locally reducible iterative systems are the reducible systems. Some class of 2-chromatic iterative systems has been defined and examined in [9]. Now we give an example of a 3-chromatic locally reducible iterative system.

3.8. **Example.** Consider the function $\omega:\{0,1\}^5 \longrightarrow \{0,1\}$ such that

$$\omega(x_1,\dots,x_5) = \begin{cases} \bar{x}_1, & \text{for } (x_1,\dots,x_5) \text{ from} \\ & \{00011,01100,10011,11100\}, \\ x_1, & \text{for the other elements of } \{0,1\}^5, \end{cases}$$

where $\bar{x}_1 = x_1 + 1$ in the Galois field F_2.

One can easily check that I_ω has the atoms $I_{\alpha_0},\dots,I_{\alpha_7}$, where:

$D_{\alpha_0} = \{00000\}$,

$D_{\alpha_1} = \{00001,00010,00100,01000,10000\}$,

$D_{\alpha_2} = \{00011,00111,01110,11100,11000,10001\}$,

$D_{\alpha_3} = \{00110,01100,11001,10011\}$,

$D_{\alpha_4} = \{00101,01010,10100,01001,10010\}$,

$D_{\alpha_5} = \{01011,10110,01101,11010,10101\}$,

$D_{\alpha_6} = \{01111,11110,11101,11011,10111\}$,

$D_{\alpha_7} = \{11111\}$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and - for each $i \in \{0,1,\ldots,7\}$ - $\alpha_i(x) = \omega(x)$, if $x \in D_{\alpha_i}$.

Each atom of I_ω is a reducible system. For instance:

$$D^*_{\alpha_2} = \{0001,0011,0111,1110,1100,1000\},$$

and

$$\alpha_2^*: \begin{cases} 0001 \longrightarrow 1 \\ 0011 \longrightarrow 1 \\ 0111 \longrightarrow 0 \\ 1110 \longrightarrow 0 \\ 1100 \longrightarrow 0 \\ 1000 \longrightarrow 1. \end{cases}$$

Thereby I_ω is a locally reducible iterative system.

In order to find a chromatic resolution of I_ω note that a subsystem of I_ω is reducible iff it has no atoms $I_{\alpha'}$ and $I_{\alpha''}$ for which $D^*_{\alpha'} \cap D^*_{\alpha''} \neq \emptyset$ (compare with Proposition 3.2). Let us consider the undirected graph G_ω, with $\alpha_0,\ldots,\alpha_7$ as the vertices, such that α_i is incident to α_j iff $\alpha_i \neq \alpha_j$ and $D^*_{\alpha'} \cap D^*_{\alpha''} \neq \emptyset$ (presented in Figure 2). The chromatic number of G_ω is identical with the chromatic number of I_ω (we think that it is an account for the notion "chromatic number of locally reducible iterative system"). We can easily check that I_ω is 3-chromatic

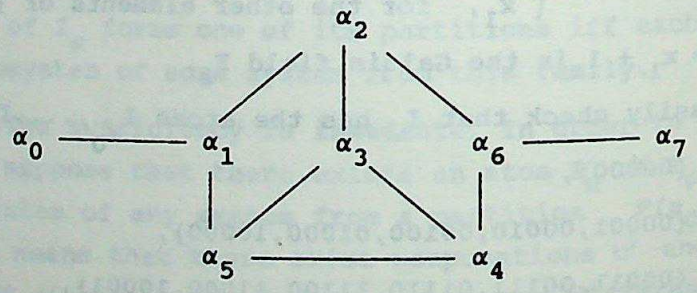


Figure 2.

and that the resolution $\mathcal{R}(\xi_0,\xi_1,\xi_2)$, where:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\xi_0(x) = \begin{cases} \alpha_1^\bullet(x), & \text{if } x \in D_{\alpha_1}^\bullet \\ \alpha_3^\bullet(x), & \text{if } x \in D_{\alpha_3}^\bullet \\ \alpha_6^\bullet(x), & \text{if } x \in D_{\alpha_6}^\bullet \\ \text{undefined, otherwise,} \end{cases}$$

$$\xi_1(x) = \begin{cases} \alpha_5^\bullet(x), & \text{if } x \in D_{\alpha_5}^\bullet \\ \text{undefined, otherwise,} \end{cases}$$

$$\xi_2(x) = \begin{cases} \alpha_0^\bullet(x), & \text{if } x \in D_{\alpha_0}^\bullet \\ \alpha_2^\bullet(x), & \text{if } x \in D_{\alpha_2}^\bullet \\ \alpha_4^\bullet(x), & \text{if } x \in D_{\alpha_4}^\bullet \\ \alpha_7^\bullet(x), & \text{if } x \in D_{\alpha_7}^\bullet \\ \text{undefined, otherwise,} \end{cases}$$

is one of its chromatic resolutions.

4. Locally reducible iterative systems with a given resolution

Each locally reducible iterative system is determined by any of its resolutions. Accordingly, we can search for conditions that must be satisfied by each family of iterative systems forming a resolution of such an iterative system. Using such families of iterative systems we can construct locally reducible iterative systems with given resolutions.

In particular cases such a construction can be easy. It deals, for example, with the functions $\varphi \in \mathcal{T}^k$ defined in [9] as follows:

$$\varphi(x_1,\ldots,x_k) = \xi(x_1,\ldots,x_{k-1}) + \xi(x_2,\ldots,x_k) + x_k \text{ (in } F_2),$$

for $\xi \in \mathcal{T}^{k-1}$ $(k>1)$. In this case φ is locally reducible and the family $\{I_\xi, I_{\bar{\xi}}\}$ is the unique chromatic resolution of I_φ, where $\bar{\xi} = \xi + 1$ in F_2.

In the case of other locally reducible iterative systems, for each of them there exist, in general, many chromatic resolutions. That is mainly what makes impossible to give likely direct formulas for dependences between locally

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

reducible iterative systems and components of corresponding resolutions. But some fundamental dependence between the components of an arbitrary resolution of a locally reducible iterative system is the same for all such systems and it will be used for construction of them.

4.1. Theorem. A family $\{I_{\xi_i} : \xi_i \in \mathcal{F}^{k-1}$ for $i \in \{0,\dots,m\}\}$, $m \geq 0$, is a resolution of a locally reducible iterative system iff for arbitrary i and j from $\{0,\dots,m\}$, such that $i \neq j$, we have

(a) $$\xi_i(x) \neq \xi_j(x),$$

for $x \in D_{\xi_i} \cap D_{\xi_j}$.

Proof. Necessity. Let $\varphi \in \mathcal{LR}^k$ and let $\mathcal{P}(\varphi_0,\dots,\varphi_m)$, $m \geq 0$, be a partition of I_φ. Then, because of Proposition 3.2 as well as (3.4.b), we have

$$\varphi_i^\cdot(x) \neq \varphi_j^\cdot(x),$$

for $i \neq j$, if $x \in D^\cdot_{\varphi_i} \cap D^\cdot_{\varphi_j}$. Thereby (a) holds.

Sufficiency. For arbitrary function $\xi_0,\dots,\xi_m$ from $\mathcal{F}^{k-1}$, fulfilling (a), we shall construct a partition of some locally reducible iterative system I_φ, and next - the function φ. For each $i \in \{0,\dots,m\}$ we set:

$$D_{\varphi_i} = \{x_1,\dots,x_{k-1}, \quad \xi_i(x_1,\dots,x_{k-1}) : x_1,\dots,x_{k-1} \in D_{\xi_i}\}$$

and

$$\varphi_i(x_1,\dots,x_k) = \xi_i(x_2,\dots,x_k),$$

for $x_1,\dots,x_k \in D_{\varphi_i}$. It immediately follows from the above construction that $\varphi_0^\cdot = \xi_0,\dots, \varphi_m^\cdot = \xi_m$, what implies (3.4.a), while (3.4.b) follows directly from (a). Thus $\{I_{\varphi_0},\dots,I_{\varphi_m}\}$ is a partition of the locally reducible iterative system I_φ, where:

$$D_\varphi = \bigcup_{i=0}^{m} D_{\varphi_i} \quad \text{and} \quad \varphi(x) = \varphi_i(x), \quad \text{for } x \in D_{\varphi_i}.$$

It completes this part of the proof. □

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4.2. Corollary. A resolution $\mathcal{R}(\xi_0,\ldots,\xi_m)$, $m \geq 0$, of a locally reducible iterative system is a chromatic resolution iff we have

$$D_{\xi_i} \cap D_{\xi_j} \neq \emptyset, \quad \text{for } i \neq j. \qquad \square$$

According to Theorem 4.1, each family $\{I_{\xi_0},\ldots,I_{\xi_m}\}$, $m \geq 0$, satisfying (4.1.a), corresponds to some locally reducible iterative system. In order to formalize the construction of such a system - described in the proof of that theorem - we make the following definition.

4.3. Definition. Let $\varphi \in \mathcal{LR}^k$. By the profile of a resolution $\mathcal{R}(\xi_0,\ldots,\xi_m)$, $m \geq 0$, of I_φ we mean the family $F = \{F_x : x \in D_\varphi^*\}$ of mappings $F_x : \{0,\ldots,m\} \longrightarrow \{0,1\}$ such that

(a) $$F_x(i) = \begin{cases} \xi_i(x), & \text{if } x \in D_{\xi_i}, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

for $i \in \{0,\ldots,m\}$.

Now we can state the following consequence of Theorem 4.1.

4.4. Corollary. If F is the profile of a resolution of a locally reducible iterative system, then each of the mappings $F_x \in F$ is one-to-one. $\square$

4.5. Theorem. Let $\varphi \in \mathcal{LR}^k$. If $F = \{F_x : x \in D_\varphi^*\}$ is the profile of a resolution of I_φ, then for all $x_1,\ldots,x_k \in D_\varphi$ we have:

(a) $$\varphi(x_1,\ldots,x_k) = F_{x_2,\ldots,x_k}(F^{-1}_{x_1,\ldots,x_{k-1}}(x_k)),$$

where F_x^{-1} denotes the inverse mapping to F_x.

Proof. Le F be the profile of $\mathcal{R}(\xi_0,\ldots,\xi_m)$, $m \geq 0$. For each $x_1,\ldots,x_k \in D_\varphi$ there exists exactly one function $\xi_i \in \{\xi_0,\ldots,\xi_m\}$ such that

$$I_\varphi(x_1,\ldots,x_k) = I_{\xi_i}(x_1,\ldots,x_{k-1}).$$

Then, for $c_1, c_2,\ldots = I_\varphi(x_1,\ldots,x_k)$ we have:

and $$c_k = x_k = \xi_i(x_1,\ldots,x_{k-1})$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$c_{k+1} = \varphi(x_1, x_2, \ldots, x_k) = \xi_i(x_2, \ldots, x_k).$$

By definition of F we obtain the following equalities:

$$\varphi(x_1, \ldots, x_k) = F_{x_2, \ldots, x_k}(i)$$

and

$$x_k = F_{x_1, \ldots, x_{k-1}}(i),$$

which imply (a). □

4.6. Example. We shall construct a locally reducible iterative system with a given resolution. For this purpose, let us consider the four functions $\xi_0, \xi_1, \xi_2, \xi_3$ from $\mathcal{F}^2$, defined in Table 1, where the symbol * informs for which arguments these functions are undefined. Since the above functions fulfill the assumptions of

x	$\xi_0(x)$	$\xi_1(x)$	$\xi_2(x)$	$\xi_3(x)$
00	0	1	*	*
01	*	0	1	*
10	*	0	1	*
11	*	*	0	1

Table 1.

u	$\varphi_0(u)$	$\varphi_1(u)$	$\varphi_2(u)$	$\varphi_3(u)$
000	0	*	*	*
001	*	0	*	*
010	*	0	*	*
011	*	*	0	*
100	*	1	*	*
101	*	*	1	*
110	*	*	1	*
111	*	*	*	1

Table 2.

Theorem 4.1, the corresponding iterative systems form a resolution of some locally reducible iterative system I_φ. Accomplishing the construction described in the sufficiency proof of that theorem we obtain the functions whose iterative systems form a partition of I_φ, presented in Table 2. Then we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\varphi(x_1,x_2,x_3) = x_1,$$

for all $x_1,x_2,x_3 \in \{0,1\}^3$. On the other hand Table 3 defines the profile of this resolution, while Table 4 defines $F^{-1} = \{F_x^{-1}: F_x \in F\}$. Now, applying Theorem 4.5 we can check that

$$\varphi(x_1,x_2,x_3) = x_1;$$

i	0	1	2	3
$F_{00}(i)$	0	1	*	*
$F_{01}(i)$	*	0	1	*
$F_{10}(i)$	*	0	1	*
$F_{11}(i)$	*	*	0	1

Table 3.

t	0	1
$F_{00}^{-1}(t)$	0	1
$F_{01}^{-1}(t)$	1	2
$F_{10}^{-1}(t)$	1	2
$F_{11}^{-1}(t)$	2	3

Table 4.

x	$\eta_0(x)$	$\eta_1(x)$
00	0	1
01	1	0
10	1	0
11	0	1

Table 5.

for instance: $\varphi(011) = F_{11}(F_{01}^{-1}(1)) = F_{11}(2) = 0$.

We conclude this example with the note that the above resolution is not chromatic one. The unique chromatic resolution has the total components I_{η_0} and I_{η_1}, defined in Table 5. Thereby, I_φ is 2-chromatic.

5. Filters of iterative systems

Now we describe a method for recognition of locally reducible iterative systems without the direct examination of the reducibility of atoms. To this purpose, we assume that q is a power of some prime number, i.e. $F_q = \{0,1,\ldots,q-1\}$ is the set of the elements of a finite field. From now on we consider the total feedback functions, only. This is a technical restriction that allows to avoid discussion on most formal details as well as to present the results in a simple form.

Note now, that it is a direct consequence of Theorem 4.5 that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(iv) $$F^{-1}_{x_1,\ldots,x_{k-1}}(x_k) = F^{-1}_{x_2,\ldots,x_k}(\varphi(x_1,\ldots,x_k)),$$

for each $x_1,\ldots,x_k \in \{0,1\}^k$. The last observation leads to the following definition.

5.1. Definition. Let $\varphi \in \mathcal{T}^k$. By a filter[1] of the iterative system I_φ we mean each total function $f:\{0,1\}^k \longrightarrow F_q$ such that

(a) $$f(x_1,\ldots,x_k) = f(x_2,\ldots,x_k,\varphi(x_1,\ldots,x_k)),$$

for $x_1,\ldots,x_k \in \{0,1\}^k$.

Note that f if is a filter of I_φ and $c_1,c_2,\ldots \in C_\varphi$, then

$$f(c_1,\ldots,c_k) = f(c_2,\ldots,c_{k+1}) = \ldots .$$

Thus, we may say that f "colours" the computations of I_φ. It is easy to see that the cardinality of R_f is not greater than the number of the atoms of I_φ. For a locally reducible iterative system we want to find filters colouring the computations of this system, according to the distributions executed by its resolutions. That is, we would like to colour all computations of an arbitrary component of this resolution with the same colour while computations of different components - with different colours. Then, for such a filter f the cardinality of R_f is not smaller than the chromatic number of I_φ. An example of such a filter is given by (iv), if we set

$$f(x_1,\ldots,x_k) = F^{-1}_{x_1,\ldots,x_{k-1}}(x_k),$$

where $F = \{F_u : u \in \{0,1\}^{k-1}\}$ is the profile of a resolution of a locally reducible iterative system I_φ. So that, Corollary 4.4 and Theorem 4.5 imply the following conclusion.

5.2. Theorem. An iterative system is locally reducible iff it has a filter f such that for arbitrary $x_1,\ldots,x_{k-1}, x'_k$ and $x_1,\ldots,x_{k-1},x''_k$ from $\{0,1\}^k$ we have

(a) if $x'_k \neq x''_k$, then $f(x_1,\ldots,x_{k-1},x'_k) \neq f(x_1,\ldots,x_{k-1},x''_k)$. □

---

[1] In this paper the word "filter" is used in the meaning connected with the notion of filter that appears in the field of digital signal processing.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In order to analyse filters of an iterative system we must define a way of representation of such functions in the finite fields. We shall use a well-known result of finite field theory that any total function $h: F_q^k \longrightarrow F_q$ can be represented by a unique polynomial in k variables, of degree at most q-1 in each variable. Accordingly, for an arbitrary partial function $h: F_q^k \longrightarrow F_q$ each q-tuple $(h_0,\ldots,h_{q-1})$ of functions such that:

$$h_i: D_h \longrightarrow F_q, \quad \text{for } i \in \{0,1,\ldots,q-1\},$$

and

$$h(x_1,\ldots,x_k) = \sum_{i=0}^{q-1} h_i(x_1,\ldots,x_{k-1})x_k^i, \quad \text{for } x_1,\ldots,x_k \in D_h,$$

is said to be a representation of h in the field F_q.
If $(h_0,\ldots,h_{q-1})$ is a representation of h in F_q, then we shall write

$$h = h_0 + h_1x_k + \ldots + h_{q-1}x_k^{q-1},$$

assuming that x_k is the symbol of the function $(x_1,\ldots,x_k) \longmapsto x_k$. Let us note yet that if h is not total in F_q^k, then it may have more than one representation. In particular, if $h:\{0,1\}^k \longrightarrow F_q$, then one of the representations has the form

$$h = h_0 + h_1x_k,$$

(with the addition and multiplication from F_q).

Let us now consider an arbitrary field F_q and the filters $f = f_0+f_1x_k$ of an iterative system I_φ. Then (5.1.a) can be written in the from

$$\text{(v)} \qquad \begin{aligned} &f_0(x_1,\ldots,x_{k-1}) - f_0(x_2,\ldots,x_k)+ \\ &+ f_1(x_1,\ldots x_{k-1})x_k - f_1(x_2,\ldots,x_k)\,\varphi(x_1,\ldots,x_k) = 0, \end{aligned}$$

for $x_1,\ldots,x_k \in \{0,1\}^k$, and it can be meant as a system of 2^k linear equations in $2\cdot 2^{k-1}$ unknowns $f_i(u)$, where the pair $(i,u) \in \{0,1,\ldots,q-1\} \times \{0,1\}^{k-1}$ is meant as the index of the unknown $f_i(u)$. If $q \geq |\{0,1\}^k/\sim|$, then Theorem 5.2 applied to the solutions f of (v) decide, whether or not, I_φ is a locally

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

reducible iterative system. In order to choice a suitable q note that

$$|\{0,1\}^k/\sim| \le \frac{1}{k} \sum_{d|k} \Phi(\frac{k}{d}) \cdot 2^d,$$

where Φ is the Euler's function, [7]. Thereby, we can state the following result.

5.3. Theorem. Let $\varphi \in \mathcal{T}^k$. The system I_φ is a locally reducible iterative system iff there exists q for which (v) has a solution $f = f_0 + f_1 x_k$ such that $f_1(u) \neq 0$, for each $u \in \{0,1\}^{k-1}$.

Proof. According to Theorem 5.2, the iterative system I_φ is locally reducible iff there exists q for which (v) has a solution $f = f_0 + f_1 x_k$ such that

$$f_0(u) + f_1(u) \cdot a' \neq f_0(u) + f_1(u) \cdot a'',$$

for each $u \in \{0,1\}^{k-1}$ and arbitrary different a' and a'' from $\{0,1\}$. Hence $f_1(u)(a'-a'') \neq 0$, what implies $f_1(u) \neq 0$. □

5.4. Example. For the function ω defined in Example 3.8 as

$$\omega(x_1,\dots,x_5) = \begin{cases} \bar{x}_1 & \text{for } (x_1,\dots,x_5) \text{ from} \\ & \{00011,01100,10011,11100\}, \\ x_1 & \text{for the other elements of } \{0,1\}^5, \end{cases}$$

after elimination of dependent equations and reduction of some unknowns, the system (v) has the form:

$f_0(0001) = f_0(0000) + f_1(0000)$,
$f_0(0010) = f_0(0000) + f_1(0000)$,
$f_0(0011) = f_0(0000) + f_1(0000) + f_1(0001) - f_1(0011)$,
$f_0(0100) = f_0(0000) + f_1(0000)$,
$f_0(0101) = f_0(0000) + f_1(0000) + f_1(0010)$,
$f_0(0110) = f_0(0000) + f_1(0000) + f_1(0001) - f_1(0011)$,
$f_0(0111) = f_0(0000) + f_1(0000) + f_1(0001)$,
$f_0(1000) = f_0(0000) + f_1(0000)$,
$f_0(1001) = f_0(0000) + f_1(0000) + f_1(0100)$,
$f_0(1010) = f_0(0000) + f_1(0000) + f_1(0010)$,
$f_0(1011) = f_0(0000) + f_1(0000) + f_1(0010) + f_1(0101)$,

CC-0. In,Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$f_0(1100) = f_0(0000) + f_1(0000) + f_1(0001),$$
$$f_0(1101) = f_0(0000) + f_1(0000) + f_1(0001) + f_1(0110) - f_1(0011),$$
$$f_0(1110) = f_0(0000) + f_1(0000) + f_1(0001),$$
$$f_0(1111) = f_0(0000) + f_1(0000) + f_1(0001) + f_1(0111),$$

and

$$f_1(0001) - f_1(1000) = 0,$$
$$f_1(0010) - f_1(0100) = 0,$$
$$f_1(0101) - f_1(1010) = 0,$$
$$f_1(0011) + f_1(1100) = 0,$$
$$f_1(0111) - f_1(1110) = 0,$$
$$f_1(1011) - f_1(1101) = 0,$$
$$f_1(0110) - f_1(0101) + f_1(1001) = 0,$$
$$f_1(0011) + f_1(1001) + f_1(0100) - f_1(0001) = 0,$$
$$f_1(1101) - f_1(1110) + f_1(0110) - f_1(0011) = 0.$$

Note that the part f_1 of a solution f is determined by the last nine equations, only. For $q = 2$ there is not any solution which satisfies the assumption of Theorem 5.3, because the unique permissible function $f_1 \equiv 1$ does not satisfy the equation:

$$f_1(0110) - f_1(0101) + f_1(1001) = 0.$$

However one of the solutions for $q = 3$ is a function $f = f_0 + f_1 x_k$, where

$$f_1(u) = \begin{cases} 1 & \text{for } u \in \{0,1\}^4 - \{0110,1001,1100,0111,1110\}, \\ 2 & \text{for } u \in \{0110,1001,1100,0111,1110\}. \end{cases}$$

So that I_ω is a locally reducible iterative system.

The last theorem implies also a conclusion dealing with the construction of locally reducible iterative systems. In order to express it, consider two functions

$$h: \{0,1\}^{k-1} \longrightarrow \{1,\ldots,q-1\},$$

and

$$\tau: \{0,1\}^{k-1} \longrightarrow \{0,1\}$$

with

$$\tau(0,\ldots,0) = 0.$$

Then we set:

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$U^{(q)}_{(\tau,h)}(x_1,\dots,x_k) =$$
$$[x_1-x_2]\cdot h(0,\dots,0) + \tau(x_2,\dots,x_k)\cdot h(x_2,\dots,x_k) +$$
$$+ [x_k - \tau(x_1,x_2,\dots,x_{k-2},x_{k-1})]\cdot h(x_1,x_2,\dots,x_{k-2},x_{k-1}) +$$
$$+ [x_{k-1} - \tau(0,x_1,x_2,\dots,x_{k-2})]\cdot h(0,x_1,x_2,\dots,x_{k-2}) +$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ [x_3 - \tau(0,\dots,0,x_1,x_2)]\cdot h(0,\dots,0,x_1,x_2) +$$
$$+ [x_2 - \tau(0,0,\dots,0,x_1)]\cdot h(0,0,\dots,0,x_1) -$$
$$- [x_k - \tau(0,x_2,\dots,x_{k-2},x_{k-1})]\cdot h(0,x_2,\dots,x_{k-2},x_{k-1}) -$$
$$- [x_{k-1} - \tau(0,0,x_2,\dots,x_{k-2})]\cdot h(0,0,x_2,\dots,x_{k-2}) -$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$[x_3 - \tau(0,\dots,0,0,x_2)]\cdot h(0,\dots,0,0,x_2),$$

in F_q, for each $x_1,\dots,x_k \in \{0,1\}^k$.

5.5. Theorem. If φ defined for $x_1,x_2,\dots,x_k \in \{0,1\}^k$ by

$$\varphi(x_1,x_2,\dots,x_k) = \begin{cases} \tau(x_2,\dots,x_k), & \text{if } x_1 = 0, \\ U^{(q)}_{(\tau,h)}(x_1,x_2,\dots,x_k)/h(x_2,\dots,x_k), & \text{if } x_1 \neq 0, \end{cases}$$

with $h(x_2,\dots,x_k) \neq 0$, is a binary function, then I_φ is a locally reducible iterative system.

Proof. In order to prove this theorem we must show that (v) has the same solution f_1 as the system

(a) $$R^0_\varphi\{f_1\}(x) = 0, \text{ for } x \in \{0,1\}^k,$$

where

$$R^0_\varphi\{f_1\}(x_1,\dots,x_k) =$$

$$\begin{cases} U^{(q)}_{(\tau,f_1)}(x_1,x_2,\dots,x_k) - \varphi(x_1,x_2,\dots,x_k) f_1(x_2,\dots,x_k), & \text{if } x_1 \neq 0, \\ \varphi(0,0,\dots,0)\ f_1(0,\dots,0), & \text{if } x_1 = 0, \end{cases}$$

with $\tau(u_1,\dots,u_{k-1}) = \varphi(0,u_1,\dots,u_{k-1})$.

For this purpose let us set:

$$L_\varphi\{f_0,f_1\}(x_1,\dots,x_k) =$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$f_0(x_1,\ldots,x_{k-1}) - f_0(x_2,\ldots,x_k) + f_1(x_1,\ldots,x_{k-1})x_k - f_1(x_2,\ldots,x_k)\varphi(x_1,x_2,\ldots,x_k),$$

$$R_\varphi\{f_1\}(x_1,\ldots,x_k) = f_1(x_1,\ldots,x_{k-1})x_k - f_1(x_2,\ldots,x_k)\varphi(x_1,x_2,\ldots,x_k)$$

and

$$L^0_\varphi\{f_0,f_1\}(x_1,\ldots,x_k) = \begin{cases} L_\varphi\{f_0,f_1\}(0,\ldots,0), \text{ if } x_1=\ldots=x_k=0 \\ \sum_{i=0}^{k-2} L_\varphi\{f_0,f_1\}(x_1,\ldots,x_1,x_2,\ldots,x_{k-i}), \\ \text{if } x_1=0,\ x_2,\ldots,x_k\neq 0,\ldots,0, \\ L_\varphi\{f_0,f_1\}(x_1,x_2,\ldots,x_k) + \\ \sum_{i=0}^{k-2}[L_\varphi\{f_0,f_1\}(0,\ldots,0,x_1,x_2,\ldots,x_{k-1-i}) - \\ -L_\varphi\{f_0,f_1\}(0,\ldots,0,x_2,\ldots,x_{k-1-i},x_{k-i})], \\ \text{if } x_1\neq 0. \end{cases}$$

Then the system of equations

(b) $$L^0_\varphi\{f_0,f_1\}(x) = 0, \text{ for } x \in \{0,1\}^k,$$

is equivalent to the following

(c) $$L_\varphi\{f_0,f_1\}(x) = 0, \text{ for } x \in \{0,1\}^k,$$

because

$$L_\varphi\{f_0,f_1\}(x_1,\ldots,x_k\} = L^0_\varphi\{f_0,f_1\}(x_1,\ldots,x_k) - L^0_\varphi\{f_0,f_1\}(0,x_1\ldots,x_{k-1}) + L^0_\varphi\{f_0,f_1\}(u_1,u_2,\ldots,u_k),$$

with

$$u_1,u_2,\ldots,u_k = \begin{cases} 0,x_2,\ldots,x_k, & \text{if } x_1 \neq 0 \text{ or } x_1=\ldots=x_k=0, \\ 0,0,\ldots,0, & \text{if } x_1 = 0 \text{ and } x_2,\ldots,x_k \neq 0,\ldots,0. \end{cases}$$

A direct computation shows that

$$L^0_\varphi\{f_0,f_1\}(x_1,\ldots,x_k) = R_\varphi\{f_1\}(x_1,\ldots,x_k) +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ \sum_{i=1}^{k-1} [R_\varphi\{f_1\}(0,\dots,0,x_1,x_2,\dots,x_{k-1-i}) -$$

$$- R_\varphi\{f_1\}(0,\dots,0,x_2,\dots,x_{k-i})] = R^0_\varphi\{f_1\}(x_1,\dots,x_k).$$

Therefore

$$L^0_\varphi\{f_0,f_1\}(x_1,\dots,x_k) = \begin{cases} R^0_\varphi\{f_1\}(x_1,\dots,x_k), & \text{if } x_1 \neq 0 \text{ or} \\ & x_1=\dots=x_k=0, \\ f_0(0,\dots,0) - f_0(x_2,\dots,x_k) + & \\ + \sum_{i=0}^{k-1} [R_\varphi\{f_1\}(x_1,\dots,x_1,x_2,\dots,x_{k-i}), & \\ & \text{otherwise.} \end{cases}$$

Then it is easy to see that (b) - as well as (c) - has the same solutions f_1 as (a). □

Let $(\tau,h)_q$ denotes the function φ defined by Theorem 5.5.

5.6. Corollary. If $\tau' \neq \tau''$, then $(\tau',h)_q \neq (\tau'',h)_q$. □

5.7. Corollary. If $\varphi = (\tau,h)_q$ then I_φ has a filter

$$f = f_0 + hx_k,$$

for some f_0. □

5.8. Example. Let us consider $h:\{0,1\}^{k-1} \longrightarrow \{0,1\}$ such that

$$h \equiv 1.$$

Then for $q = 2$ and for each permissible $\tau:\{0,1\}^{k-1} \longrightarrow \{0,1\}$ the function $\varphi = (\tau,h)_q$ is binary and it has filter of the form $f_0 + x_k$. Such filters determine the chromatic resolutions of I_φ, thereby I_φ is 2-chromatic. So that, we obtain the class of elementary iterative systems, defined in [9], with the chromatic resolutions of the form $\{I_\xi, I_{\bar\xi}\}$, for $\xi \in \mathcal{F}^{k-1}$ and $\bar\xi = \xi + 1$ in F_2 (see also Example 4.6). Then φ has a simple form, namely:

$$\varphi(x_1,\dots,x_k) = \xi(x_1,\dots,x_{k-1}) + \xi(x_2,\dots,x_k) + x_{k'}$$

with the addition from F_2.

At the end note that we can easily extend the definition of the locally reducible iterative systems onto the iterative systems in an arbitrary alphabet A. Then, if for some τ and h

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the function $\varphi = (\tau,h)_q$ is not binary then we can extend its domain to A^k - with τ and h extended to A^{k-1} - for a minimal set A for which $R_\varphi = A$. In this way we can construct all locally reducible iterative systems having filters of the form $f_0 + f_1 x_k$.

REFERENCES

[1] G. Birkhoff, T.C. Bartee: Modern applied algebra. McGraw-Hill Company, 1970.

[2] H. Fredricksen: A survey of full length nonlinear shift register cycle algorithms, SIAM Rev.,24(1982),195-221.

[3] S. W. Golomb: Shift register sequences, Holden-Day, San Francisco, 1967.

[4] R. Lidl, H. Niederreiter: Introduction to finite fields and their applications, Cambridge Univ. Press, 1986.

[5] M. Latko: The similitude of shift registers, Demonstratio Math. 19(1986), 469-485.

[6] M. Latko, J. Zurawiecki: Strong similitudes of the de de Bruijn sequences, (unpublished).

[7] J. Mykkeltveit: A proof of Golomb's conjecture for the de Bruijn graph, J.Combin. Theory 13(1975), 40-45.

[8] J. Zurawiecki: The maximal k-machines, Demonstratio Math. 11(1978), 509-518.

[9] J. Zurawiecki: Elementary k-iterative systems (the Binary Case), J. Inf. Process. Cybern. EIK 24(1988) 1/2, 51-64.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY,
20-950 LUBLIN, POLAND
Received April 26, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Valeriu Popa

SOME PROPERTIES OF ALMOST FEEBLY CONTINUOUS FUNCTIONS

In [4] Levine defines a set in a topological space X to be semiopen if there exists an open set U such that $U \subset A \subset Cl(U)$, where Cl(U) denotes the closure of U. A set is semi-closed if its complement is semi-open. The intersection of all the semi-closed sets containing a set A is the semi-closure of A, denoted by sCl(A). In a topological space a set A is feebly-open if there exists an open set U such that $U \subset A$ sCl(U) [6]. A set S of a topological space is α-set if $S \subset$ Int (Cl(Int(S))). It is known that S is α-set if and only if S is feebly-open.

A mapping $f: \longrightarrow Y$ is feebly continuous [6] if the inverse image by f of every open set of Y is feebly-open in X. A mapping $f:X \longrightarrow Y$ is almost continuous [19] if for each $x \in X$ and each open set $V \subset Y$ containing f(x), there exists an open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(V))$. In [7] Maheshawari, Chae and Jain introduced the concept of almost feebly continuous functions as a generalization of almost continuity and feebly continuity. In [16] Noiri introduced the notion of almost α-continuous function and proved that the function $f:X \longrightarrow Y$ is almost α-continuous if and only if is almost feebly continuous.

Definition 1. A function $g:X \longrightarrow Y$ is almost feebly continuous if the inverse image of every regular open set of Y is feebly-open in X [7].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 1. A function $g:X \longrightarrow Y$ is almost feebly continuous if for each $x \in X$ and each open set $V \subset Y$ containing $g(x)$, there exists a feebly-open set $U \subset X$ containing x such that $g(U) \subset sCl(U)$.

Proof. Let V be any open set containing $g(x)$, then by Theorem 3.2 of [16] there exists a feebly-open set $U \subset X$ containing x such that $g(U) \subset Int(Cl(V))$. V being an open set by Proposition 2.7(a) of [3] $Int(Cl(V)) = sCl(V)$ and thus $g(U) \subset sCl(V)$.

Conversely, let $V \subset Y$ be an open set containing $g(x)$. Then there exists a feebly-open set U containing x such that $g(U) \subset sCl(V) = Int(Cl(V)$ by Proposition 2.7(a) of [3]. By Theorem 3.2 of [16] g is almost feebly continuous.

Definition 2. The semi-frontier of a subset A of the topo-logical space X is the set $sFr(A) = sCl(A) \cap sCl(X-A)$ [1].

Theorem 2. If $g:X \longrightarrow Y$ is an almost feebly continuous function and $g^{-1}(sFr(G))$ is closed in X for each open set $G \subset Y$, then g is feebly continuous.

Proof. Let G be any open set of Y and $x \in X$ such that $g(x) \in G$. Then by Theorem 1, there exists a feebly-open set $V \subset X$ such that $x \in V$ and $g(V) \subset sCl(V)$. Let us consider the set $U = V-g^{-1}(sFr(G)) = V-(X-g^{-1}(sFr(G)))$. As $g^{-1}(sFr(G))$ is closed, then $X-g^{-1}(sFr(G))$ is open and by Lemma 3.3 of [13] U is feebly-open. As $sFR(G) = sCl(G) \cap sCl(X-G) = sCl(G) \cap (X-G)$ $= sCl(G)-G$, then $U = V \cap (X-g^{-1}(sCl(G)-G))$. As $x \in V$ and $g(x) \in G$ there follows that $x \in U$. If $y \in U$ then $y \in V$ and $y \bar{\in} g^{-1}(sCl(G)-G)$, thus $g(y) \in sCl(G)$ and then $g(y) \bar{\in} sCl(G)-G$. Therefore $g(y) \in G$. As U is feebly-open and contains x, it follows by Theorem 1.1 of [9] that f is feebly continuous.

Definition 3. A topological space X is said to be rim-compact if each point of X has a base of neighbourhoods with compact frontiers.

Theorem 3. If Y is a rim-compact space and $f:X \longrightarrow Y$ is an almost feebly continuous function with the closed graphs, then f is feebly continuous.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. Suppose $x \in X$ and V is any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set W such that $f(x) \in W \subset V$ and the frontier Fr(W) is compact. It is obvious that $f(x) \bar{\in} Fr(W)$. Thus, for each $y \in Fr(W)$, we have $(x,y) \bar{\in} G(f)$. Since G(f) is closed by Lemma of [5] there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x) \cap V(y) = \emptyset$. The family $\{V(y): y \in Fr(W)\}$ is a cover of Fr(W) by open sets of Y. Since Fr(W) is compact, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in Fr(W) such that $Fr(W) \subset \bigcup \{V(y_i): 1 \leq i \leq n\}$. Since f is almost feebly continuous by Theorem 3.2 of [16] there exists a feebly-open set $U_0 \subset X$ containing x such that $f(U_0) \subset Int(Cl(W))$. Put $U = U_0 \cap \{\cap U_{y_i}(x): 1 \leq i \leq n\}$, then by Lemma 3.3 of [13] U is a feebly-open set containing x and $f(U) \subset Int(Cl(W))$. Then $f(u) \cap (Y-W) = f(U) \cap Fr(W) \subset \bigcup_{i=1}^{n} f(U) \cap V(y_i) \subset \bigcup_{i=1}^{n} f(U_{y_i}(x)) \cap V(y_i) = \emptyset$. This shows that $f(U) \subset V$ and by Theorem 1.1 (ii) of [9] f is feebly continuous.

Theorem 4. If Y is rim-compact Hausdorff and $f: X \longrightarrow Y$ is almost feebly continuous then f is continuous.

Proof. From Theorem 4 of [12] Y is regular. Then by Proposition 2 of [7] f is feebly continuous and by Remark [9] f is continuous.

Definition 4. A subset S of a topological space X is said to be an $\mathcal{A}$-set if $S = U - V$, where U is an open set and V is a regular open set [20].

Lemma 1. Let X be a topological space. Then a subset of X is open if only if is both feebly-open and $\mathcal{A}$-set (Theorem 3.2 [20]).

Definition 5. A mapping $f: X \longrightarrow Y$ is said almost $\mathcal{A}$-continuous if for each regular open set $V \subset Y, f^{-1}(V)$ is an $\mathcal{A}$-set in X.

Remark 1. Every almost continuous is almost $\mathcal{A}$-continuous, the converse need not be true.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Example 1. (Ex.4.2 [20]). Let $I = [0,1]$ with the usual topology, $X = I \times I$, $Y = I$, $E = (1/3, 2/3)$. Let $f:X \longrightarrow Y$ defined as follows:

$$f((x,y)) = \begin{cases} 0 & \text{if } (x,y) \in I \times E \\ x & \text{if } (x,y) \in I \times (I-E) \end{cases}$$

Then f is almost $\mathcal{A}$-continuous but is not almost continuous.

Remark 2. Every almost continuous function is almost feebly continuous, the converse need not be true.

Example 2. (Ex.3.10 [14]). Let $X = \{a,b,c,d\}$ be and $T = \{\emptyset, \{c\}, \{d\}, \{a,c\}, \{c,d\}, \{a,c,d\}, X\}$. Let $Y = \{x,y,z\}$ be and $T = \{\emptyset, \{x\}, \{y\}, \{x,y\}, Y\}$. Let $f(X,T) \longrightarrow (Y,Q)$ be defined as follows $f(a) = z$ and $f(b) = f(c) = f(d) = \{y\}$. Then f is almost feebly continuous but is not almost continuous.

Theorem 5. A mapping $f:X \longrightarrow Y$ is almost continuous if and only if is both almost feebly continuous and almost $\mathcal{A}$-continuous.

Proof. Follows by Lemma 1 and Theorem 2.2 of [19].

Definition 6. Let X be a topological space and S be a subset of X. S is said to be preopen if $S \subset \text{Int}(\text{Cl}(S))$ [8].

Definition 7. The function $f:X \longrightarrow Y$ is H-almost continuous at $x \in X$ if for each open set $V \subset Y$ containing $f(x)$, the closure of $f^{-1}(V)$ is a neighbourhood of x. The function $f:X \longrightarrow Y$ is H-almost continuous if it has this property in any point $x \in X$ [2].

Definition 8. A mapping $f:X \longrightarrow Y$ is almost quasi continuous at $x \in X$ if for any open set $V \subset X$ containing x and for any open set $G \subset Y$ containing $f(x)$, there exist a non-empty open set $U \subset V$ such that $f(U) \subset \text{Int}(\text{Cl}(G))$.

The mapping $f:X \longrightarrow Y$ is almost quasi continuous if it has tis property in any point $x \in X$ [18].

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Definition 9. A function $f:X \longrightarrow Y$ is said be semiweakly continuous if for each open set V containing $f(x)$, there exists a semi-open set $U \subset X$ containing x such that $f(U) \subset sCl(V)$ [15].

Definition 10. A function $f:X \longrightarrow Y$ is said to be almost semi-continuous if for every $x \in X$ and for every regularly open set $G \subset Y$ containing $f(x)$, there exists a semi-open set $U \subset X$ containing x such that $f(U) \subset G$ [10].

By Theorem 3.1 of [17] and Corollary 3.3 of [17] follows that almost quasi continuity, almost semi-continuity and semi-weak continuity are equivalent.

Remark 3. By Theorem 1 follows that every almost feebly continuous mapping is almost quasi continuous, the converse need not be true.

Example 3. (Ex.5.12 [17]). Let $X = \{a,b,c\}$, $T = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $Q = \{\emptyset, X, \{a\}, \{b,c\}\}$. Let $f:(X,T) \longrightarrow (X,Q)$ be the identity function. Then f is almost quasi continuous but is not almost feebly continuous.

Example 4. (Ex.2.1 [19]). Let R the set of real numbers and T consists of $\emptyset$, R and the complements of all countable subsets of R. Let $X = \{a,b\}$ and $Q = \{X, \emptyset, \{a\}\}$. Let $f:(R,T) \longrightarrow (X,Q)$ be defined as follows

$$f(x) = \begin{cases} a & \text{if } x \text{ is rational} \\ b & \text{if } x \text{ is irrational} \end{cases}$$

Then f is almost feebly continuous, but f is not H-almost continuous.

Theorem 6. If a function $f:X \longrightarrow Y$ is H-almost continuous and almost quasi continuous, then f is almost feebly continuous.

Proof. Let V be regular open set of Y, then V is open. As f is H-almost ccntinuous by Theorem 1 of [8], $f^{-1}(V)$ is a preopen set of X. As f is almost quasi continuous by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 3.1 of [18], $f^{-1}(V)$ is semi-open in X. By Lemma 3.1 of [14], $f^{-1}(V)$ is feebly-open and by Definition 1, f is almost feebly continuous.

REFERENCES

[1] K. K. Dube, O. S. Panwar: Some properties of "s-connectedness between sets" and "set s-connected mappings", Indian J.Pure Appl. Math. 15(4) (1984) 343-354.

[2] T. Husain: Almost continuous mappings, Prace Mat. 10 (1960) 1-7.

[3] D. S. Janković: A note on mappings of extremally disconnected spaces, Acta Math. Hung. 46(1-2) (1985) 83-92.

[4] N. Levine: Semi-open sets semi-continuity in topological spaces, Amer. Math. Monthly 70(1963) 36-41.

[5] P. E. Long: Functions with closed graphs, Amer. Math. Monthly (1969), 830-932.

[6] S. N. Maheshwari, P. C. Jain: Some new mappings, Mathematica-Revue d'analise numerique et de la théorie de l'approximation, Mathematica, 24(47) 1-2(1982) 53-55.

[7] S. N. Maheshwari, Chae Gyu Ihn, P. C. Jain: Almost feebly continuous functions, U.I.T.Report, 13, 1(1982), 195-197.

[8] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb: On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt 53(1982) 47-53.

[9] A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb: α-continuous and α-open mappings, Acta Math.Hung. 41(3-4) (1983) 213-218.

[10] B. M. Munshi, D. S. Bassan: Almost semi-continuous mappings, Math. Student 49,3 (1981) 239-248.

[11] O. Njåstad: On some classes of nearly open sets, Pacific J.Math. 15(1965) 961-970.

[12] T. Noiri: Weak continuity and closed graphs, Časopis Pěst. Mat., 10 (1976) 379-382.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[13] **T. Noiri:** A function which preserves connected spaces, Časopis Pěst. Mat. 107(1982) 393-396.

[14] **T. Noiri:** On α-continuous functions, Časopis Pěst. Mat. 109 (1984) 117-126.

[15] **T. Noiri, B. Ahmad:** On semi-weakly continuous mappings, Kyungpook Math. J. 25,2 (1985) 123-126.

[16] **T. Noiri:** Almost α-continuous functions, Kyungpook Math. J. 28,1(1988) 71-77.

[17] **T. Noiri:** Almost quasi continuous functions (Submitted).

[18] **V. Popa:** On the decomposition of the quasicontinuites in topolgical spaces (Romanian), Stud. Cerc. Mat. 30(1978) 31-35.

[19] **M. K. Singal, A. R. Singal:** Almost continuous mappings, Yokohama Math. J., 16(1968) 63-73.

[20] **J. C. Tonga:** A decomposition of continuity, Acta Math. Hung. 48(1-2), (1986) 11-15.

HIGHER EDUCATION INSTITUTE, 5500 BACAU, ROMANIA
Received May 2nd, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Monika Kozak

AN ABSTRACT NONLINEAR TEMPORALLY INHOMOGENEOUS EQUATION

1. Introduction

The aim of this paper is to investigate the abstract semi-linear initial value problem

$$\frac{du}{dt} = A(t)u + f(t,u), \qquad t\in(0,T], \tag{1}$$

$$u(0) = u_0. \tag{2}$$

Let X be a real reflexive Banach space and for each $t\in[0,T]$ let $A(t):X \longrightarrow X$ denote a closed densely defined linear operator. Let u be a function from $\mathbb{R}$ into X, f be a nonlinear function from $\mathbb{R}\times X$ into X and $u_0\in X$.

Our task is to prove the theorem on the existence and uniqueness of the solution of the problem (1)-(2). We shall use the existence and uniqueness-theorem for the linear problem corresponding with the problem (1)-(2). In the paragraph 3 we shall formulate and prove such a theorem on existence and uniqueness for the linear problem which is similar to the one proved in [4] but using a bit weaker assumptions.

This paper was inspired by the work [5] where the analogous considerations were carried out for the operator A which was t independent.

2. Preliminaries and lemmas

Let us make the following assumptions about the family $\{A(t)\}$, $t\in[0,T]$.

Assumption A1. For each $t\in[0,T]$ A(t) is the generator of C_0-semigroup (see [4] chapter 3 or [6]), their domains

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$D(A(t))=D$ do not depend on t and D is dense in X.

Assumption A2. The $\{A(t)\}, t\in[0,T]$, is stable what means that there exist real numbers $M\geq 1$ and β such that

$$\left\| \prod_{j=1}^{k} (A(t_j)-\lambda)^{-1} \right\| \leq M(\lambda-\beta)^{-k}$$

for all $\lambda>\beta$, $0\leq t_1\leq t_2\ldots\leq t_k\leq T$, $k\in N$.

Assumption A3. For each $v\in D$ the function $t \longrightarrow A(t)v$ is of class C^1 on $[0,T]$.

Under these assumptions there exists (see [4], chapter 4) the unique function $U(t,s)$, called a fundamental solution of a equation (1), defined on the triangle $\Delta_T=\{(t,s) : 0\leq s\leq t\leq T\}$ and having the following properties:

a) $U(t,s)\in B(X)$ and $\|U(t,s)\| \leq Me^{\beta(t-s)}$.

b) $U(t,s)$ is strongly continuous on Δ_T, $U(s,s)=I$ and $U(t,r)U(r,s) = U(t,s)$ for $0\leq s\leq t\leq T$.

c) $U(t,s)$ maps D into D.

d) For each $v\in D$ $U(t,s)v$ is continuously differentiable on Δ_T and $\frac{\partial}{\partial t} U(t,s)v = A(t)U(t,s)v$, $\frac{\partial}{\partial s} U(t,s)v = - U(t,s)A(s)v$.

Lemma 1. Using assumptions A1, A2, A3 and additionally if:

1° for each $t\in[0,T]$ 0 belongs to the resolvent set of the operator $A(t)$,

2° the function $f : [0,T] \longrightarrow X$ satisfies the Lipschitz condition with a constant $L>0$,

then for almost all $s\in[0,T]$ there exists $\frac{d}{ds}(A^{-1}(s)f(s))$.

Proof. The assumption 1° implies that for each $s\in[0,T]$ there exists $A(s)^{-1}$ $(A(s)^{-1}: X \longrightarrow D)$ and because of A3 $A(s)^{-1}$ is strongly continuously differentiable and

$$\frac{d}{ds}(A(s)^{-1}) = - A(s)^{-1}(\frac{d}{ds} A(s))A(s)^{-1} \quad \text{(see [4])}.$$

The assumption of the reflexivity of X and 2° imply that for almost all $s\in[0,T]$ there exists $\frac{d}{ds}f(s)$ (see [5]). Hence, for almost all $s\in[0,T]$ there exists $\frac{d}{ds}(A(s)^{-1}f(s))$ and

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\frac{d}{ds}(A(s)^{-1}f(s)) = \frac{d}{ds}(A(s)^{-1})f(s)+A(s)^{-1}\frac{d}{ds}f(s). \tag{3}$$

Lemma 2. Under assumptions of Lemma 1 there exist Bochner integrals

$$\int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}f(s)\,ds \quad \text{and} \quad \int_0^t A(t)U(t,s)A(s)^{-1}\frac{d}{ds}f(s)\,ds.$$

Proof. The definition and properties of Bochner integral can be found e.g. in [5], [6]. Especially Lemma 4.1 in [5] implies the existence of Bochner integral $\int_0^t \frac{d}{ds}f(s)\,ds$.

Next, the functions $U(t,s)A(s)^{-1}$ and $A(t)U(t,s)A(s)^{-1}$ defined on Δ_T with values in $B(X)$ (for $A(t)U(t,s)A(s)^{-1}\in B(X)$ see e.g. [4]) are strongly continuous so $\|U(t,s)A(s)^{-1}\|$ and $\|A(t)U(t,s)A(s)^{-1}\|$ are bounded on Δ_T. This fact and the theorem that a function is Bochner integrable if and only if its norm is (Lebesque) integrable (see [6]) result of Lemma 2.

Lemma 3. Let $g : \Delta_T \longrightarrow X$. If:

1. For almost all $s\in[0,t]$ the function $[0,T]\ni t \longrightarrow g(t,s)$ is continuous;
2. For each $t\in[0,T]$, $g(t,\cdot)$ is summable over $[0,t]$;
3. There exists $N>0$ such that for $(t,s)\in\Delta_T$ $\|g(t,s)\|\leq N$

then the function $G : [0,T] \longrightarrow X$ defined as

$$G(t) := \int_0^t g(t,s)\,ds \tag{4}$$

is continuous.

Proof. Let $\Phi : \Delta_T \ni (t,u) \longrightarrow \Phi(t,u)\in X$ be defined as follows

$$\Phi(t,u) := \int_0^u g(t,s)\,ds\,, \quad u\leq t\leq T.$$

Let $(t,u)\in\Delta_T$ be fixed and let $t_n \longrightarrow t$ as $n \longrightarrow \infty$. Let for almost all $s : 0\leq s\leq u\leq t_n\leq T$, $g_n(s) := g(t_n,s)$. Condition 1. implies $g_n(s) \longrightarrow g(t,s)$ almost everywhere in $[0,u]$ and under condition 3. $\|g_n(s)\| \leq N$. Thus, on the basis of the Lebesgue theorem,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\Phi(t_n,u) = \int_0^u g_n(s)\,ds \longrightarrow \int_0^u g(t,s)\,ds = \Phi(t,u), \quad \text{as} \quad n \longrightarrow \infty.$$

That means, that $[0,T] \ni t \longrightarrow \Phi(t,u)$ is continuous for the fixed $u \in [0,t]$.

Next, let $u_1, u_2 \in [0,t]$, $t \in [0,T]$. Then

$$\|\Phi(t,u_1)-\Phi(t,u_2)\| \leq \left|\int_{u_1}^{u_2} \|g(t,s)\|ds\right| \leq N|u_1-u_2|.$$

It follows that $\Phi(t,\cdot) : [0,t] \longrightarrow X$ is uniformly continuous with respct to $t \in [0,T]$. Hence, the function $\Phi : \Delta_T \longrightarrow X$ is continuous and because of $G(t) = \Phi(t,t)$, $G : [0,T] \longrightarrow X$ is continuous.

3. Linear problem corresponding with (1)-(2)

Let X, as it was assumed in Introduction, be a real reflexive Banach space, for each $t \in [0,T]$ $A(t)$ be a closed linear operator from X into $X, u_0 \in X$. Let $f : [0,T] \longrightarrow X$ be a continuous function.

Let us consider the following linear problem:

$$\frac{du}{dt} = A(t)u+f(t), \quad t \in (0,T], \tag{5}$$

$$u(0) = u_0 . \tag{6}$$

Definition 1. A function $u : [0,T] \longrightarrow X$ is said to be a solution of the problem (5)-(6) if:

(i) for each $t \in (0,T]$ $u(t) \in D(A(t))$,

(ii) u is continuous on $[0,T]$ and of class C^1 on $(0,T]$,

(iii) for each $t \in (0,T]$ $\frac{du}{dt}(t) = A(t)u(t) + f(t)$ and $u(0) = u_0$.

Theorem 1. If the assumptions A1,A2,A3 are satisfied and if:

1. for each $t \in [0,T]$ 0 belongs to the resolvent set of $A(t)$,
2. $f:[0,T] \longrightarrow X$ satisfies the Lipschitz condition (with $L>0$),
3. $u_0 \in D$,

then the problem (5)-(6) has exactly one solution given by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds. \tag{7}$$

Proof. The properties of $U(t,s)$ imply (see [4]) that if the problem (5)-(6) has a solution $u(t)$ then that solution is given by (7) because

$$\frac{\partial}{\partial s}(U(t,s)u(s)) = U(t,s)\frac{du}{ds}(s) - U(t,s)A(s)u(s) = U(t,s)f(s)$$

and integrating both sides from 0 to t we obtain the formula (7). It is also easy to verify that the function $U(t,0)u_0$ (thanks to the assumption 3.) is a solution of the homogenous equation $\frac{du}{dt} = A(t)u$ with the initial condition (6).

So, the only fact left to prove is that the function

$$v(t) = \int_0^t U(t,s)f(s)\,ds \tag{8}$$

is the solution of the equation (5) with the initial condition $v(0) = 0$.

Now, we will try to get the function v in such a shape which is more convenient for investigating its properties.

Lemma 1 and the properties of $U(t,s)$ imply that for almost all $s\in[0,T]$ there exists $\frac{d}{ds}((U(t,s)A(s)^{-1}f(s))$ and

$$\frac{d}{ds}(U(t,s)A(s)^{-1}f(s)) = \tag{9}$$

$$= -U(t,s)f(s) - U(t,s)A(s)^{-1}\left(\frac{d}{ds}A(s)\right)A(s)^{-1}f(s) + U(t,s)A(s)^{-1}\frac{d}{ds}f(s).$$

The first and the second terms on the right-hand side of (9) are the continuous functions (of the variable s) so they are Bochner integrable (from 0 to t) and the integrability of the third term has been shown in Lemma 2. Thus, there also exists the Bochner integral $\int_0^t \frac{d}{ds}(U(t,s)A(s)^{-1}f(s))\,ds$.

Let us mark $U(t,s)A(s)^{-1}f(s) = F(s)$. We shall show that

$$\int_0^t \frac{d}{ds}F(s)\,ds = F(t) - F(0). \tag{10}$$

The proof of this formula can be made using the properties of Bochner integral and the construction of the sequence of the simple functions, approximating (almost everywhere) $\frac{d}{ds}F(s)$, which is analogous to the one used in the proof of Lemma 4.1 in [5]. It is sufficient to show that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(11) $$\frac{\|F(\xi)-F(\tau)\|}{\xi-\tau} \le \text{const} \quad \text{for} \quad \xi>\tau, \quad \xi,\tau\in[0,t]$$

$$\frac{\|U(t,\xi)A(\xi)^{-1}f(\xi) - U(t,\tau)A(\tau)^{-1}f(\tau)\|}{\xi-\tau} \le$$

$$\le \frac{\|U(t,\xi)A(\xi)^{-1}\|\|f(\xi) - f(\tau)\|}{\xi-\tau} +$$

$$+ \frac{\|U(t,\xi)A(\xi)^{-1} - U(t,\tau)A(\tau)^{-1}\|\|f(\tau)\|}{\xi-\tau} .$$

For each fixed $t\in[0,T]$ the function $U(t,s)A(s)^{-1}$ is strongly continuously differentiable on $[0,t]$ so, in particular, it satisfies the Lipschitz condition. The formula for the derivative implies that the Lipschitz constant for this function (with respect to s) can be (commonly) estimated by the constant which is t-independent. From that and using the assumption that the function f satisfies the Lipschitz condition one obtains the required inequality (11).

So, (9) and (10) give

$$\int_0^t \frac{d}{ds}(U(t,s)A(s)^{-1}f(s))ds = -\int_0^t U(t,s)f(s)ds -$$

$$-\int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}(A(s))A(s)^{-1}f(s)ds + \int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}f(s)ds,$$

$$\int_0^t \frac{d}{ds}(U(t,s)A(s)^{-1}f(s))ds = U(t,t)A(t)^{-1}f(t) - U(t,0)A(0)^{-1}f(0)$$

and using (8) finally we arrive to

(12) $$v(t) = U(t,0)A(0)^{-1}f(0) - A(t)^{-1}f(t) -$$

$$-\int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}(A(s))A(s)^{-1}f(s)ds + \int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}f(s)ds.$$

Thanks to this representation of v it is easily seen that for each $t\in[0,T]$ $v(t)\in D$ what is directly implied by the properties of the operators $A(s)^{-1}$, $U(t,s)$ and in case of integrals also by Lemma 4.3 in [5] and Lemma 2. So, let us find $A(t)v(t)$.

(13) $$A(t)v(t) = A(t)U(t,0)A(0)^{-1}f(0) - f(t) -$$

$$- A(t)\int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}(A(s))A(s)^{-1}f(s)ds +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ A(t)\int_0^t U(t,s)A(s)^{-1}\frac{d}{ds}f(s)ds.$$

Now, we will show that the function $t \longrightarrow A(t)v(t)$ is continuous on $[0,T]$. In order to do that, taking into account that the operators $A(t)U(t,s)A(s)^{-1}$ are bounded and using once again (with respect to terms with integrals) Lemma 4.3 in [5], it is sufficient to show the continuity of the functions:

$$t \longrightarrow \int_0^t A(t)U(t,s)A(s)^{-1}\frac{d}{ds}(A(s))A(s)^{-1}f(s)ds,$$

$$t \longrightarrow \int_0^t A(t)U(t,s)A(s)^{-1}\frac{d}{ds}f(s)ds,$$

and this follows from Lemma 3.

The last thing to do is to calculate $\frac{d}{dt}v(t)$. First, we will show the existence of the right-hand side derivative. Let $t,t+h \in (0,T]$, $h>0$

$$\frac{v(t+h)-v(t)}{h} = \frac{1}{h}\int_0^{t+h} U(t+h,s)f(s)ds - \frac{1}{h}\int_0^t U(t,s)f(s)ds =$$

$$= \frac{1}{h}[U(t+h,t)-I]\int_0^t U(t,s)f(s)ds + \frac{1}{h}\int_t^{t+h} U(t+h,s)f(s)ds.$$

Since $v(t)\in D$ so, the first term converges to $A(t)v(t)$ as $h \longrightarrow 0$. The second term converges to $f(t)$ because

$$\left\| \frac{1}{h}\int_t^{t+h} U(t+h,s)f(s)ds - f(t) \right\| =$$

$$= \left\| \frac{1}{h}\int_t^{t+h} U(t+h,s)[f(s)-f(t)]ds + \frac{1}{h}\int_t^{t+h} [U(t+h,s)-I]f(t)ds \right\| \leq$$

$$\leq CL\,\frac{1}{h}\int_t^{t+h}(s-t)ds + \sup\left\{ \|[U(t+h,s)-I]f(t)\| : s\in[t,t+h]\right\}$$

(where $C := \sup\{\|U(t,s)\|: (t,s)\in\Delta_T\}$, L is the Lipschitz constant for f) and these two terms converge to 0 as $h \longrightarrow 0$.

So, we have obtained the formula

$$\frac{d^+v(t)}{dt} = A(t)v(t) + f(t) \quad \text{for } t\in(0,T]. \tag{14}$$

But the right-hand side of (14) is the continuous function so,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the identical formula is correct also for the derivative $\frac{dv(t)}{dt}$.

4. Existence and uniqueness for problem (1)-(2)

Let us go back to our basic problem i.e. the nonlinear problem (1)-(2), mentioned in the Introduction. With regard to the family $\{A(t)\}$ let us accept the same assumptions as in the part 2. Let $f:[0,T]\times X \longrightarrow X$ be continuous and $u_0 \in D$.

The solution of the problem (1)-(2) we define in the same manner as for the linear one in Definition 1.

Also, as for the linear case, one can be shown that if u is the solution of (1)-(2) then it is also the solution of the following integral equation

$$(15) \qquad u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s,u(s))ds,$$

where $U(t,s)$ is the fundamental solution described in the part 2.

Theorem 2. If

1. The family $\{A(t)\}$ satisfies assumptions A1, A2, A3,
2. $f : [0,T]\times X \longrightarrow X$ is continuous and there exists a constant $L>0$ such that $\|f(t,x)-f(t,y)\| \leq L\|x-y\|$ for $t\in[0,T]$, $x,y\in X$,

then for any $u_0 \in X$ there exists exactly one continuous solution of the integral equation (15) on $[0,T]$.

Since the proof of the above theorem is analogous to the proof of Theorem 4.5. in [5] it will be omitted here.

Theorem 3. If the family $\{A(t)\}$ satisfies assumptions A1, A2, A3 and for each $t\in[0,T]$ 0 belongs to the resolvent set of $A(t)$ and if:

1. $u:[0,T] \longrightarrow X$ is a continuous solution of the integral equation (15),
2. for each $t\in[0,T]$ $f(t,u(t)) \in D$ and $A(t)f(t,u(t))$ is summable over $[0,T]$,
3. $f : [0,T]\times X \longrightarrow X$ satisfies the Lipschitz condition,
4. $u_0 \in D$,

then u is the solution of the problem (1)-(2).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. First let us notice that the assumption 2 results in the existence of the integral $\int_0^t A(t)U(t,s)f(s,u(s))ds$ because (compare with the proof of Lemma 2)

$$(17)\qquad \int_0^t \|A(t)U(t,s)f(s,u(s))\|ds \le \int_0^t \|A(t)U(t,s)A(s)^{-1}\| \cdot$$

$$\cdot \|A(s)f(s,u(s))\|ds \le C_1 \int_0^t \|A(s)f(s,u(s))\|ds \le C_2$$

(where $C_1 := \sup\{ \|A(t)U(t,s)A(s)^{-1}\| : (t,s)\in\Delta_T \}$).

So, using Lemma 4.3 in [5], the assumptions 1 and 4 and the formula (15) we see that for each $t\in[0,T]$ $u(t)\in D$.

Now, we will show that the function u satisfies the Lipshitz condition in $[0,T]$. Let $t,t+h\in[0,T]$, $h>0$.

$$\frac{u(t+h)-u(t)}{h} = \frac{[U(t+h,0)-U(t,0)]u_0}{h} +$$

$$+ \frac{U(t+h,t)-I}{h}\int_0^t U(t,s)f(s,u(s))ds + \frac{1}{h}\int_t^{t+h} U(t+h,s)f(s,u(s))ds.$$

As $h \longrightarrow 0$ the first term tends to $A(t)U(t,0)u_0$ (because $u_0\in D$), the second term tends to $A(t)\int_0^t U(t,s)f(s,u(s))ds$ (from 2.) and the third to $f(t,u(t))$ because

$$\left\| \frac{1}{h}\int_t^{t+h} U(t+h,s)f(s,u(s))ds - f(t,u(t)) \right\| \le$$

$$\le \left\| \frac{1}{h}\int_t^{t+h} U(t+h,s)[f(s,u(s))-f(t,u(t))]ds \right\| +$$

$$+ \left\| \frac{1}{h}\int_t^{t+h} [U(t+h,s)-I]f(t,u(t))ds \right\| \le$$

$$\le C \sup\{ \|f(s,u(s))-f(t,u(t))\| : s\in[t,t+h] \} +$$

$$+ \sup\{ \|[U(t+h,s)-I]f(t,u(t))\| : s\in[t,t+h] \},$$

(C as defined in the proof of Theorem 1.) and it is obvious that this sum tends to 0 as $h\longrightarrow 0$.

So, we have shown that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\frac{d^+u(t)}{dt} = A(t)U(t,0)u_0 + A(t)\int_0^t U(t,s)f(s,u(s))ds + f(t,u(t)).$$

From the above formula, the continuity of the functions $A(t)U(t,0)u_0$ and $f(t,u(t))$ in $[0,T]$ and the inequality (17) (and again Lemma 4.3 in [5]) we conclude that there exists a constant $K>0$ such that $\left\|\frac{d^+u(t)}{dt}\right\| \le K$ and using now the conclusion 3.2.2. in [1] we obtain

(18) $$\|u(t_1)-u(t_2)\| \le K|t_1-t_2| \quad \text{for} \quad t_1,t_2\in[0,T].$$

The inequality (18) and the assumption 3. imply that the function $t \longrightarrow f(t,u(t))$ satisfies the Lipshitz condition in $[0,T]$ so, using Theorem 1. the problem

$$\begin{cases} \frac{dw(t)}{dt} = A(t)w(t) + f(t,u(t)), & t\in(0,T], \\ w(0) = u_0 \end{cases}$$

has exactly one solution given by the formula

$$w(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s,u(s))ds \quad \text{for} \quad t\in[0,T].$$

Since u is the solution of (15) hence, $w(t)=u(t)$ for $t\in[0,T]$ thus, u is solution of the problem (1)-(2).

As a conclusion of Theorem 2 and Theorem 3 we have

Theorem 4. If the family $\{A(t)\}$ satisfies assumptions A1, A2, A3 and for each $t\in[0,T]$ 0 belongs to the resolvent set of A(t) and if:

1. for each $t\in[0,T]$ and $x\in X$ $f(t,x)\in D$ and $A(t)f(t,x)$ is summable over $[0,T]$,
2. $f:[0,T]\times X \longrightarrow X$ satisfies the Lipshitz condition,
3. $u_0\in D$,

then there exists exactly one solution of the problem (1)-(2).

Acknowledgments. The author is greatly indebted to Professor J. Bochenek for his stimulating comments and discussions.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] H. Cartan: Calcul differentiel. Formes differentielles. Hermann, Paris, 1967.

[2] R.E. Edwards: Functional Analysis (Theory and Application). Holt Rinehart and Winston, New York, Chicago, San Francisco, Toronto, London, 1965.

[3] S.G. Krein: Lineinye differentsial'nye uravneniia v banakhovom prostranstve. Nauka, Moscow, 1963 (in Russian).

[4] H. Tanabe: Equations of Evolution. Pitman, London, San Francisco, Melbourne, 1979.

[5] T. Winiarska: Differential Equation with Parameter. Monograph 68, Technical University of Cracow, Cracow, 1988.

[6] K. Yosida: Functional Analysis. Springer-Verlag, Berlin, Heidelberg, New York, 1974.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF CRACOW,
31-155 KRAKÓW, POLAND

Received May 8, 1989.

130274

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Romuald Małecki

ESTIMATES OF THE SOLUTION OF THE BOUNDARY-VALUE PROBLEM WITH A COMPLEX PARAMETER FOR A CERTAIN ELLIPTIC SYSTEM

In this paper we construct a solution of the boundary--value problem with a complex parameter (having a large absolute value) for a certain elliptic system of partial differential equations of order 2s and obtain suitable bounds for it. These estimates play an important role in methods of contour integration for parabolic equations with time-dependent boundary conditions (see [1], [2]). For s=1 this problem was investigated in [2].

1. Notation and statement of the problem

Let D be a domain in the Euclidean space E^n, $n \geq 3$, bounded by a closed Lyapunov's surface S with the exponent $\kappa \in (0,1>$.

Let us denote by

$$R_\delta = \{ \lambda : |\arg\lambda| \leq \frac{\pi+2\delta}{4s} , |\lambda| \geq R>0 \} \tag{1}$$

the region of complex λ-plane, R being a sufficiently large number, $\delta \in (0, \frac{\pi}{4})$ and $s \in \mathbb{N}$.

Let us consider the following boundary-value problem

$$A(x,\frac{\partial}{\partial x})u(x,\lambda) - \lambda^{2s}u(x,\lambda) = -\varphi(x) \quad \text{for } x \in D,\ \lambda \in R_\delta , \tag{2}$$

$$\lim_{D \ni x \to z \in S} B(z,\lambda^{2s},\frac{\partial}{\partial x})u(x,\lambda) = \psi(z,\lambda) \quad \text{for } z \in S,\ \lambda \in R_\delta , \tag{3}$$

where

$$A(x,\frac{\partial}{\partial x}) = A^0(x,\frac{\partial}{\partial x}) + A^1(x,\frac{\partial}{\partial x}) , \tag{4}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(5) $$A^0(x,\frac{\partial}{\partial x}) =$$

$$= (-1)^s a^0(x) \sum_{i_1,..,i_{2s}=1}^{n} a_{i_1 i_2}(x)...a_{i_{2s-1} i_{2s}}(x) \frac{\partial^{2s}}{\partial x_{i_1}...\partial x_{i_s}},$$

(6) $$A^1(x,\frac{\partial}{\partial x}) = \sum_{k=0}^{2s-1} a^1_{k_1...k_n}(x) \frac{\partial^k}{\partial x_1^{k_1}...\partial x_n^{k_n}},$$

(7) $$B(z,\lambda^{2s},\frac{\partial}{\partial x}) = (b^0(z)+\lambda^{2s}b^1(z))D_{T_z}^{2s-1} + B_1(z,\lambda^{2s},\frac{\partial}{\partial x}),$$

(8) $$B_1(z,\lambda,\frac{\partial}{\partial x}) = \sum_{k=0}^{2s-2} \sum_{l=0}^{1} \lambda^{2sl} b^l_{k_1...k_n}(z) \frac{\partial^k}{\partial x_1^{k_1}...\partial x_n^{k_n}},$$

(9) $$D_{T_z}^{2s-1} = \sum_{i_1,..,i_{2s}=1}^{n} a_{i_1 i_2}(x)...$$

$$...a_{i_{2s-1} i_{2s}}(x)\cos(n_z,x_{i_{2s}}) \frac{\partial^{2s-1}}{\partial x_{i_1}...\partial x_{i_{2s-1}}}$$

with $k:=k_1+...+k_n$ in (6),(8).

Here, $a^0(x), a^1_{k_1...k_n}(x), b^0(z), b^1(z), b^1_{k_1...k_n}(z)$ denote the $N\times N$ functional matrices, $N\geq 1$; $\varphi(x),\psi(z,\lambda),u(x,\lambda)$ are N-th order vector-functions and $a_{i,j}(x)$, $i,j = 1,....n$, are real functions. In (3), the point $x\in D$ tends to $z\in S$ along a curve satisfying the condition (see [7], p.115)

(10)[1] $$\lim_{\|z-z_x\|\to 0} \|z-z_x\|^{\kappa} \log\|x-z_x\| = 0,$$

where $\|x-z_x\| = \inf_{\xi\in S} \|x-\xi\|$ and $\kappa\in(0,1>$.

We make the following assumptions:

(I) The matrix $a(x) = [a_{ij}(x)]_{N\times N}$ is symmetric and posi-

[1]Here, and in the sequel, $\|\cdot\|$ denotes the standard norm in E^n.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

tively defined at any point $x \in \overline{D} = D \cup S$.

(II) The coefficients of the operator (4) are determined in the domain $D_0 \supset \overline{D}$ and $a^0, a \in C^{2s}(D_0)$,

$$a^1_{k_1 \ldots k_n} \in C^{k_1+\ldots+k_n}(D_0), \quad 0 \le k_1+\ldots+k_n \le 2s-1.$$

(III) The matrices b^0, b^1, $b^1_{k_1 \ldots k_n}$, $l = 0,1$, are determined and continuous on the boundary S.

(IV) The vector-function $\varphi \in C^1(D)$ satisfies the condition: there exists a domain D_1 such that $\overline{D}_1 \subset D$ and $\varphi(x) = 0$ for $x \in D \setminus D_1$.

(V) The vector-function $\psi(z,\lambda)$ is continuous with respect to $z \in S$, analytic with respect to $\lambda \in R_\delta$ and $\lim_{\lambda \to \infty} \psi(z,\lambda) = 0$ uniformly with respect to $\arg \lambda$ for $\lambda \in R_\delta$.

(VI) The matrices $[b^0(z) + \lambda^{2s} b^1(z)]^{-1}$,

$$[b^0(z) + \lambda^{2s} b^1(z)]^{-1} \sum_{l=0}^{1} b^l_{k_1 \ldots k_n}(z) \lambda^{2sl}, \quad k = 0,1,..,2s-2,$$

are bounded for $z \in S$ and $\lambda \in R_\delta$.

(VII)[2] $$\det\left[\left(\sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \right)^s a^0(x) - \lambda^{2s} I\right] \neq 0$$

for $\lambda \in R_\delta$ and any $\alpha = (\alpha_1,..,\alpha_n) \in E^n$.

A solution $u(x,\lambda)$ of the problem (2), (3) is inquired in class of vector-functions in E^N which have in D continuous derivatives of the 2s-th order with respect to x and are analytic with respect to λ for any $\lambda \in R_\delta$.

The fundamental solution of the homogeneous system

(11) $$A(x, \frac{\partial}{\partial x})u(x,\lambda) - \lambda^{2s} u(x,\lambda) = 0$$

was constructed by Mamedov in [3].

Let us assume that

[2] Here, and in the sequel, I denotes the unit matrix.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(12)\qquad A^0(\xi,\frac{\partial}{\partial x}) = (-1)^s a^0(\xi)\Big(\sum_{i,j=1}^{n} a_{ij}(\xi)\frac{\partial^2}{\partial x_i \partial x_j}\Big)^s,$$

and

$$(13)\qquad P_0(x,\xi,\lambda) =$$

$$= (2\pi)^{-n}\int_{E^n} e^{i(x-\xi,\alpha)}\times\Big[\Big(\sum_{i,j=1}^{n} a_{ij}(\xi)\alpha_i\alpha_j\Big)^s a^0(\xi)-\lambda^{2s}I\Big]^{-1} d\alpha.$$

Under the assumption (VII), we can conclude from the results of [3] that (13) presents a fundamental solution of the system

$$(14)\qquad A^0(\xi,\frac{\partial}{\partial x})\ u(x,\lambda) - \lambda^{2s}u(x,\lambda) = 0$$

and the following estimates hold for $k = 0,1,\ldots,2s$; $\lambda \in R_\delta$.

$$(15)^3\qquad \left|\frac{\partial^k P_0(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-2s+k}}$$

A fundamental solution of (11) was constructed in [3] by using Levi's "parametrix" method, in the form

$$(16)\qquad P(x,\xi,\lambda) = P_0(x,\xi,\lambda) + P_1(x,\xi,\lambda)$$

where

$$(16')\qquad P_1(x,\xi,\lambda) = \int_D P_0(x,\eta,\lambda)h(\eta,\xi,\lambda)d\eta$$

and $h(\eta,\xi,\lambda)$ is an unknown $N\times N$ functional matrix; besides in [3], by the assumptions (II) and (VII), the following relations were shown

$$(17)\qquad |h(x,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-1}},$$

---

[3] Here, and in the sequel, the estimates are valid for all elements of matrix on the left-hand side of (15). C and ε denote some positive constants: C-sufficiently large numbers and ε-sufficiently small ones.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(18)\qquad \left|\frac{\partial^k P_1(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-1-2s+k}},$$

$$(19)\qquad \left|\frac{\partial^k P(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-2s+k}}$$

for $k = 0,1,\ldots,2s$, $\lambda\in R_\delta$ and

$$(20)\qquad [A(x,\frac{\partial}{\partial x}) - \lambda^{2s}I]\int_D P(x,\xi,\lambda)\varphi(\xi)d\xi = -\varphi(x)$$

for φ satisfying the assumption (IV).

2. Auxiliary theorems

Lemma 1. If the conditions (I), (II) and (VII) are satisfied and

$$(21)\qquad D_0(x,\xi,\lambda) = \Big(\sum_{i,j=1}^{n} a_{ij}(\xi)\frac{\partial^2}{\partial x_i\partial x_j}\Big)^{s-1} P_0(x,\xi,\lambda),$$

where $P_0(x,\xi,\lambda)$ is given by (13), then

$$(22)\qquad D_0(x,\xi,\lambda) = D_{01}(x.\xi,\lambda) + D_{02}(x,\xi,\lambda),$$

where

$$(23)^4\qquad D_{01}(x,\xi,\lambda) = \frac{(-1)^{s-1}[\det a(\xi)]^{-\frac{1}{2}}[a^0(\xi)]^{-1}}{(n-2)|S_1|\big(a^{-1}(\xi)(x-\xi),x-\xi\big)^{\frac{n}{2}-1}},$$

$$(24)\qquad \left|\frac{\partial^k D_{02}(x,\xi,\lambda)}{\partial x_{i_1}^{k_1}\partial x_{i_2}^{k_2}}\right| \le \frac{C(\lambda)}{\|x-\xi\|^{n-3+k}}$$

for $k = k_1+k_2 = 0,1,2$, $\lambda\in R_\delta$, $x \ne \xi$, and $C(\lambda)$ denoting some positive constant dependent only on λ.

---

[4] The symbol $(\cdot,\cdot)$ denotes here, and in the sequel, the standard product in E^n i.e. $(x,y) = \sum_{i=1}^{n} x_iy_i$, where $x = (x_1\ldots x_n)$, $y = (y_1\ldots y_n)$; $|S_1| = 2(\sqrt{\pi})^n[\Gamma(n/2)]^{-1}$ is the area of the unit sphere.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Proof. On the basis of the definitions (21), (13) we get

$$D_0(x,\xi,\lambda) = \frac{(-1)^{s-1}}{(2\pi)^n}\int_{E^n} e^{i(x-\xi,\alpha)}\times$$

$$\times(a(\xi)\alpha,\alpha)^{s-1}[a^0(\xi)(a(\xi)\alpha,\alpha)^s-\lambda^{2s}I]^{-1}d\alpha.$$

From the assumption (I) there exists the square root of the matrix $a(\xi)$. Let $\beta = a^{1/2}(\xi)\alpha$ for $\alpha \in E^n$. Hence $d\beta = [\det a(\xi)]^{1/2}d\alpha$, $(a(\xi)\alpha,\alpha) = \|\beta\|^2$ and

$$(25)\qquad D_0(x,\xi,\lambda) = \frac{(-1)^{s-1}}{(2\pi)^n(\det a(\xi))^{1/2}}\times$$

$$\times\int_{E^n} e^{i(x-\xi,a^{-1/2}(\xi)\beta)}\|\beta\|^{2s-2}[a^0(\xi)\|\beta\|^{2s}-\lambda^{2s}I]^{-1}d\beta =$$

$$= \frac{(-1)^{s-1}}{(2\pi)^n(\det a(\xi))^{1/2}}\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)}\times$$

$$\times\|\beta\|^{2s-2}[a^0(\xi)\|\beta\|^{2s}-\lambda^{2s}I]^{-1}d\beta = \frac{(-1)^{s-1}[a^0(\xi)]^{-1}}{(2\pi)^n(\det a(\xi))^{1/2}}\times$$

$$\times\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)}\|\beta\|^{-2}[I-[a^0(\xi)]^{-1}\|\beta\|^{-2s}\lambda^{2s}]^{-1}d\beta.$$

Keeping in mind the identity $[I-B]^{-1} = I + [B^{-1}-I]^{-1}$ for $B = [a^0(\xi)]^{-1}\|\beta\|^{-2s}\lambda^{2s}$, we get

$$(26)\qquad D_0(x,\xi,\lambda) = D_{01}(x,\xi,\lambda) + D_{02}(x,\xi,\lambda),$$

where

$$D_{01}(x,\xi,\lambda) = \frac{(-1)^{s-1}[a^0(\xi)]^{-1}}{(2\pi)^n(\det a(\xi))^{1/2}}\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)}\|\beta\|^{-2}d\beta,$$

$$(27)\qquad D_{02}(x,\xi,\lambda) = \frac{(-1)^{s-1}\lambda^{2s}[a^0(\xi)]^{-1}}{(2\pi)^n(\det a(\xi))^{1/2}}\times$$

$$\times\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)}\|\beta\|^{-2}[a^0(\xi)\|\beta\|^{2s}-\lambda^{2s}I]^{-1}d\beta.$$

It can be shown (see [5], p.122) that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$D_{01}(x,\xi,\lambda) = \frac{(-1)^{s-1}\ (\det a(\xi))^{-1/2}}{(n-2)|S_1|\ \|a^{-1/2}(\xi)(x-\xi)\|^{n-2}}\ [a^0(\xi)]^{-1},$$

and hence we get (23). To estimate the elements of the matrix $\frac{\partial^k D_{02}(x,\xi,\lambda)}{\partial x_{i_1}^{k_1}\ \partial x_{i_2}^{k_2}}$ for $k = k_1+k_2 = 0,1,2$, $i_1,i_2 = 1,\dots,n$, denote

(28) $$f_k(\beta,\xi,\lambda) =$$

$$= (a^{-1/2}(\xi)\beta)_{i_1}^{k_1}(a^{-1/2}(\xi)\beta)_{i_2}^{k_2}\|\beta\|^{-2}[a^0(\xi)\|\beta\|^{2s}-\lambda^{2s_I}]^{-1},$$

where $(a^{-1/2}(\xi)\beta)_{i_r}^{k_r}$ is the k_r-th power of i_r-th coordinate of the vector $a^{-1/2}(\xi)\beta$. Let us observe that the function f_k is "practically of the type σ" (see [4], definition 1, p.33) with $\sigma = -2s-2+k$ and, hence, on the basis of the suitable theorem (see [4], theorem 13, p.50), we can prove the following inequalities

(29) $$\left|\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)}\ f_k(\beta,\xi,\lambda)\ d\beta\right| \le$$

$$\le \begin{cases} \dfrac{C(\lambda)}{\|a^{-1/2}(\xi)(x-\xi)\|^{n-2s-2+k}} & \text{for } n-2s-2+k>0, \\ -C(\lambda)\ \log\|a^{-1/2}(\xi)(x-\xi)\| & \text{for } n = 2s+2-k, \\ C(\lambda) & \text{for } n-2s-2+k<0. \end{cases}$$

Since $n-2s-2+k < n-3+k$, from the above estimate we have

(30) $$\left|\int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)} f_k(\beta,\xi,\lambda)\,d\beta\right| \le \frac{C(\lambda)}{\|a^{-1/2}(\xi)(x-\xi)\|^{n-3+k}}.$$

The matrix $a^{-1/2}(\xi)$ is positively defined. Then there exist positive constants C_1, C_2 such that the inequalities $C_1\|x-\xi\| \le \|a^{-1/2}(\xi)(x-\xi)\| \le C_2\|x-\xi\|$ hold.

Finally, by (30), (27) and the above inequalities, we get (24), and the proof of Lemma 1 is completed.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Lemma 2. If the assumptions (I), (II) and (VII) are satisfied, then the transversal derivative of the fundamental solution (16) of the system (11) has the following estimate

$$(31)\quad |D_{T_z}^{2s-1} P(x,\xi,\lambda)| \le \frac{C \exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-1}} \{|\cos(n_z, x-\xi)| + \|x-\xi\|\}$$

Proof. From (16) we get

$$(32)\quad D_{T_z}^{2s-1} P(x,\xi,\lambda) = D_{T_z}^{2s-1} P_0(x,\xi,\lambda) + D_{T_z}^{2s-1} P_1(x,\xi,\lambda).$$

Keeping in mind the definition (9) of operator $D_{T_z}^{2s-1}$ and the definition (21) of $D_0(x,\xi,\lambda)$, we can write

$$(33)\quad D_{T_z}^{2s-1} P_0(x,\xi,\lambda) = \sum_{i_1,\dots i_{2s}=1}^{n} a_{i_1 i_2}(x)\dots a_{i_{2s-1} i_{2s}}(x) \times$$

$$\times \frac{\partial^{2s-1} P_0(x,\xi,\lambda)}{\partial x_{i_1}\dots\partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) = \sum_{i_1,\dots i_{2s}=1}^{n} [a_{i_1 i_2}(x)\dots a_{i_{2s-1} i_{2s}}(x) -$$

$$- a_{i_1 i_2}(\xi)\dots a_{i_{2s-1} i_{2s}}(\xi)] \frac{\partial^{2s-1} P_0(x,\xi,\lambda)}{\partial x_{i_1}\dots\partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) +$$

$$+ \sum_{i,j=1}^{n} a_{ij}(\xi)\cos(n_z, x_j) \frac{\partial D_0(x,\xi,\lambda)}{\partial x_i} = I_1(x,\xi,\lambda) + I_2(x,\xi,\lambda).$$

From the assumption (II) it follows that the elements of the matrix $a(x)$ satisfy the Lipschitz condition. By (15) for $k = 2s-1$, we obtain

$$(34)\quad |I_1(x,\xi,\lambda)| \le \frac{C \exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-2}}.$$

To estimate $I_2(x,\xi,\lambda)$ observe that, by (25), we have

$$(35)\quad D_0(x,\xi,\lambda) = (-1)^{s-1}(2\pi)^{-n}(\det a(\xi))^{-1/2} \times$$

$$\times \int_{E^n} e^{i(a^{-1/2}(\xi)(x-\xi),\beta)} \|\beta\|^{2s-2} [a^0(\xi)\|\beta\|^{2s} - \lambda^{2s} I]^{-1} d\beta.$$

Put $\gamma = M\beta$ in (35), where M is the orthogonal matrix to that of rotation of the system of coordinates in which the n-

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

th axis coincides with the direction of the unit-vector $\frac{a^{-1/2}(\xi)(x-\xi)}{\|a^{-1/2}(\xi)(x-\xi)\|}$.

Hence $d\gamma = d\beta$, $\|\gamma\| = \|\beta\|$, $\gamma_n = \frac{(a^{-1/2}(\xi)(x-\xi),\beta)}{\|a^{-1/2}(\xi)(x-\xi)\|}$ and

$$(36)\quad D_0(x,\xi,\lambda) = (-1)^{s-1}(2\pi)^{-n}(\det a(\xi))^{-1/2}\times$$

$$\times\int_{E^n} e^{i\|a^{-1/2}(\xi)(x-\xi)\|\gamma_n\|\gamma\|^{2s-1}}\,[a^0(\xi)\|\gamma\|^{2s}-\lambda^{2s}I]^{-1}d\gamma.$$

Let

$$(37)\quad \rho = \rho(x,\xi) := \|a^{-1/2}(\xi)(x-\xi)\|^2 = (a^{-1}(\xi)(x-\xi),x-\xi).$$

From (36) we obtain

$$(38)\quad D_0(x,\xi,\lambda) = \bar{D}_0(\rho(x,\xi),\xi,\lambda)\ ,$$

where

$$(39)\quad \bar{D}_0(\rho(x,\xi),\xi,\lambda) = (-1)^{s-1}(2\pi)^{-n}(\det a(\xi))^{-1/2}\times$$

$$\times\int_{E^n} e^{i\rho^{1/2}\gamma_n}\|\gamma\|^{2s-2}[a^0(\xi)\|\gamma\|^{2s}-\lambda^{2s}I]^{-1}d\gamma.$$

Differentiating (38), by (37), we have

$$(40)\quad \frac{\partial D_0}{\partial x_r} = \frac{\partial \bar{D}_0}{\partial \rho}\frac{\partial \rho}{\partial x_r} = 2\frac{\partial \bar{D}_0}{\partial \rho}\sum_{j=1}^{n} a^{rj}(\xi)(x_j-\xi_j)\ ,\quad r=1,\ldots,n,$$

where $a^{rj}(\xi)$ are the elements of matrix $a^{-1}(\xi)$.

Multiplying now both sides of (40) suitably by $(x_r-\xi_r)$ and summing up after r we get

$$(41)\quad \sum_{r=1}^{n}(x_r-\xi_r)\frac{\partial D_0}{\partial x_r} = 2\frac{\partial \bar{D}_0}{\partial \rho}\sum_{r,j=1}^{n} a^{rj}(\xi)(x_j-\xi_j)(x_r-\xi_r) = 2\frac{\partial \bar{D}_0}{\partial \rho}\rho$$

and hence

$$(42)\quad \frac{\partial \bar{D}_0}{\partial \rho} = \sum_{r=1}^{n}\frac{x_r-\xi_r}{2\rho}\frac{\partial D_0}{\partial x_r}.$$

By (40) and (42), we can write $I_2(x,\xi,\lambda)$ from (33) in the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(43)\qquad I_2(x,\xi,\lambda) = \sum_{r,p=1}^{n} a_{rp}(\xi)\cos(n_z,x_p)\frac{\partial D_0(x,\xi,\lambda)}{\partial x_r} =$$

$$= \sum_{r,p=1}^{n} a_{rp}(\xi)\cos(n_z,x_p)\frac{\partial \bar{D}_0(\rho,\xi,\lambda)}{\partial \rho}\, 2\sum_{j=1}^{n} a^{rj}(\xi)(x_j-\xi_j) =$$

$$= 2\frac{\partial \bar{D}_0(\rho,\xi,\lambda)}{\partial \rho}\sum_{p,j=1}^{n}\Big[\sum_{r=1}^{n} a_{rp}(\xi)\; a^{rj}(\xi)\Big](x_j-\xi_j)\cos(n_z,x_p) =$$

$$= 2\frac{\partial \bar{D}_0(\rho,\xi,\lambda)}{\partial \rho}\sum_{j=1}^{n}(x_j-\xi_j)\cos(n_z,x_j) = 2\|x-\xi\|\frac{\partial \bar{D}_0(\rho,\xi,\lambda)}{\partial \rho}\cos(n_z,x-\xi) =$$

$$= \|x-\xi\|\Big[\sum_{r=1}^{n}\frac{x_r-\xi_r}{\partial \rho}\,\frac{\partial D_0(x,\xi,\lambda)}{\partial x_r}\Big]\cos(n_z,x-\xi)\ .$$

The matrix $a^{-1}(\xi)$ is positively defined. Then there exist positive constants C_1', C_2' such that the inequalities

$$(44)\qquad C_1'\|x-\xi\|^2 \le \rho(x,\xi) \le C_2'\|x-\xi\|^2$$

hold. The matrix $\dfrac{\partial D_0(x,\xi,\lambda)}{\partial x_r}$ (see (21)) is a combination of derivatives of order 2s-1. Thus, by (42), (43) and (15), we get

$$(45)\qquad |I_2(x,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-1}}|\cos(n_z,x-\xi)|.$$

From the inequality (18) for k = 2s-1 we have

$$(46)\qquad |D_{T_z}^{2s-1}P_1(x,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-2}}.$$

Finally, combining (32)-(34), (45), (46), we obtain (31).

Lemma 3. Let $S = \partial D$ be a closed Lyapunov's surface with the exponent $\kappa\in(0,1>$ and

$$(47)\qquad u(x,\lambda) = \int_S P(x,\xi,\lambda)\Phi(\xi,\lambda)\,d\xi,$$

where $P(x,\xi,\lambda)$ is the fundamental solution of the system (11). If the function $\Phi(\xi,\lambda)$ is continuous on S for all $\lambda\in R_\delta$ and the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

assumptions (I), (II), (VII) are satisfied, then the transversal derivative of the potential (47) has, for $\lambda \in R_\delta$, the following limit property

$$(48)\qquad \lim_{D \ni x \to z \in S} D_{T_z}^{2s-1} u(x,\lambda) =$$

$$= \frac{(-1)^s}{2} [a^0(z)]^{-1} \Phi(z,\xi) + \int_S D_{T_z}^{2s-1} P(x,\xi,\lambda) \Phi(\xi,\lambda) d\xi,$$

where x tends to z along a curve satisfying the condition (10).

Proof. By (16), we can write

$$(49)\qquad D_{T_z}^{2s-1} u(x,\lambda) = D_{T_z}^{2s-1} \int_S P_0(x,\xi,\lambda) \Phi(\xi,\lambda) d\xi +$$

$$+ \int_S D_{T_z}^{2s-1} P_1(x,\xi,\lambda) \Phi(\xi,\lambda) d\xi = I_1^*(x,\lambda) + I_2^*(x,\lambda)$$

for $x \in D$, $\lambda \in R_\delta$. It follows from (46) that

$$(50)\qquad \lim_{x \to z \in S} I_2^*(x,\lambda) = I_2^*(z,\lambda).$$

According to Lemma 1, we can write

$$(51)\qquad I_1^*(x,\lambda) = \int_S \sum_{i_1,\dots i_{2s}=1}^{} [a_{i_1 i_2}(x) \dots a_{i_{2s-3} i_{2s-2}}(x) -$$

$$- a_{i_1 i_2}(\xi) \dots a_{i_{2s-3} i_{2s-2}}(\xi)] a_{i_{2s-1} i_{2s}}(x) \frac{\partial^{2s-1} P_0(x,\xi,\lambda)}{\partial x_{i_1} \dots \partial x_{i_{2s-1}}} \cos(n_z, x_{i_{2s}}) \times$$

$$\times \Phi(\xi,\lambda) d\xi + \int_S \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial D_{02}(x,\xi,\lambda)}{\partial x_i} \cos(n_z, x_j) \Phi(\xi,\lambda) d\xi +$$

$$+ \int_S \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial D_{01}(x,\xi,\lambda)}{\partial x_i} \cos(n_z, x_j) \Phi(\xi,\lambda) d\xi =$$

$$= I_{11}^*(x,\lambda) + I_{12}^*(x,\lambda) + I_{13}^*(x,\lambda).$$

From the assumption (II) and the inequalities (15), (24) we conclude that the integrals $I_{11}^*(x,\lambda)$, $I_{12}^*(x,\lambda)$ are continuous on $\bar{D}$. The integral $I_{13}^*(x,\lambda)$ is the first-order transversal derivative of a quasi-potential (see (23)) with the density $\frac{(-1)^{s-1} [a^0(\xi)]^{-1} \Phi(\xi,\lambda)}{(\det a(\xi))^{1/2} (n-2) |S_1|}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Making use of the appropriate result (see[6],p.270) we get

$$\lim_{D\ni x\to z\in S} I^*_{13}(x,\lambda) = \frac{(-1)^s}{2}[a^0(z)]^{-1}\Phi(z,\xi) + I^*_{13}(z,\lambda). \tag{52}$$

Finally, combining (49)-(52), we have the limit property (48).

3. Estimates of the solution of the boundary-value problem

We seek a solution of the problem (2),(3) in the form

$$u(x,\lambda) = \int_S P(x,\xi,\lambda)\Phi(\xi,\lambda)d\xi. \tag{53}$$

By (3) and Lemma 3, we get the integral equation

$$\frac{(-1)^s}{2}[b^0(z) + \lambda^{2s}b^1(z)]\,[a^0(z)]^{-1}\Phi(z,\xi) + \tag{54}$$

$$+ \int_S B(z,\lambda^{2s},\frac{\partial}{\partial z})P(z,\xi,\lambda)\Phi(\xi,\lambda)d\xi = \psi(z,\lambda).$$

Under the assumption (VI), we can define the following functions

$$\mu(z,\lambda) = 2(-1)^s a^0(z)[b^0(z)+\lambda^{2s}b^1(z)]^{-1}\psi(z,\lambda), \tag{55}$$

$$K(z,\xi,\lambda) = 2(-1)^{s-1}a^0(z)\{D^{2s-1}_{T_z} + \tag{56}$$

$$+ [b^0(z)+\lambda^{2s}b^1(z)]^{-1}B_1(z,\lambda^{2s},\frac{\partial}{\partial z})\}P(z,\xi,\lambda).$$

Thus, we can rewrite (54) in the form

$$\Phi(z,\lambda) = \mu(z,\lambda) + \int_S K(z,\xi,\lambda)\Phi(\xi,\lambda)d\xi, \quad z\in S, \ \lambda\in R_\delta. \tag{57}$$

By (II), (VI), Lemma 2 and (19), we get

$$|K(z,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\,\|z-\xi\|)}{\|z-\xi\|^{n-1-\kappa}} \tag{58}$$

for $\lambda\in R_\delta$, $z,\xi\in S$, $z\neq\xi$ and $\kappa\in(0,1>$ (cf. Lemma 3).

It can be proved by the method used in [1] (see Lemma 2, p. 385) that there is a unique solution of (57) of the form

$$\Phi(z,\lambda) = \mu(z,\lambda) + \int_S R(z,\xi,\lambda)\mu(\xi,\lambda)d\xi \tag{59}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and the resolvent $R(z,\xi,\lambda) = \sum_{r=1}^{\infty} K_r(z,\xi,\lambda)$ ($K_1 = K$ and K_r is the r-th iteration of the kernel K), has the following estimate

$$(60) \qquad |R(z,\xi,\lambda)| \leq \frac{C \exp(-\varepsilon|\lambda|\|z-\xi\|)}{\|z-\xi\|^{n-1-\kappa}},$$

for $z,\xi \in S$, $z \neq \xi$, $\lambda \in R_\delta$ (cf.(1)). It results from (60),(55) and (VI) that the function Φ given by (59) satisfies the inequalities

$$(61) \qquad |\Phi(z,\lambda)| \leq |\mu(z,\lambda)| + C_1 \sup_{\xi\in S}|\mu(\xi,\lambda)| \leq$$

$$\leq C_2 \sup_{\xi\in S}|\mu(\xi,\lambda)| \leq C \sup_{\xi\in S}|\psi(\xi,\lambda)|$$

for $z \in S$ and $\lambda \in R_\delta$. Thus, by (19) and (61), we get

$$(62) \qquad \left|\frac{\partial^k u(x,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \leq C \sup_{\xi\in S}|\psi(\xi,\lambda)| \frac{\exp(-\varepsilon|\lambda|d(x))}{d(x)^{n-2s+k}},$$

for $x \in D$; $\lambda \in R_\delta$; $k = 0,1,\ldots,2s$, where $d(x) = \inf_{\xi\in S}\|x-\xi\|$, and

$$(63) \qquad \left|\frac{\partial^k u(z,\lambda)}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}}\right| \leq C \sup_{\xi\in S}|\psi(\xi,\lambda)|$$

for $z \in S$, $\lambda \in R_\delta$, $k = 0,1,\ldots,2s-2$. By Lemma 2, we have

$$(64) \qquad |D_{T_z}^{2s-1} P(z,\xi,\lambda)| \leq \frac{C \exp(-\varepsilon|\lambda|\|z-\xi\|)}{\|z-\xi\|^{n-1-\kappa}}$$

for $z,\xi \in S$, $z \neq \xi$, $\lambda \in R_\delta$. Hence, we obtain

$$(65) \qquad |D_{T_z}^{2s-1} u(z,\lambda)| \leq C \sup_{\xi\in S}|\psi(\xi,\lambda)| \quad \text{for } \lambda \in R_\delta .$$

Finally, we can formulate the following theorem.

Theorem 1. If the assumptions (I)-(III), (V)-(VII) are satisfied and S is the Lyapunov surface with exponent $\kappa \in (0,1>$, then the problem (2),(3) (for $\varphi(x) \equiv 0$) has the solution given by the formula (53). This solution and its derivatives have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

the estimates (62),(63), (65).

4. The Green matrix for the boundary-value problem

We seek the Green matrix for the problem (2),(3) in the form

$$G(x,\xi,\lambda) = P(x,\xi,\lambda) - Q(x,\xi,\lambda), \tag{66}$$

where $P(x,\xi,\lambda)$ is the fundamental solution of (11), and the regular part $Q(x,\xi,\lambda)$ can be represented as the potential of a single layer

$$Q(x,\xi,\lambda) = \int_S P(x,y,\lambda)h_1(y,\xi,\lambda)dy\ . \tag{67}$$

Thus, $Q(x,\xi,\lambda)$ is the solution of (11) and satisfies the boundary condition

$$\lim_{D\ni x\to z\in S} B(z,\lambda^{2s},\frac{\partial}{\partial x})Q(x,\xi,\lambda) = g(z,\xi,\lambda), \tag{68}$$

where

$$g(z,\xi,\lambda) = B(z,\lambda^{2s},\frac{\partial}{\partial z})P(z,\xi,\lambda) \quad \text{for } \xi\in D,\ \lambda\in R_\delta. \tag{69}$$

According to Lemma 3, similarly as in section 2, it can be proved that the matrix $h_1(y,\xi,\lambda)$ satisfies the integral equation

$$h_1(z,\xi,\lambda) = \mu_1(z,\xi,\lambda) + \int_S K(z,y,\lambda)h_1(y,\xi,\lambda)dy \tag{70}$$

with

$$\mu_1(z,\xi,\lambda) = 2(-1)^s a^0(z)[b^0(z) + \lambda^{2s}b^1(z)]^{-1}g(z,\xi,\lambda) \tag{71}$$

and kernel (56) satisfying (60). Hence, (70) has the unique solution

$$h_1(z,\xi,\lambda) = \mu_1(z,\xi,\lambda) + \int_S R(z,y,\lambda)\mu_1(y,\xi,\lambda)dy. \tag{72}$$

By (71),(69),(19),(31) and (VI), we get

$$|\mu_1(z,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|z-\xi\|)}{|z-\xi|^{n-1}}|\cos(n_z,z-\xi)| \tag{73}$$

for $z\neq\xi$, $z\in S$, $\xi\in\bar{D}$, $\lambda\in R_\delta$. From (73), (60), using the method analogous to that of [2] (see Lemma 3, p. 173), we get

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(74)\qquad \left|\int_S R(z,y,\lambda)\mu_1(y,\xi.\lambda)dy\right| \le \frac{C\exp(-\varepsilon|\lambda|\|z-\xi\|)}{\|z-\xi\|^{n-1-\kappa}}$$

for $z \ne \xi$, $z\in S$, $\xi\in\bar{D}$, $\lambda\in R_\delta$. Combining (70), (73) and (74), we obtain

$$(75)\qquad |h_1(z,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|z-\xi\|)}{\|z-\xi\|^{n-1}}\left(|\cos(n_z,z-\xi)| + \|z-\xi\|^{\kappa}\right)$$

for $z \ne \xi$, $z\in S$, $\xi\in\bar{D}$, $\lambda\in R_\delta$. Hence, by the method of [2] (see Theorem 4, p.196, Lemma 3,p.176, Lemma 2,p.175) we can prove the following theorem.

Theorem 2. If the assumptions (I)-(III), (V)-(VII) are satisfied and S is the Lyapunov surface with the exponent $\kappa\in(0,1>$, then, for all $\lambda\in R_\delta$, the problem (2),(3) (for $\psi(z,\lambda)=0$) has the solution

$$(76)\qquad u(x,\lambda) = \int_D G(x,\xi,\lambda)\varphi(\xi)d\xi\ .$$

The regular part (67) of the Green function $G(x,\xi,\lambda)$ and its derivatives have the following estimates

$$(77)\qquad \left|\frac{\partial^k Q(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{d(x)^{n-2s+k}},\quad k=0,1,\ldots,2s,$$

for $x \ne \xi$, $x\in D_1 \subset D$, $\xi\in\bar{D}$, with $d(x) = \inf_{y\in S}\|x-y\|$,

$$(78)\qquad \left|\frac{\partial^k Q(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{\|x-\xi\|^{n-2s+k}},\quad k=0,1,\ldots,2s-2,$$

for $x \ne \xi$, $x,\xi\in\bar{D}$,

$$(79)\qquad \left|\frac{\partial^k Q(x,\xi,\lambda)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\right| \le \frac{C\exp(-\varepsilon|\lambda|\|x-\xi\|)}{|\lambda|^{2s-1-k}},\quad k=0,1,\ldots,2s-2,$$

for $x \ne \xi$, $x\in D$, $\xi\in D_1$ and

$$(80)\qquad |D_{T_z}^{2s-1} Q(z,\xi,\lambda)| \le \frac{C\exp(-\varepsilon|\lambda|\|z-\xi\|)}{\|z-\xi\|^{n-1-\kappa}}$$

for $z\in S$, $\xi\in D$, $\lambda\in R_\delta$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] **M. L. Rasulov:** Methods of contour integration, Amsterdam 1967.

[2] М. Л. Расулов: Применения метода контурного интеграла, Москва 1975.

[3] Н. М. Манедов: Применение метода контурного интеграла к решению задач Коши, Diff. Ur. 3 (1978) 492-498.

[4] **S. Bochner:** Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients, Ann. of Math. 57 (1953) 32-56.

[5] С. Мизохата: Теориа уравнений с частными производными, Москва 1977.

[6] **W. Pogorzelski:** Étude de la solution fondamentale de l'équation elliptique et des problèmes aux limites, Ann. Polon. Math. 3 (1957) 247-284.

[7] **W. Pogorzelski:** Integral equations and their applications, Warszawa 1958 .

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, PŁOCK BRANCH, 09-400 PŁOCK, POLAND

Received May 8, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

H. Dębiński, T. Poreda, A. Szadkowska

ON THE STABILITY OF THE GENERALIZED LINEAR DIFFERENTIAL EQUATION OF FIRST ORDER IN BANACH SPACES

1. Introduction

Let X,Y be real Banach spaces and L(X,Y) denote the space of continuous linear mappings from Y into Y. In this paper we are occupied with study of the stability of solutions for the generalized linear differential equation of the first order.

Definition 1. Let a be a nonzero fixed point of X, g be a mapping from X into Y, A be a mapping from X into L(Y,Y). The differential equation

$$\nabla_a f(x) = A(x)f(x) + g(x) \quad \text{for } x \in X \tag{1}$$

will be called the generalized linear differential equation of the first order, and a function f (differentiable at each point of the set X in the direction of the vector a) defined on X with values in Y, satisfying equation (1), will be called its solution.

In paper [8] the theorem about the existence and the uniqueness of the solution of equation (1) is proved. The formulation of this theorem needs following denotations.

Let $a \neq 0$ be a point of X (fixed in our further considerations) and $L_a = \{\kappa a;\ \kappa \in \mathbb{R}\}$. By X_a we denote any fixed complementary subspace to L_a (for the definition of complementarity see [1], p. 61) and by X^+ we denote the set defined in the following way

$X^+ = \{x;\ x = \overline{x} + \tau a,\ \text{where } \overline{x} \in X_a,\ \tau \in [0,\infty)\}$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Definition 2. Mappings $y: X \to X_a$, $t: X \to \mathbb{R}$ such that

$$x = y(x) + t(x)a \quad \text{for } x \in X \tag{2}$$

will be called projection operators. (These mappings are obvious the linear operators).

In the next part of this paper we shall assume that X_a is a closed subspace of X.

By $B(X_a,Y)$ we denote the space of bounded functions which map X_a into Y with sup norm. The ball in $B(X_a,Y)$ with a centre at the point f_0 and with a radius δ will be denoted by $K(f_0,\delta)$.

Let x be a fixed point in X, X_a be a certain space complementary to L_a, y and t be projection operators and A be a continuous mapping from X into L(Y,Y). Let $R(.,.,y(x)): \mathbb{R}^2 \to L(Y,Y)$ be a solution of the equation

$$\frac{d}{d\tau} R(\tau,s,y(x)) = A(y(x) + at(x))R(\tau,s,y(x)) \tag{3}$$

fulfilling the initial condition $R(s,s,y(x)) = I_Y$ where I_Y is the identity operator on Y (from theorem IX.10 from [7] it follows that there exists exactly one such a function).

By the above denotations we can formulate

Theorem A ([8]). Let a be a nonzero element of X and f_0 be a mapping from X_a into Y. If the mappings A and g are continuous then there exists exactly one solution of the equation

$$\nabla_a f(x) = A(x)f(x) + g(x) \quad \text{for } x \in X \tag{4}$$

fulfilling the condition

$$f(x) = f_0(x) \quad \text{for } x \in X_a \tag{5}$$

and it has a form

$$f(x) = R(t(x),0,y(x))f_0(y(x)) + \\ + R(t(x),0,y(x)) \int_0^{t(x)} R(0,s,y(x))g(y(x) + as)ds$$

for $x \in X$.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Remark 1. The solution f of equation (1) satisfying the condition $f(x) = f_0(x)$ for $x \in X$ will be denoted by $f = f(.,f_0)$.

Remark 2. The equation

$$\nabla_a f(x) = A(x)f(x) \quad \text{for } x \in X \tag{1'}$$

will be called the homogeneous equation corresponding to equation (1).

2. Stability of the solutions for the generalized linear differential equation of the first order

The further considerations make the generalization of the classical theory of the stability which is presented in books [2], [3], [4].

Let now $a \in X$ and $a \neq 0$, A, g be the continuous mappings on X and $f_0 \in B(X_a, Y)$.

Definition 3. The solution $f = f(x,f_0)$ of the differential equation

$$\nabla_a f(x) = A(x)f(x) + g(x) \quad \text{for } x \in X \tag{6}$$

fulfilling the condition

$$f(x) = f_0(x) \quad \text{for } x \in X_a$$

will be called stable in X^+ if for every $\varepsilon > 0$ there exists $\delta > 0$ that for every $\tilde{f}_0 \in K(f_0,\delta)$ the following inequality holds

$$\sup_{x \in X^+} \| f(x,f_0) - f(x,\tilde{f}_0) \| < \varepsilon.$$

Definition 4. Differential equation (6) will be called stable in X^+ if each of its solution $f = f(.,f_0)$ is stable in X^+.

Theorem 1. The necessary and sufficient condition for equation (6) to be stable in X^+ is that the trivial solution of the homogeneous equation corresponding to equation (6) to be stable in X^+.

Proof. Let $f(.,f_0)$ be a certain stable solution of equation (6) and let ε be a fixed positive number. Therefore there exists a number $\delta > 0$ such that for every solution

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$f(.,\tilde{f}_0)$, where $\tilde{f}_0 \in K(f_0,\delta)$, the following inequality takes place

$$\sup_{x\in X^+} \| f(x,f_0) - f(x,\tilde{f}_0) \| < \varepsilon. \tag{7}$$

Let, furthermore, $F_0 \in K(0,\delta)$ and $F(.,F_0)$ be the solution of the initial problem (1'), $F(x) = F_0(x)$, $x\in X$. Note that the function

$$\tilde{f}(x,F_0 + f_0) = F(x,F_0) + f(x,f_0) \quad \text{for } x\in X^+$$

is the solution of equation (6) and $F_0 + f_0 \in K(f_0,\delta)$. Then taking account of inequality (7) we have that $\sup_{x\in X^+} \|F(x,F_0)\| < \varepsilon$ which proves the stability of the trivial solution of the homogeneous equation.

Now let the trivial solution of the homogeneous equation to be stable in X^+ and ε be a fixed positive number. Then there exists such a number $\delta>0$ that for $F_0 \in K(0,\delta)$ the following inequality holds

$$\sup_{x\in X^+} \| F(x,F_0) \| < \varepsilon \tag{8}$$

where $F(.,F_0)$ is a solution of the homogeneous equation. Furthermore, let $f(.,f_0)$ be a solution of equation (6). Let us notice that if $f(.,\tilde{f}_0)$ is the solution of equation (6) and $f_0 \in K(f_0,\delta)$ then

$$F(x,\tilde{f}_0 - f_0) = f(x,\tilde{f}_0) - f(x,f_0) \quad \text{for } x\in X^+$$

is the solution of the homogeneous equation and $\tilde{f}_0 - f_0 \in K(0,\delta)$. Hence taking account of inequality (8) we obtain

$$\sup_{x\in X^+} \| f(x,\tilde{f}_0) - f(x,f_0) \| < \varepsilon.$$

which from the free choice of ε completes the proof.

Remark 3. Theorem 1 is analogous to the results obtained in classical theory of the stability (see [3], [4]).

Definition 5. The family of the solution $f = f(.,f_0)$ for $f_0 \in B(X_a,Y)$ of equation (1') will be called almost commonly bounded if for every $\delta>0$ there exists such M that for every $f_0 \in K(0,\delta)$ the inequality

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(i) $$\sup_{x\in X^+} \| f(x,f_0) \| \leq M$$

takes place.

Theorem 2. Linear homogeneous equation (1′) is stable in X^+ if and only if the solutions of this equation are almost commonly bounded.

Proof. First, let us assume that equation (1′) is stable in X^+. Let ε_0 be a fixed positive number. Hence there exists such $\delta_0 > 0$ that for $f_0 \in K(0,\delta_0)$ the inequality

$$\sup_{x\in X^+} \| f(x,f_0) \| < \varepsilon_0.$$

takes place.

Let, furthermore, δ be any positive number and f_0 belong to the ball $K(0,\delta)$. Then the function $\frac{\delta_0}{\delta} f_0$ belongs to the ball $K(0,\delta_0)$ hence the following inequality holds

$$\sup_{x\in X^+} \| f(x,\frac{\delta_0}{\delta} f_0) \| < \varepsilon_0.$$

Since $f(.,\frac{\delta_0}{\delta} f_0) = \frac{\delta_0}{\delta} f(.,f_0)$ then for $M = \frac{\varepsilon_0\delta}{\delta_0}$ we obtain the inequality

$$\sup_{x\in X^+} \| f(x,f_0) \| \leq M.$$

and as a consequence the solutions of this equation are almost commonly bounded.

In the analogous way we can prove that the almost common boundedness of the solutions of (1′) implies the stability of (1′).

Remark 4. In the classical theory of the stability there is not the analogue to the notion introduced by Definition 5 because in the case of the infinite dimensional Banach spaces the boundedness of solutions of the linear equations and almost common boundedness of this solutions are equivalent. So Theorem 2 makes the generalization of the results obtained in the classical theory of the stability (see [3], [4], [5]).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorem 3. Equation (1) is stable in X^+ if and only if for $x \in X^+$ the mapping $R(t(x),0,y(x))$ is bounded on X^+ and $R(.,0,y(x))$ is the solution of the equation

$$\frac{d}{d\tau} R(\tau,0,y(x)) = A(y(x) + a\tau)\, R(\tau,0,y(x))$$

fulfilling the condition $R(0,0,y(x)) = I_y$.

Proof. Let equation (1) be stable in X^+. Then from Theorem 1 it follows that the solutions $f(.,f_0)$ of equation (1′) are almost commonly bounded. It means that for every $\delta>0$ there exists M such that for $f_0 \in K(0,\delta)$ the inequality

$$\sup_{x \in X^+} \| f(x,f_0) \| \leq M$$

takes place.

By Theorem A (see Introduction) the following equality holds

$$f(x,f_0) = R(t(x),0,y(x)) f_0(y(x)) \quad \text{for } x \in X^+ \tag{9}$$

and as a consequence

$$\sup_{x \in X^+} \| R(t(x),0,y(x)) f_0(y(x)) \| \leq M. \tag{10}$$

Let us notice furthermore that for $y_0 \in Y$ such that $\|y_0\| < \delta$ the following inequality is true

$$\sup_{x \in X^+} \| R(t(x),0,y(x)) \frac{y_0}{\delta} \| \leq \frac{M}{\delta}, \tag{10′}$$

therefore from the definition of the norm of the operator we have

$$\sup_{x \in X^+} \| R(t(x),0,y(x)) \| \leq \frac{M}{\delta}.$$

Now let the mapping $R(t(.),0,y(.))$ be bounded on X^+. Then from Theorem 1 and from equality (9) it follows that equation (1) is stable in X^+.

Remark 5. In the case $X=\mathbb{R}$, the theorem analogous to the above one can be found in paper [2] (see Lemma 3.1 in chapter III).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Asymptotic stability of the generalized linear differential equation of the first order

In this part of our paper we shall occupied in the generalization of the asymptotic stability (see [2], [3]) for the case of the generalized linear differential equations of the first order and in the conditions for equation (1′) to be asymptotic stable.

Similarly, as in part 2 of this paper, our considerations connected with equation (1) will concern the case where $a \in X$ and $a \neq 0$ and A, g are continuous on X, $f_0 \in B(X_a, Y)$ and X_a, L_a are closed subspaces of X.

Definition 6. The solution $f=f(.,f_0)$ of equation (1) will be called asymptotically stable in X^+ if

(i) this solution is stable in X^+,

(ii) there exists $\Delta>0$ such that for every $\varepsilon>0$ there exists such $t_0>0$ that for $\tilde{f}_0 \in K(f_0,\Delta)$ and for $x \in X$ such that $t(x) > t_0$ the following inequality is true

$$\| f(x,f_0) - f(x,\tilde{f}_0) \| < \varepsilon. \tag{11}$$

Definition 7. Differential equation (1) will be called asymptotically stable in X^+ if each of its solutions is asymptotically stable in X^+.

Similarly as Theorem 1, we can prove

Theorem 4. The necessary and sufficient condition for equation (1) to be asymptotically stable in X^+ is that the trivial solution of the homogeneous equation corresponding to equation (1) is asymptotically stable in X^+.

The next theorem presenting the sufficient condition for equation (1) to be asymptotically stable in X^+ will be proceded by the remarks about the semi-inner product.

The class of all real continuous linear functionals on Y regarded as a real linear space is denoted by Y^*. For each $x \in Y-\{0\}$, we define the set

$$T(x) = \{x^* \in Y^*; \quad \|x^*\| = 1, \quad x^*(x) = \|x\|\};$$

the Hahn-Banach theorem guarantees that $T(x)$ is non-empty.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In the space Y we introduce the semi-inner product as follows. Let us choose one nonzero element with the norm equal to 1 from each line in Y containing the point x=0 and denote the set of all chosen elements by Y_0. Then, to each $y \in Y_0$ let us assign any functional $J_0(y) \in T(y)$. We have thus defined the map $J_0: Y_0 \to Y^*$. Let us extend that map onto Y by putting $J(\lambda y) = \lambda J_0(y)$ for each $y \in Y_0$ and $\lambda \in \mathbb{R}$. Now, we can define the semi-inner product denoted by $\langle .,. \rangle$. For $x,y \in Y$, we put $\langle x,y \rangle = J(y)(x)$.

It has the following properties:

a) it maps $Y \times Y$ into $\mathbb{R}$,

b) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$,
$\langle \lambda x,y \rangle = \lambda \langle x,y \rangle$, $\langle x,\lambda y \rangle = \lambda \langle x,y \rangle$ for $x,y,z \in Y$ and $\lambda \in \mathbb{R}$,

c) $\langle x,x \rangle = \|x\|^2$ for each $x \in Y$,

d) $|\langle x,y \rangle|^2 \leq \langle x,x \rangle \langle y,y \rangle$ for $x,y \in Y$.

Now, we shall present a sufficient condition for linear equation (1) to be asymptotically stable. In the case $X=\mathbb{R}$, it implies the dependence of the stability of a linear equation on the condition that all eigenvalues of the operator A lie in the left complex half-plane, very well known in the classical theory of stability (see [3], [4]).

Theorem 5. If there exists such a constant $C>0$ that

$$\langle A(x)y,y \rangle \leq -C\|y\|^2 \quad \text{for } y \in Y,\ y \neq 0 \text{ and } x \in X^+ \tag{12}$$

then equation (1) is asymptotically stable in X^+.

Proof. Taking account of Theorem 4 it is sufficient to prove that the trivial solution of equation (1′) is asymptotically stable. Let $f_0 \in B(X_a, Y)$. Then each solution f of the equation (1′) fulfilling the initial condition $f(x,f_0) = f_0(x)$ for $x \in X_a$ can be presented in the form (see Theorem A)

$$f(x,f_0) = R(t(x),0,y(x))f_0(y(x)) \quad \text{for } x \in X^+ \tag{13}$$

where $R(.,0,y(x))$ for $x \in X^+$ is the solution of the equation

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(14) $$\frac{d}{d\tau} R(\tau,0,y(x)) = A(y(x) + a\tau)R(\tau,0,y(x))$$

fulfilling the condition $R(0,0,y(x)) = I_Y$.

From equality (13) it follows that for $\tau \geq 0$ and $x \in X^+$

$$f(y(x) + a\tau, f_0) = R(\tau,0,y(x))f_0(y(x)).$$

Since the function $f(y(x) + a\tau, f_0)$ is continuously differentiable with respect to the variable τ for $\tau \geq 0$, then the function $\| f(y(x) + a\tau, f_0) \|$ is absolutely continuous with respect to the variable τ in each interval $[0,\tau_0]$ (see [6]); therefore differentiable almost everywhere for $\tau \geq 0$.

Taking account to this fact and Lemma 1.3 from [5], we obtain that for almost every $\tau \geq 0$

(15) $$\frac{d}{d\tau} (\| f(y(x) + a\tau, f_0) \|^2) =$$

$$= 2\langle A(y(x) + a\tau)f(y(x) + a\tau, f_0), f(y(x) + a\tau, f_0)\rangle.$$

By applying inequality (12) it follows from (15) that

$$\frac{d}{d\tau} (\| f(y(x) + a\tau, f_0) \|^2) \leq - 2C \| f(y(x) + a\tau, f_0) \|^2$$

for almost every $\tau \geq 0$. Hence

(16) $$e^{2C\tau} \frac{d}{d\tau} (\| f(y(x) + a\tau, f_0 \|^2) \leq$$

$$\leq - 2Ce^{2C\tau} \| f(y(x) + a\tau, f_0 \|^2$$

thus

(17) $$\frac{d}{d\tau} (e^{2C\tau} \| f(y(x) + a\tau, f_0 \|^2) \leq 0$$

for almost every $\tau \geq 0$.

Taking account of the absolute continuity of the function $\| f(y(x) + a\tau, f_0 \|$ it follows from (17) and Theorem 5 from [6] that the function $e^{2C\tau} \| f(y(x) + a\tau, f_0 \|^2$ is decreasing for $\tau \geq 0$.

Futhemore, let $\tau = t(x)$. By applying the facts that $x = y(x) + a\tau$ for $x \in X$, $f(x,f_0) = f_0(x)$ for $x \in X_a$, we obtain the inequality

(18) $$\| f(x,f_0) \| \leq \| f_0(y(x)) \| e^{-Ct(x)} \quad \text{for } x \in X.$$

The simple consequence of inequality (18) is the asymptotic

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

stability of the trivial solution of equation (1′).

Example. Let $X = Y = l$ (where l is the space of the absolutely summable sequences), $a = (1,0,0,\ldots)$ and X_a be a set of such absolutely summable sequences $x = (\alpha_n)$ $(n=1,2,\ldots)$ for which $\alpha_1=0$. Naturally, $X = L_a \oplus X_a$ and the projection operators on L_a and X_a are continuous.

Let A be a linear operator mapping l into l defined by the formula

$$Ax = \left(-\frac{\alpha_n}{n}\right) \quad \text{where} \quad x = (\alpha_n) \in l \quad \text{for } n=1,2,\ldots$$

Let us consider the differential equation

$$Df(x)(a) = Af(x) \quad \text{for } x=X^+. \tag{19}$$

Let $x\in X$. Let us notice that by the denotations assumed in Theorem 3, we have

$$R(x)u = e^{-\frac{1}{n}t(x)} (u_n) \quad \text{where } u = (u_n) \in l.$$

Therefore $\| R(x) \| \le 1$ for $x\in X^+$, hence from Theorem 3, equation (19) is stable in X^+. After the calculation of the form of semi-inner product in l it is not difficult to verify that

$$\langle Ay,y\rangle \le 0 \quad \text{for } y\in l.$$

After simple calculations it appears that though the operator A fulfils the above inequality nevertheless, equation (19) is not asymptotically stable. This fact shows that the assumption $C>0$ cannot be weakened to $C\ge 0$.

REFERENCES

[1] **A. Alexiewicz:** Functional Analysis. PWN, Warszawa 1969, (in Polish).

[2] **J. Dalecki, M.G. Krein:** The stability of the solutions of the differential equations in Banach space. Moscow 1970, (in Russian).

[3] **B. Demidovič:** The mathematical theory of the stability. WNT, Warsaw 1972, (in Polish).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[4] R. Gutowski: The ordinary differential equations, WNT, Warsaw, 1971, (in Polish).

[5] T. Kato: Nonlinear semigroups and evolution equations. J.Math.Soc. Japan 19 (1967), 508-520.

[6] S. Łojasiewicz: Introduction to the theory of the real functions. PWN, Warsaw 1976, (in Polish).

[7] K. Maurin: Analysis, Part I. PWN, Warsaw 1971, (in Polish).

[8] T. Poreda: Generalized differential equations for maps of Banach space into Banach space, Comment. Math. 30 (1) 1990, 13-18.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF ŁÓDŹ,
90-924, ŁÓDŹ, POLAND
Received May 9, 1989.

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARDWAR

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

T

1.
Th
approx
(1)
(1a)
(1b)
in any
U(x,t)
and Q(
H(x)=[
assume
it is
te<0,t
heat e
(2)
and int
solutio
work [

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Halina Grabarska

THE APPROXIMATION OF SOLUTIONS OF THE FOURIER PROBLEM OF THE FIRST KIND FOR SYSTEMS OF PARABOLIC EQUATIONS

1. Introduction

The purpose of this work is a construction of an approximate solution of the problem

(1) $$U_{xx}+Q(x,t)U=U_t, \qquad (x,t)\in R,$$

(1a) $$U(-1,t)=F^1(t), \quad U(1,t)=F^2(t), \qquad t\in<0,t_o),$$

(1b) $$U(x,0)=H(x), \qquad x\in<-1,1>$$

in any set R_o contained in $R=\{(x,t):|x|<1,0<t<t_o\}$, where $U(x,t)=\left[U_1(x,t),\ldots,U_n(x,t)\right]^T$ is an unknown vector function, and $Q(x,t)=\left[q^{ij}(x,t)\right]_{n\times n}$, $F^i(t)=\left[F_1^i(t),\ldots,F_n^i(t)\right]^T$, $i=1,2$, $H(x)=\left[H_1(x),\ldots,H_n(x)\right]^T$ are matrices of given functions. We assume that Q is a matrix of class $C^1(R)$ and for every $x\in<-1,1>$ it is a matrix of analytic functions with respect to t for $t\in<0,t_o>$, while F^1, F^2 and H are continuous functions.

We will base upon an approximation of the solution of the heat equations system

(2) $$w_{xx} = w_t , \qquad w(x,t)=\left[w_1(x,t),\ldots,w_n(x,t)\right]^T$$

and integral operators transforming solutions of (2) into solutions of (1). We will apply the results obtained in the work [1] by D. Colton for skalar case and make use of [2],

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3], where a construction of integral operators is given.

By $\|\cdot\|$ we will denote the Euclidean vector or matrix norm, i. e.

$$\|w\| = \left(\sum_{i=1}^{n} w_i^2 \right)^{\frac{1}{2}}, \qquad \text{if} \quad w=\left[w_1,\ldots,w_n\right]^T$$

or

$$\|A\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \qquad \text{if} \quad A=\left[a_{ij}\right]_{n\times n}.$$

2. Approximation of a solution of the heat equations system

Let

$$h_j(x,t) = j! \sum_{k=0}^{\left[\frac{j}{2}\right]} \frac{x^{j-2k} t^k}{(j-2k)!\,k!}, \quad j=0,1,\ldots .$$

Lemma 1 ([1], Lemma 2.1). Let $h(x,t)$ be a regular solution of the equation $h_{xx}=h_t$ in a domain R (i.e. of class C^2 with respect to x and of class C^1 with respect to t) and continuous in $\bar{R} = \left\{(x,t): |x|\le 1,\ 0\le t\le t_0\right\}$. Then for every $\varepsilon>0$ there exist numbers $a_0,\ldots,a_N$ such that

$$\max_{(x,t)\in\bar{R}} \left| h(x,t) - \sum_{j=0}^{N} a_j h_j(x,t) \right| < \varepsilon.$$

Using the above lemma, we will prove the following one.

Lemma 2. Let $w(x,t)$ be a solution of (2), regular in R and continuous in $\bar{R}$. Then for every $\varepsilon>0$ there exists a matrix $A = \left[a_{ij}\right]_{nx(N+1)}$ such that

$$\max_{(x,t)\in \bar{R}} \|w(x,t) - Ah^N(x,t)\| < \varepsilon,$$

where $h^N(x,t) = \left[h_0(x,t),\ldots,h_N(x,t)\right]^T$.

Proof. If $w(x,t) = \left[w_1(x,t),\ldots,w_n(x,t)\right]^T$ is a solution of (2), then from Lemma 1 we get that for every $\varepsilon_1>0$ and every

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$i = 1,2,\ldots,n$ there exist numbers $a_{i_0},\ldots,a_{i_{N_i}}$ such that

$$\max_{(x,t)\in\bar{R}}\left|w_i(x,t)-\sum_{j=0}^{N_i} a_{ij}\,h_{ij}(x,t)\right|<\varepsilon_1.$$

Let $N=\max\limits_{1\le i\le n} N_i$ and $A=\left[a_{ij}\right]_{nx(N+1)}$, where $a_{ij}=0$ for $j>N_i$. Since

$$w(x,t)-A\cdot h^N(x,t)=$$

$$=\left[w_1(x,t)-\sum_{j=0}^{N_1} a_{1j}h_j(x,t),\ldots,w_n(x,t)-\sum_{j=0}^{N_n} a_{nj}h_j(x,t)\right]^T,$$

then

$$\max_{(x,t)\in\bar{R}}\|w(x,t)-A\cdot h^N(x,t)\|=$$

$$=\max_{(x,t)\in\bar{R}}\left(\sum_{i=1}^{n}\left(w_i(x,t)-\sum_{j=0}^{N_i}a_{ij}h_j(x,t)\right)^2\right)^{\frac{1}{2}}\le\left(\sum_{i=1}^{n}\varepsilon_1^2\right)^{\frac{1}{2}}=\varepsilon_1\sqrt{n}\,.$$

By setting $\varepsilon_1=\varepsilon/\sqrt{n}$, we obtain the thesis of the Lemma 2.

3. Approximation of the solutions of a system of parabolic equations

Let $U(x,t)$ be a regular solution of (1) in R such that $U(0,t)=0$. In [2] it has been demonstrated that $U(x,t)$ can be represented in the form

$$U(x,t)=w(x,t)+\int_0^x K(s,x,t)w(s,t)ds,$$

where $w(x,t)$ is a solution of (2) in R, satisfying $w(0,t)=0$. If instead of the condition $U(0,t)=0$ we have $U_x(0,t)=0$, then we can represent $U(x,t)$ in the form

$$U(x,t)=w(x,t)+\int_0^x L(s,x,t)w(s,t)ds,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $w(x,t)$ is a solution of (2) in R, satisfying $w_x(0,t)=0$. The matrices $K(s,x,t) = [K_{ij}(s,x,t)]_{nxn}$, and $L(s,x,t) = [L_{ij}(s,x,t)]_{nxn}$ are defined in [2].

Proceeding as in [1], one can prove that any regular solution of (1) in R can be represented in the form

$$U(x,t)=T_0w(x,t)\equiv w(x,t)+\frac{1}{2}\int_0^x\Big[K(s,x,t)+L(s,x,t)\Big]w(s,t)ds,$$

where $w(x,t)$ is a regular solution of (2) in R.

Since for $y(x)=\Big[y_1(x),\ldots,y_n(x)\Big]^T$ we have

$$\Big\|\int_a^b y(x)dx\Big\| \le (b-a)\max_{a\le x\le b}\| y(x)\|,$$

then

$$\max_{(x,t)\in\bar{R}}\|T_0w(x,t)\|\le(1+M)\max_{(x,t)\in\bar{R}}\|w(x,t)\|, \tag{3}$$

where

$$M = \max_{\substack{(x,t)\in\bar{R}\\ |s|\le 1}}\|K(s,x,t)+L(s,x,t)\|.$$

Let $w^{ij}(x,t) = [0,\ldots,0,h_j(x,t),0,\ldots,0]^T = e_ih_j(x,t)$, $i=1,\ldots,n$, $j = 0,\ldots,N$, where $e_i = [0,\ldots,0,1,0,\ldots0]^T$ is a i-th versor in the n-dimensional space and let

$$v^{ij}(x,t)=T_0w^{ij}(x,t), \quad i=1,\ldots,n, \quad j=0,1,\ldots N. \tag{4}$$

Theorem 1. Let $U(x,t)$ be a solution of the system (1) regular in R and continuous in $\bar{R}$. Then for every $\varepsilon>0$ there exists a matrix $A=[a_{ij}]_{nx(N+1)}$ such that

$$\max_{(x,t)\in\bar{R}}\Big\| U(x,t)-\sum_{i=1}^{n}\sum_{j=0}^{N}a_{ij}v^{ij}(x,t)\Big\|<\varepsilon.$$

Proof: In the paper [2] was shown the existence of a vector function $w(x,t)$ being a solution of (2) and $U(x,t)=T_0w(x,t)$ regular in R and continuous in $\bar{R}$. By Lemma 2, there exists a matrix $a = [a_{ij}]_{nx(N+1)}$ such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(5)\qquad \max_{(x,t)\in\bar{R}} \|w(x,t)-A\cdot h^N(x,t)\| < \frac{\varepsilon}{1+M}.$$

From the definitions of v and T_0, we get

$$U(x,t)-\sum_{i=1}^{n}\sum_{j=0}^{N} a_{ij}\, v^{ij}(x,t)=T_0w(x,t)-\sum_{i=1}^{n}\sum_{j=0}^{N} a_{ij}T_0w^{ij}(x,t)=$$

$$= T_0\Big(w(x,t) -\sum_{i=1}^{n}\sum_{j=0}^{N} a_{ij}e_ih_j(x,t)\Big) = T_0\big(w(x,t)-Ah^N(x,t)\big).$$

Hence by the inequalities (3) and (5), we obtain

$$\max_{(x,t)\in\bar{R}} \|U(x,t)-\sum_{i=1}^{n}\sum_{j=0}^{N} a_{ij}\, v^{ij}(x,t)\| \le$$

$$\le (1+M) \max_{(x,t)\in\bar{R}} \|w(x,t)-Ah^N(x,t)\|<\varepsilon.$$

Remark. Since $h_{2j}(-s,t)=h_{2j}(s,t)$, $h_{2j+1}(-s,t)= -h_{2j+1}(s,t)$ and (see [2])

$$L(-s,x,t) = L(s,x,t),\ K(-s,x,t) = -K(s,x,t),$$

then

$$\int_{-x}^{x} L(s,x,t)w^{i,2j}(s,t)ds = 2\int_{0}^{x} L(s,x,t)w^{i,2j}(s,t)ds,$$

$$\int_{-x}^{x} L(s,x,t)w^{i,2j+1}(s,t)ds = 0,$$

$$\int_{-x}^{x} K(s,x,t)w^{i,2j}(s,t)ds = 0,$$

$$\int_{-x}^{x} K(s,x,t)w^{i,2j+1}(s,t)ds = 2\int_{0}^{x} K(s,x,t)w^{i,2j+1}(s,t)ds.$$

Applying the above formulae, one can write the functions $v^{ij}(x,t)$ in a simpler form

$$v^{i,2j}(x,t) = w^{i,2j}(x,t)+\int_{0}^{x} L(s,x,t)w^{i,2j}(s,t)\,ds,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$v^{i,2j+1}(x,t) = w^{i,2j+1}(x,t) + \int_0^x K(s,x,t) w^{i,2j+1}(s,t)\, ds,$$

$i=1,\dots,n; \quad j=0,1,\dots$.

4. Construction of the approximate solution of the Fourier problem

Theorem 1 states that the solution of Eq. (1) can be approximated with any accuracy by functions (4). Now we will construct of the approximate solution of the problem (1), (1a), (1b) in any compact set R_0 contained in R.

Let $C = \left[c_{ij}\right]_{\substack{i=1,\dots,n \\ j=0,\dots,N}}$ and let

$$S(C) = \int_0^{t_0} \|F^1(\tau) - \sum_{i=1}^{n}\sum_{j=0}^{N} c_{ij}\, v^{ij}(-1,\tau)\|^2 d\tau +$$

$$+ \int_0^{t_0} \|F^2(\tau) - \sum_{i=1}^{n}\sum_{j=0}^{N} c_{ij}\, v^{ij}(1,\tau)\|^2 d\tau +$$

$$+ \int_{-1}^{1} \|H(\xi) - \sum_{i=1}^{n}\sum_{j=0}^{N} c_{ij}\, v^{ij}(\xi,0)\|^2 d\xi.$$

Our aim will be to find a matrix C for which the functional S attains its minimum. This is a least-squares approximation problem (see for example [4]) and it can be reduced to the problem of solving the system

$$\frac{\partial S}{\partial c_{ij}} = 0, \qquad i=1,\dots,n, \quad j=0,\dots,N,$$

which leads to the following system of linear equations

$$\sum_{i=1}^{n}\sum_{j=0}^{N} d_{kl}^{ij}\, c_{ij} = b_{kl}, \quad k=1,\dots,n, \; l=0,\dots,N, \tag{6}$$

where

$$d_{kl}^{ij} = \sum_{p=1}^{n} \left\{ \int_0^{t_0} \left[v_p^{ij}(-1,\tau) v_p^{kl}(-1,\tau) + v_p^{ij}(1,\tau) v_p^{kl}(1,\tau) \right] d\tau + \right.$$

$$\left. + \int_{-1}^{1} v_p^{ij}(\xi,0) v_p^{kl}(\xi,0)\, d\xi \right\},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$b_{kl} = \sum_{p=1}^{n} \left\{ \int_0^{t_0} \left[F_p^1(\tau) v_p^{kl}(-1,\tau) + F_p^2(\tau) v_p^{kl}(1,\tau) \right] d\tau + \right.$$

$$\left. + \int_{-1}^{1} H_p(\xi) v_p^{kl}(\xi,0)\, d\xi \right\}.$$

Theorem 2. Let $U(x,t)$ be a regular solution of the system (1) in R continuous in $\bar{R}$ and satisfying the initial-boundary data (1a) and (1b). Let R_0 be a compact subset of R. Then for every $\varepsilon>0$ there exists a positive integer N such that

$$\max_{(x,t)\in R_0} \| U(x,t) - \sum_{i=1}^{n} \sum_{j=0}^{N} c_{ij}\, v^{ij}(x,t) \| < \varepsilon,$$

where $C = \left[c_{ij}\right]_{nx(N+1)}$ is the solution of the system (6).

Proof. Let $G(x,t;\xi,\tau)$ be a Green function of first kind for equation (1) and let

$$m = \max_{(x,t)\in R_0} \left\{ \int_0^{t_0} \| \frac{\partial}{\partial \xi} G(x,t;-1,\tau) \|^2 d\tau + \int_0^{t_0} \| \frac{\partial}{\partial \xi} G(x,t;1,\tau) \|^2 d\tau + \right.$$

$$\left. + \int_{-1}^{1} \| G(x,t;\xi,0) \|^2 d\xi \right\}.$$

By Theorem 1 there exists a constant N and a matrix $A = \left[a_{ij}\right]_{nx(N+1)}$ such that for

$$Z(x,t) = \sum_{i=1}^{n} \sum_{j=0}^{N} a_{ij}\, v^{ij}(x,t)$$

the following inequality holds

$$\max_{(x,t)\in \bar{R}} \| U(x,t) - Z(x,t) \| < \frac{\varepsilon}{\sqrt{2m(t_0+1)}}. \tag{7}$$

Let $C = \left[c_{ij}\right]_{nx(N+1)}$ be the solution of the system (6) and let

$$V(x,t) = \sum_{i=1}^{n} \sum_{j=0}^{N} c_{ij}\, v^{ij}(x,t).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From the definition of C it follows that

$$(8)\qquad \int_0^{t_0}\|F^1(\tau)-V(-1,\tau)\|^2d\tau+\int_0^{t_0}\|F^2(\tau)-V(1,\tau)\|^2d\tau+$$
$$+\int_{-1}^{1}\|H(\xi)-V(\xi,0)\|^2d\xi \le \int_0^{t_0}\|F^1(\tau)-Z(-1,\tau)\|^2d\tau +$$
$$+\int_0^{t_0}\|F^2(\tau)-Z(1,\tau)\|^2d\tau+\int_{-1}^{1}\|H(\xi)-Z(\xi,0)\|^2d\xi.$$

The functions $v^{ij}(x,t)$ for $i=1,\dots,n$, $j=0,\dots,N$ satisfy (1), because they are t_0 images of the functions $w^{ij}(x,t)$. So $V(x,t)$ satisfies (1), too. Hence

$$U(x,t)-V(x,t) = \int_0^t \frac{\partial}{\partial\xi}G(x,t;-1,\tau)\left[F^1(\tau)-V(-1,\tau)\right]d\tau+$$
$$+\int_0^t \frac{\partial}{\partial\xi}G(x,t;1,\tau)\left[F^2(\tau)-V(1,\tau)\right]d\tau+\int_{-1}^{1}G(x,t;\xi,0)\left[H(\xi)-V(\xi,0)\right]d\xi.$$

Since for a matrix function $B(x)=\left[b_{ij}(x)\right]_{nxn}$ and a vector function $y(x) = [y_1(x),\dots,y_n(x)]^T$ there is

$$\left\|\int_a^b B(x)y(x)dx\right\| \le \left(\int_a^b\|B(x)\|^2dx\right)^{\frac{1}{2}}\left(\int_a^b\|y(x)\|^2dx\right)^{\frac{1}{2}},$$

then

$$\|U(x,t)-V(x,t)\| \le \left\|\int_0^t \frac{\partial}{\partial\xi}G(x,t;-1,\tau)\left[F^1(\tau)-V(-1,\tau)\right]d\tau\right\|+$$
$$+\left\|\int_0^t \frac{\partial}{\partial\xi}G(x,t;1,\tau)\left[F^2(\tau)-V(1,\tau)\right]d\tau\right\|+$$
$$+\left\|\int_{-1}^{1}G(x,t;\xi,0)\left[H(\xi)-V(\xi,0)\right]d\xi\right\| \le$$
$$\le \left(\int_0^{t_0}\frac{\partial}{\partial\xi}G(x,t;-1,\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^{t_0}\|F^1(\tau)-V(-1,\tau)\|^2d\tau\right)^{\frac{1}{2}}+$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+\left(\int_0^{t_0}\left\|\frac{\partial}{\partial\xi}G(x,t;1,\tau)\right\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^{t_0}\|F^2(\tau)-V(1,\tau)\|^2 d\tau\right)^{\frac{1}{2}}+$$

$$+\left(\int_{-1}^{1}\|G(x,t;\xi,0)\|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\|H(\xi)-V(\xi,0)\|^2 d\xi\right)^{\frac{1}{2}}.$$

Hence and from the Cauchy inequality we have

$$\|U(x,t)-V(x,t)\|^2 \le \left\{\int_0^{t_0}\left\|\frac{\partial}{\partial\xi}G(x,t;-1,\tau)\right\|^2 d\tau+\int_0^{t_0}\left\|\frac{\partial}{\partial\xi}G(x,t;1,\tau)\right\|^2 d\tau\right.$$

$$\left.+\int_{-1}^{1}\|G(x,t;\xi,0)\|d\xi\right\}\left\{\int_0^{t_0}\|F^1(\tau)-V(-1,\tau)\|^2 d\tau+\right.$$

$$\left.+\int_0^{t_0}\|F^2(\tau)-V(1,\tau)\|^2 d\tau+\int_{-1}^{1}\|H(\xi)-V(\xi,0)\|^2 d\xi\right\}.$$

Then by the inequalities (7) and (8) we obtain

$$\max_{(x,t)\in R_0}\|U(x,t)-V(x,t)\|^2 \le$$

$$\le m\left(t_0\frac{\varepsilon^2}{2m(t_0+1)}+t_0\frac{\varepsilon^2}{2m(t_0+1)}+2\frac{\varepsilon^2}{2m(t_0+1)}\right)=\varepsilon^2.$$

which implies the thesis of Theorem 2.

REFERENCES

[1] D. Colton: The approximation of solutions to initial boundary value problems for parabolic equations in one space variable, Quarterly of Applied Math, (1976) 377-386.

[2] H. Grabarska: Integral operators for systems of parabolic equations, Demonstratio Math., (1983) 787-796.

[3] H. Grabarska: Application of integral operators to the study of the Fourier problem of first kind, Demonstratio Math. 18 (1984) 127-140.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[4] J. i M. Jankowscy: Przegląd metod i algorytmów numerycznych, część 1, WNT, Warszawa 1981.

[5] M. Dryja, J.i M. Jankowscy: Przegląd metod i algorytmów numerycznych, część 2, WNT Warszawa, 1982.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received May 31, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Zbigniew Binderman

COMPLEX R-SHIFTS FOR RIGHT INVERTIBLE OPERATORS

In the books [1], [2] of D. Przeworska-Rolewicz real R-shifts for right invertible operators have been defined and studied. Here a complex extension of R-shifts for right invertible operators is defined. Fundamental algebraic properties of these R-shifts are given.

Let X be a linear space over the field $\mathbb{C}$ of the complex numbers. Denote by L(X) the set of all linear operators which domains and ranges are linear subsets of X. In particular, we shall write: $L_o(X) = \{A \in L(X): \text{dom } A=X\}$. An operator $D \in L(X)$ is said to be right invertible if there exists an operator $R \in L(X)$ such that $DR=I$. The set of all right invertible operators belonging to L(X) will be denote by R(X). If $D \in R(X)$ then we denote by $\mathcal{R}_D$ the all right inverses of D. In sequel we shall assume that dim ker $D \neq 0$ and that all right inverses belong to $L_o(X)$. An operator $F \in L_o(X)$ is said to be initial for the operator D corresponding to an $R \in \mathcal{R}_D$ if

$$F^2 = F, \quad FX = \ker D \text{ and } FR = 0.$$

This definition implies that

$$F = I - RD \quad \text{on} \quad \text{dom } D.$$

Denote by P(R) the linear span of the set of all D polynomials, i.e. $P(R) = \text{lin}\,(R^k z: z \in \ker D, \ k \in \{0\} \cup \mathbb{N})$. Elements $R^k z$ $(z \neq 0, k=0,1,2,\ldots)$ are linearly independent [2].

Let $\mathcal{F}_{CG}$ be the family of all nonempty subsets of the space of complex numbers $\mathbb{C}$ which are groups with respect to the addition in $\mathbb{C}$. By $\mathcal{F}_{CS}$ we denote the family of all nonempty

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

subsets of $\mathbb{C}$ which are semigroups with respect to the addition and are not members of the family $\mathcal{F}_{\mathbb{C}G}$.

For example, the family $\mathcal{F}_{\mathbb{C}G}$ contains $\mathbb{C}$ and sets $\{w \in \mathbb{C}: w = aw_0, \ a \in \mathbb{R}\}$, where $0 \neq w_0 \in \mathbb{C}$ is an arbitrary fixed complex number. The family $\mathcal{F}_{\mathbb{C}S}$ contains sets: $\mathbb{C}_{(\Phi_1,\Phi_2)} = \{w \in \mathbb{C}: \Phi_1 < \arg w < \Phi_2, \quad 0 \le \Phi_2 - \Phi_1 \le \Pi, \quad \mathbb{C}_{(\Phi_1,\Phi_2)} \setminus K_\rho$, where $K_\rho = \{w \in \mathbb{C}: |w| < \rho, \rho \ge 0\}$, $0 \le \Phi_2 - \Phi_1 < \frac{2}{3}\Pi$, $\{w \in \mathbb{C}: \operatorname{Im} w = 0, \operatorname{re} w > b > 0\} \cup \{0\}$.

In further considerations we shall deal with algebraic properties of these sets only. Here and in the sequel we admit that $0^0 = 1$, and we shall write $A(\mathbb{C})$ instead of sets belonging to families $\mathcal{F}_{\mathbb{C}G}$, $\mathcal{F}_{\mathbb{C}S}$. If for all $h \in A(\mathbb{C})$, $\operatorname{Im} h = 0$, then we shall write $A(\mathbb{R})$ instead of $A(\mathbb{C})$.

In [1], [2] real R-shifts for right invertible operators have been defined. Here in a similar way, we shall define complex R-shifts.

Definition 1. Suppose that $D \in R(X)$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. We say that the family of linear operators $S_{A(\mathbb{C})} = \{S_h\}_{h \in A(\mathbb{C})} \subset L_o(X)$ is family of R- shifts for D if the following conditions are satisfied:

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F \quad \text{for all } h \in A(\mathbb{C}), \ k \in \{0\} \cup \mathbb{N}.$$

By definition, if $0 \in A(\mathbb{C})$ then $S_0 R^k F = R^k F$.

In a similar way as in [1], [2] we can prove the following

Theorem 1. We assume that $D \in R(X)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Suppose that we are given a family $S_{A(\mathbb{C})} = \{S_h\}_{h \in A(\mathbb{C})}$ of R-shifts. Then:

(i) For all $h \in A(\mathbb{C})$, $z \in \ker D$, $k = 0,1,2,\ldots$

$$S_h R^k z = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j z$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

in particular S_h preserve constants, i.e. $S_h z = z$ for all $z \in \ker D$, $h \in A(\mathbb{C})$.

(ii) If $A(\mathbb{C}) \in \mathcal{F}_{\mathbb{CS}}$ then $S_{A(\mathbb{C})}$ is a comutative semigroup, if $A(\mathbb{C}) \in \mathcal{F}_{\mathbb{CG}}$ then $S_{A(\mathbb{C})}$ is a Abelian group, with respect to the superposition of operators, i.e.

$S_{h_1} S_{h_2} = S_{h_2} S_{h_1} = S_{h_1+h_2}$ on the set $P(R)$, for all $h_1, h_2 \in A(\mathbb{C})$.

(iii) For all $h \in A(\mathbb{C})$ the operators S_h are commutative with the operator D i.e. $S_h D = D S_h$ on $P(R)$ and uniquely determined on the set $P(R)$.

(iv) If X is a complete linear metric space, $\overline{P(R)} = X$ and S_h are continuous for all $h \in A(\mathbb{C})$ then S_h are uniquely determined on the whole space X.

Definition 2. We assume that all assumptions of the Definition 1 are satisfied. Define families $c_{A(\mathbb{C})} = \{c_h\}_{h \in A(\mathbb{C})}$, $ch_{A(\mathbb{C})} = \{ch_h\}_{h \in A(\mathbb{C})}$, $s_{A(\mathbb{C})} = \{s_h\}_{h \in A(\mathbb{C})}$, $sh_{A(\mathbb{C})} = \{sh_h\}_{h \in A(\mathbb{C})}$ of linear operators of space $L_o(X)$ in the following way:

$$c_h R^k F = \sum_{j=0}^{k_1} \frac{(-1)^j h^{2j}}{(2j)!} R^{k-2j} F, \quad ch_h R^k F = \sum_{j=0}^{k_1} \frac{h^{2j}}{(2j)!} R^{k-2j} F,$$

$$s_h R^k F = \begin{cases} 0 & \text{for } \kappa = 0 \\ \sum_{j=0}^{k_2} \frac{(-1)^j h^{2j+1}}{(2j+1)!} R^{k-2j-1} F \end{cases},$$

$$s_h R^k F = \begin{cases} 0 & \text{for } k = 0 \\ \sum_{j=0}^{k_2} \frac{h^{2j+1}}{(2j+1)!} R^{k-2j-1} F, & k \in \mathbb{N}, \end{cases}$$

where $k_1 = [k/2]$, $k_2 = [(k-1)/2]$, $[p]$ - is the so-called entier of $p \in R$, i.e. the greatest integer which does not exceed p. We say that the families $c_{A(\mathbb{C})}$, $ch_{A(\mathbb{C})}$, $s_{A(\mathbb{C})}$, $sh_{A(\mathbb{C})}$ determine cosine, cosine hyperbolic, sine, sine hyperbolic R-shifts for D and that c_h, ch_h, s_h, sh_h are a cosine, a cosine hyperbolic, a sine , a sine hyperbolic R-shift on h for an arbitrary $h \in A(\mathbb{C})$, respectively.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

If is easy to verify two following propositions

Proposition 1. We assume that $D \in R(X)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Suppose that we are given families $S_{A(\mathbb{C})}$, $c_{A(\mathbb{C})}$, $ch_{A(\mathbb{C})}$, $s_{A(\mathbb{C})}$, $sh_{A(\mathbb{C})}$ of R-shifts. Then on the set $P(R)$

(i) $ch_h = c_{ih}$, $sh_h = -is_{ih}$ for $h \in A(\mathbb{C})$ together with ih,

(ii) $c_h = \frac{1}{2}(S_{ih}+S_{-ih})$, $s_h = \frac{1}{2i}(S_{ih}-S_{-ih})$ for $h \in A(\mathbb{C})$ together with ih and $-ih$,

(iii) $c_h = c_{-h}$, $s_h = -s_{-h}$,

$ch_h = \frac{1}{2}(S_h+S_{-h})$, $sh_h = \frac{1}{2}(S_h-S_{-h})$

for $h \in A(\mathbb{C})$ together with $-h$.

Proposition 2. We assume that $D \in R(X)$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Suppose that we are given families $c_{A(\mathbb{C})}$, $ch_{A(\mathbb{C})}$, $s_{A(\mathbb{C})}$, $sh_{A(\mathbb{C})}$ of R-shifts. Then for all $h \in A(\mathbb{C})$:

(i) $$c_h R^k F = \sum_{j=0}^{k} \frac{h^j \cos(j\Pi/2)}{j!} R^{k-j} F,$$

$$s_h R^k F = \sum_{j=0}^{k} \frac{h^j \sin(j\Pi/2)}{j!} R^{k-j} F,$$

$$ch_h R^k F = \sum_{j=0}^{k} \frac{h^j |\cos(j\Pi/2)|}{j!} R^{k-j} F,$$

$$s_h R^k F = \sum_{j=0}^{k} \frac{h^j |\sin(j\Pi/2)|}{j!} R^{k-j} F, \quad k=0,1,2,\ldots .$$

(ii) For all $z \in \ker D$, $k \in \mathbb{N}$

$$c_h R^k z = \sum_{j=0}^{k_1} \frac{(-1)^j h^{2j}}{(2j)!} R^{k-2j} z, \quad ch_h R^k z = \sum_{j=0}^{k_1} \frac{h^{2j}}{(2j)!} R^{k-2j} z,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$s_h R^k z = \sum_{j=0}^{k_2} \frac{(-1)^j h^{2j+1}}{(2j+1)!} R^{k-2j-1} z, \quad sh_h R^k z = \sum_{j=0}^{k_2} \frac{h^{2j+1}}{(2j+1)!} R^{k-2j-1} z,$$

$$c_h z = ch_h z = z \quad , \qquad shz = sh_h z = 0,$$

where $k_1 = [k/2]$, $k_2 = [(k-1)/2]$.

Theorem 2. Suppose that all assumptions of the Proposition 2 are satisfied. Then for all $h \in A(\mathbb{C})$ the operators c_h, s_h, ch_h, sh_h, are uniquely determined on the set $P(R)$ and commute on $P(R)$ with D.

Proof. Let $h \in A(\mathbb{C})$, $k \in \mathbb{N}$, $z \in \ker D$ are arbitrary fixed. From the Proposition 2 (ii) we have

$$(Dc_h - c_h D)R^k z = D \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h^j R^{k-j} z - c_h R^{k-1} z$$

$$= \frac{\cos(k\Pi/2)}{k!} DR^0 z + \sum_{j=0}^{k-1} \frac{\cos(j\Pi/2)}{j!} h^j R^{k-j-1} z$$

$$- \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h^j R^{k-1-j} z = \frac{\cos(k\Pi/2)}{k!} Dz = 0,$$

similarly for the operators sine, cosine hyperbolic and sine hyperbolic. For $k=0$: $c_h=I$, $s_h=0$, $ch_h=I$, $sh_h=0$. This implies that under $k=0$ $(Dc_h-c_hD)R^k z=0$, as for s_h, ch_h, sh_h, too.

The second part of the theorem follows directly from Definition 2 and Proposition 2.

Theorem 3. Suppose that all assumptions of Proposition 1 are satisfied. Then for arbitrary h_1, $h_2 \in A(\mathbb{C})$ on the set $P(R)$

$$c_{h_1} s_{h_2} = s_{h_2} c_{h_1}, \quad s_{h_1} s_{h_2} = s_{h_2} s_{h_1},$$

$$ch_{h_1} s_{h_2} = s_{h_2} ch_{h_1}, \quad sh_{h_1} s_{h_2} = s_{h_2} sh_{h_1}.$$

Proof. We assume that $k \in \{0\} \cup \mathbb{N}$; $h_1, h_2 \in A(\mathbb{C})$ are arbitrarily fixed. Then on $P(R)$ we have

$$c_{h_1} s_{h_2} R^k F = c_{h_1} \sum_{j=0}^{k} \frac{h_2^{k-j}}{(k-j)!} R^j F = \sum_{j=0}^{k} \frac{h_2^{k-j}}{(k-j)!} c_{h_1} R^j F$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= \sum_{j=0}^{k} \frac{h_2^{k-j}}{(k-j)!} \sum_{m=0}^{j} \frac{h_1^m \cos(m\Pi/2)}{m!} R^{j-m_F}$$

$$= \sum_{m=0}^{k} \frac{h_1^m \cos(m\Pi/2)}{m!} \sum_{j=m}^{k} \frac{h_2^{k-j}}{(k-j)!} R^{j-m_F}$$

$$= \sum_{m=0}^{k} \frac{h_1^m \cos(m\Pi/2)}{m!} \sum_{n=0}^{k-m} \frac{h_2^{k-m-n}}{(k-m-n)!} R^{n_F}$$

$$= \sum_{m=0}^{k} \frac{h_1^m \cos(m\Pi/2)}{m!} S_{h_2} R^{k-m_F}$$

$$= S_{h_2} \sum_{m=0}^{k} \frac{h_1^m \cos(m\Pi/2)}{m!} R^{k-m_F} = S_{h_2} c_{h_1} R^{k_F}.$$

Similarly, we can prove other equalities .

Proposition 3. Suppose that all assumptions of the Proposition 2 are satisfied. Then on the set P(R) :

(i) $c_{h_1+h_2} = c_{h_1} c_{h_2} - s_{h_1} s_{h_2}$,

$s_{h_1+h_2} = s_{h_1} c_{h_2} + c_{h_1} s_{h_2}$; $h_1, h_2 \in A(\mathbb{C})$.

(ii) $c_h^2 + s_h^2 = I$, $h \in A(\mathbb{C})$.

(iii) $c_{h_1} c_{h_2} = c_{h_2} c_{h_1}$, $s_{h_1} s_{h_2} = s_{h_2} s_{h_1}$, $c_{2h} = c_h^2 - s_h^2$,

$s_{2h} = c_h s_h + s_h c_h$; $h, h_1, h_2 \in A(\mathbb{C})$.

(iv) $c_h F = F$, $c_h RF = RF$, $s_h F = 0$, $s_h RF = hF$, $h \in A(\mathbb{C})$.

Proof. As an example we prove the first formulae of (i) and (ii). We have on P(R) :

$$c_{h_1+h_2} R^{k_F} = \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} (h_1+h_2)^j R^{k-j_F}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} \sum_{m=0}^{j} \binom{j}{m} h_1^{j-m} h_2^{m} R^{k-j}F$$

$$= \sum_{j=0}^{k} \left(\sum_{m=j}^{k} \frac{\cos(m\Pi/2)}{(m-j)!} h_1^{m-j} R^{k-m}F \right) \frac{1}{j!} h_2^{j}$$

$$= \sum_{j=0}^{k} \left(\sum_{n=0}^{k-j} \frac{\cos[(n+j)\Pi/2]}{n!} h_1^{n} R^{k-n-j}F \right) \frac{1}{j!} h_2^{j}$$

$$= \sum_{j=0}^{k} \left(\sum_{n=0}^{k-j} \frac{\cos(n\Pi/2)}{n!} h_1^{n} R^{k-n-j}F \right) \frac{\cos(j\Pi/2)}{j!} h_2^{j}$$

$$- \sum_{j=0}^{k} \left(\sum_{n=0}^{k-j} \frac{\sin(n\Pi/2)}{n!} h_1^{n} R^{k-n-j}F \right) \frac{\sin(j\Pi/2)}{j!} h_2^{j}$$

$$= \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h_2^{j} c_{h_1} R^{k-j}F - \sum_{j=0}^{k} \frac{\sin(j\Pi/2)}{j!} h_2^{j} s_{h_1} R^{k-j}F$$

$$= c_{h_1} \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h_2^{j} R^{k-j}F - s_{h_1} \sum_{j=0}^{k} \frac{\sin(j\Pi/2)}{j!} h_2^{j} R^{k-j}F$$

$$= \left(c_{h_1} c_{h_2} - s_{h_1} s_{h_2} \right) R^{k}F \; ; \quad k=0,1,2,\ldots \; ; \; h_1, h_2 \in A(\mathbb{C}),$$

$$\left(c_h^2 + s_h^2 \right) R^{k}F = c_h \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h^{j} R^{k-j}F + s_h \sum_{j=0}^{k} \frac{\sin(j\Pi/2)}{j!} h^{j} R^{k-j}F$$

$$= \sum_{j=0}^{k} \left\{ \sum_{n=0}^{k-j} \frac{\cos[(j-n)\Pi/2]}{n!} h^{n} R^{k-n-j}F \right\} \frac{1}{j!} h^{j}$$

$$= \sum_{j=0}^{k} \left\{ \sum_{m=j}^{k} \frac{\cos[(2j-m)\Pi/2)]}{(m-j)!} h^{m-j} R^{k-m}F \right\} \frac{1}{j!} h^{j}$$

$$= \sum_{j=0}^{k} \frac{\cos(j\Pi/2)}{j!} h^{j} \left\{ \sum_{m=0}^{j} (-1)^{m} \binom{j}{m} \right\} R^{k-j}F$$

$$= R^{k}F \; ; \quad k=0,1,2,\ldots; \; h \in A(\mathbb{C}).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Definition 3. Suppose that we are given families $S_{A(\mathbb{C})}$, $S_{A_1(\mathbb{R})}$, $C_{A_2(\mathbb{R})}$, $S_{A_2(\mathbb{R})}$ of R-shifts for $D \in R(X)$. We assume that for all $h \in A(\mathbb{C})$: $\operatorname{Re} h \in A_1(\mathbb{R})$, $\operatorname{Im} h \in A_2(\mathbb{R})$. Then the linear operators

$$\operatorname{Re} S_h = S_a C_b \quad , \quad \operatorname{Im} S_h = S_a S_b \quad ,$$

where $h=a+ib \in A(\mathbb{C})$, $S_a \in S_{A_1(\mathbb{R})}$, $C_b \in C_{A_2(\mathbb{R})}$, $S_b \in S_{A_2(\mathbb{R})}$, are called real and imaginary parts of the complex R-shifts $S_h \in S_{A(\mathbb{C})}$, respectively.

Proposition 5. Suppose that all assumptions of the Definition 3 are satisfied. Then every $S_h \in S_{A(\mathbb{C})}$

$$S_h = \operatorname{Re} S_h + i \operatorname{Im} S_h \quad \text{on the set } P(R),$$

is independent on a choice of the families $S_{A_1(\mathbb{R})}$, $C_{A_2(\mathbb{R})}$, $S_{A_2(\mathbb{R})}$.

Proof. Let $h=a+ib \in A(\mathbb{C})$, $r=|h|$, $t=\arg h$. Then on $P(R)$ we have

$$S_h R^k F = \sum_{j=0}^{k} \frac{h^{k-j}}{(k-j)!} R^j F = \sum_{j=0}^{k} \frac{r^j e^{ijt}}{j!} R^{k-j} F$$

$$= \sum_{j=0}^{k} \frac{r^j \cos jt}{j!} R^{k-j} F + i \sum_{j=0}^{k} \frac{r^j \sin jt}{j!} R^{k-j} F \quad .$$

On the other hand by Theorem 1 and Definition 2, we have

$$S_h R^k F = S_{a+ib} R^k F = S_a \Big[\sum_{j=0}^{k} \frac{i^j b^j}{j!} R^{k-j} F \Big] = \Big(S_a C_b + i S_a S_b \Big) R^k F,$$

where S_a, C_b, S_b belong to families $S_{A_1(\mathbb{R})}$, $C_{A_2(\mathbb{R})}$, $S_{A_2(\mathbb{R})}$, respectively. This implies that

$$S_a C_b R^k F = \sum_{j=0}^{k} \frac{r^j \cos jt}{j!} R^{k-j} F,$$

$$S_a S_b R^k F = \sum_{j=0}^{k} \frac{r^j \sin jt}{j!} R^{k-j} F; \quad k=0,1,2,\ldots \ .$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

By the definition of the operators $\operatorname{Re} S_h$ and $\operatorname{Im} S_h$ and Proposition 5 we have following

Proposition 6. Suppose that all assumptions of Proposition 5 are satisfied. Then on the set $P(R)$:

(i) $S_h = \operatorname{Re} S_h$, $\operatorname{Im} S_h = 0$ for $h \in A(\mathbb{C})$: $\operatorname{Im} h=0$.

(ii) $\operatorname{Re} S_0 = I$ if $0 \in A(\mathbb{C})$, $S_i = c_1 + is_1$ if $i \in A(\mathbb{C})$.

(iii) $\operatorname{Re} S_{ib} = c_b$, $\operatorname{Im} S_{ib} = s_b$ if $ib \in A(\mathbb{C})$, $(b \in \mathbb{R})$.

Observe that Theorem 1 implies that on $P(R)$

$$S_{h_1} S_{h_2} = S_{h_1+h_2} = \operatorname{Re} S_{h_1+h_2} + i\operatorname{Im} S_{h_1+h_2};\ h_1,h_2 \in A(\mathbb{C}).$$

Therefore it is easy to prove the following

Proposition 7. Suppose that all assumptions of Proposition 5 are satisfied. Then on the set $P(R)$:

(i) $S_{nh} = S_h^n = S_a^n\left(c_{nb} + is_{nb}\right)$, $n \in \mathbb{N}$, $h=a+ib \in A(\mathbb{C})$,

where S_a, c_b, s_b belong to families $S_{A_1(\mathbb{R})}$, $C_{A_2(\mathbb{R})}$, $S_{A_2(\mathbb{R})}$, respectively. It is an analogue of the de Moivre formula.

(ii) $$\operatorname{Re} S_{h_1+h_2} = \operatorname{Re} S_{h_1} \operatorname{Re} S_{h_2} - \operatorname{Im} S_{h_1} \operatorname{Im} S_{h_2}$$
$$= \operatorname{Re} S_{h_2} \operatorname{Re} S_{h_1} - \operatorname{Im} S_{h_2} \operatorname{Im} S_{h_1},$$
$$\operatorname{Im} S_{h_1+h_2} = \operatorname{Re} S_{h_1} \operatorname{Im} S_{h_2} + \operatorname{Im} S_{h_2} \operatorname{Re} S_{h_2}$$
$$= \operatorname{Re} S_{h_2} \operatorname{Im} S_{h_1} + \operatorname{Im} S_{h_2} \operatorname{Re} S_{h_1},$$

for all h_1, $h_2 \in A(\mathbb{C})$.

Proof. (i) We assume that $n \in \mathbb{N}$, $h=a+ib \in A(\mathbb{C})$ are arbitrary fixed. Theorem 1 and Proposition 5 imply

$$S_{nh} = S_{na+inb} = S_{na}c_{nb} + iS_{na}s_{nb} = S_a^n\left(c_{nb} + is_{nb}\right), \text{ on } P(R),$$

where S_a, c_b, s_b belong to families $S_{A_1(\mathbb{R})}$, $C_{A_2(\mathbb{R})}$, $S_{A_2(\mathbb{R})}$, respectively.

(ii) Let $h_1=a_1+ib_1$, $h_2=a_2+ib_2$, $h_1,h_2 \in A(\mathbb{C})$. Then the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Theorems 1,3 and Proposition 3 imply

$$\operatorname{Re} S_{h_1+h_2} = S_{a_1+a_2} c_{b_1+b_2} = S_{a_1} S_{a_2} \left(c_{b_1} c_{b_2} - s_{b_1} s_{b_2} \right)$$

$$= S_{a_1} S_{a_2} c_{b_1} c_{b_2} - S_{a_1} S_{a_2} s_{b_1} s_{b_2}$$

$$= S_{a_1} c_{b_1} S_{a_2} c_{b_2} - S_{a_1} s_{b_1} S_{a_2} s_{b_2}$$

$$= \operatorname{Re} S_{h_1} \operatorname{Re} S_{h_2} - \operatorname{Im} S_{h_1} \operatorname{Im} S_{h_2}, \text{ on } P(R),$$

where $S_{a_1}, S_{a_2} \in S_{A_1(\mathbb{R})}$; $c_{b_1}, c_{b_2} \in c_{A_2(\mathbb{R})}$; $s_{b_1}, s_{b_2} \in s_{A_2(\mathbb{R})}$.

Similarly, we can prove other equalities.

Theorem 4. We assume that all assumptions of the Theorem 3 are satisfied. Then the operators $\operatorname{Re} S_h$, $\operatorname{Im} S_h$, $h \in A(\mathbb{C})$ commute with D on P(R).

Proof. Let $h = a+ib \in A(\mathbb{C})$, from Definition 3 and Theorems 1,2 on P(R) we have

$$\operatorname{Re} S_h D - D \operatorname{Re} S_h = S_a c_b D - D S_a c_b = S_a D c_b - S_a D c_b = 0,$$

$$\operatorname{Im} S_h D - D \operatorname{Im} S_h = S_a s_b D - D S_a s_b = S_a D s_b - S_a D s_b = 0.$$

It is easy to prove the following fact

Proposition 8. Suppose that we are given families $S_{A(\mathbb{C})}$, $c_{A_1(\mathbb{R})}$, $s_{A_1(\mathbb{R})}$, $ch_{A_2(\mathbb{R})}$, $sh_{A_2(\mathbb{R})}$ of R-shifts for $D \in R(X)$. We assume that for all $h \in A(\mathbb{C})$: $\operatorname{Re} h \in A_1(\mathbb{R})$, $\operatorname{Im} h \in A_2(\mathbb{R})$. Then on the set P(R):

$$c_h = ch_b c_a - ish_b s_a,$$

$$s_h = ch_b s_a + ish_b c_a,$$

where $h=a+ib \in A(\mathbb{C})$, $c_a \in c_{A_1(\mathbb{R})}$, $s_a \in s_{A_1(\mathbb{R})}$, $ch_b \in ch_{A_2(\mathbb{R})}$, $sh_b \in sh_{A_2(\mathbb{R})}$.

Definition 4. Suppose that all assumptions of the Definition 3 are satisfied. Then the operator

$$\bar{S}_h = \operatorname{Re} S_h - i \operatorname{Im} S_h, \quad h \in A(\mathbb{C}),$$

is called a conjugate shift operator for the operator S_h.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Observe that Theorem 4 implies

Proposition 9. Suppose that all assumptions of Definition 3 are satisfied. Then for all $h \in A(\mathbb{C})$ the operators $\bar{S}_h$ commute with the operator D on the set P(R).

Proposition 10. Suppose that the all assumptions of Definition 3 are satisfied. Let $h \in A(\mathbb{C})$ together with $\bar{h}$. Then on the set P(R)

(i) $S_{\bar{h}} = \bar{S}_h$,

(ii) $\bar{\bar{S}}_h = S_h$,

(iii) $S_h \bar{S}_h = \bar{S}_h S_h = S_h S_{\bar{h}} = S_{2a} = S_a^2$,

where $a = \operatorname{Re} h$, $S_a \in S_{A_1(\mathbb{R})}$.

Proof. (i) Let $h=a+ib \in A(\mathbb{C})$, from Definition 4 and Propositions 1,5 on P(R) we have

$$S_{\bar{h}} = S_{a-ib} = S_a c_{-b} + iS_a s_{-b} = S_a c_b - iS_a s_b = \operatorname{Re} S_h - i \operatorname{Im} S_h = \bar{S}_h,$$

where $S_a \in S_{A_1(\mathbb{R})}$, $c_b \in c_{A_2(\mathbb{R})}$, $s_b \in s_{A_2(\mathbb{R})}$.

The equality (i) implies (ii). The equality (i) with Theorem 1 imply (iii).

Examples of complex R-shifts are given by the author in [3].

REFERENCES

[1] **D. Przeworska-Rolewicz:** Shift and periodicity for right invertible operators, Research Notes Series 43, Pitman Advanced Publish. Program, Boston-London-Melbourne, 1980.

[2] **D. Przeworska-Rolewicz:** Algebraic Analysis, D. Reidel Publishing Company and PWN-Polish Scientific Publishers, Dordrecht-Warszawa, 1987.

[3] **Z. Binderman:** Some properties of complex differentiation and shifts, (to appear).

DEPARTMENT OF COMPUTER SCIENCE AND ECONOMETRICS, MAIN SCHOOL OF FARMING - ACADEMY OF AGRICULTURE, 02-766 WARSZAWA.

Received June 14, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMO
Vol.

I
or C)
domai
dom A
ble o
inver
prese
the w
T
sequer
(1)

Th

X = (

(2)

n' =1,2

ii)

iii)

(3)

1≤s≤m

iv)

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Abderrahim Kalfat

REMARKS ON VOLTERRA RIGHT INVERSES FOR WEIGHTED DIFFERENCE OPERATOR OF HIGH DEGREE

Let X be a linear space over a field $\mathfrak{P}$ of scalars ($\mathfrak{P} = \mathbb{R}$ or $\mathbb{C}$). By $L(X)$ we denote the set of linear operators with domains and ranges contained in X and $L_0(X) = \{A : A\in L(X): \operatorname{dom} A = X\}$. By $R(X)$ we denote the set of all right invertible operators belonging to $L(X)$. The set of all right inverses for a given $D\in R(X)$ will be denoted by $\mathfrak{R}_D$. In the present paper we shall extend the result obtained in [1] to the weighted difference operator of high degree.

To begin with denote by $X = (S)_{\mathfrak{P}}$ the space of all sequences over $\mathfrak{P}$. Consider the operator given by

$$D_{m,p}x = \{x_{n+m} - px_n\} \quad \text{where} \quad x = \{x_n\},\ p\in\mathfrak{P};\ m,n\in\mathbb{N}. \tag{1}$$

Theorem 1. i) The operator $D_{m,p}$ is right invertible in $X = (S)_{\mathfrak{P}}$ and its right inverse $R^{(1)}_{m,p}$ is given by

$$R^{(1)}_{m,p}x = y, \quad y = \{y_n\} \quad \text{where} \quad y_n = y_{mn'+s} = \sum_{k=0}^{n'-1} p^{n'-1-k}x_{mk+s} \tag{2}$$

$n' = 1,2,\ldots$ $1\le s\le m$ and $y_s = 0$ for $n' = 0$;

ii) $R^{(1)}_{m,p}$ is a Volterra operator;

iii) $\operatorname{Ker} D_{m,p}$ is m-dimensional and is given by

$$\operatorname{Ker} D_{m,p} = \left\{ z = \{z_n\},\ n = mn'+s,\ z_{mn'+s} = p^{n'}x_s,\ n' = 0,1,\ldots \right. \tag{3}$$
$$\left. 1\le s\le m \right\}.$$

iv) The exponential elements are given by

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(4) $e_\lambda(z)=y$, $n=mn'+s$, $y_{mn'+s}=(\lambda+p)^{n'}x_s$, $n'=0,1,\ldots$ $1\le s\le m$.

Proof. i) We have dom $D_{m,p}=X$ and then $R^{(1)}_{m,p}X\subset$dom $D_{m,p}=X$. Let us prove that $D_{m,p}R^{(1)}_{m,p}=I$. If we put $R^{(1)}_{m,p}x=y$ then $D_{m,p}R^{(1)}_{m,p}x=D_{m,p}y=w$ where for $n=mn'+s$, $(n'=1,2,\ldots)$, $1\le s\le m$

$$w_{mn'+s}=y_{m(n'+1)+s}-py_{mn'+s}=$$

$$=\sum_{k=0}^{n'}p^{n'-k}x_{mk+s}-p\sum_{k=0}^{n'-1}p^{n'-1-k}x_{mk+s}=$$

$$=x_{mn'+s}+\sum_{k=0}^{n'-1}p^{n'-k}x_{mk+s}-p\sum_{k=0}^{n'-1}p^{n'-k}x_{mk+s}\,,\quad w_{mn'+s}=x_{mn'+s}=x_n$$

and for $n'=0$

$$w_s=y_{m+s}-py_s=x_s$$

then $w_n=x_n$ i.e. $D_{m,p}R^{(1)}_{m,p}x=x$,

and by the arbitrariness of x we conclude that $D_{m,p}R^{(1)}_{m,p}=I$.

ii) To prove that $R^{(1)}_{m,p}$ is a Volterra operator we prove that $(I-\lambda R^{(1)}_{m,p})$ is invertible for every $\lambda\in\mathbb{C}$.

By induction we show that for $n=mn'+s$

$$x_{mn'+s}=y_{mn'+s}+\lambda\sum_{k=0}^{n'-1}(\lambda+p)^{n'-1-k}y_{mk+s}\,, \tag{5}$$

$(n'=1,2,3,\ldots)$; $1\le s\le m$

are solutions of the equations

$$x_{mn'+s}-\lambda\sum_{k=0}^{n'-1}p^{n'-1-k}x_{mk+s}=y_{mn'+s} \tag{6}$$

for $n'=1$ we have $x_{m+s}-\lambda x_s=y_{m+s}$ but $x_s=y_s$.
Hence $x_{m+s}=y_{m+s}+\lambda y_s$.

Suppose that formulae (5) are true for n'. Then for $n'+1$ we have

$$x_{m(n'+1)+s}=y_{m(n'+1)+s}+\lambda\sum_{k=0}^{n'}p^{n'-k}x_{mk+s}=$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$= y_{m(n'+1)+s} + \lambda\sum_{k=0}^{n'-1} p^{n'-k} x_{mk+s} + \lambda x_{mn'+s} =$$

$$= y_{m(n'+1)+s} + \lambda p\sum_{k=0}^{n'-1} p^{n'-1-k} x_{mk+s} + \lambda x_{mn'+s} =$$

$$= y_{m(n'+1)+s} + p(x_{mn'+s} - y_{mn'+s}) + \lambda x_{mn'+s} =$$

$$= y_{m(n'+1)+s} + (\lambda+p)x_{mn'+s} - py_{mn'+s} =$$

$$= y_{m(n'+1)+s} + (\lambda+p)\left[y_{mn'+s} + \lambda\sum_{k=0}^{n'-1}(\lambda+p)^{n'-1+k} y_{mk+s}\right] - py_{mn'+s} =$$

$$= y_{m(n'+1)+s} + \lambda y_{mn'+s} + \lambda\sum_{k=0}^{n'-1}(\lambda+p)^{n'-k} y_{mk+s} =$$

$$= y_{m(n'+1)+s} + \lambda\sum_{k=0}^{n'}(\lambda+p)^{n'-k} y_{mk+s} .$$

Thus we see from (5) that for every $y\in X=(s)_p$, x exists. To complete the proof of the invertibility of $(I - \lambda R^{(1)}_{m,p})$ we have to show also by induction that the solutions

$$x_{mn'+s} = \lambda p\sum_{k=0}^{n'-1} p^{n'-1-k} x_{mk+s} \quad \text{are} \quad x_{mn'+s} = 0,$$

where $(n'=1,2,\ldots)$, $1\le s\le m$,

i.e. that $\mathrm{Ker}(I - \lambda R^{(1)}_{m,p}) = 0$.

First for $n'=0$, we obtain by (2) that $x_s=0$. Now for $n'=1$, $x_{m+s}=\lambda x_s$ which implies that $x_{m+s}=0$. Suppose now that $x_{mn'+s}=0$ for n' and let us prove that for $n'+1$, $x_{m(n'+1)+s}$ is also zero. Indeed,

$$x_{mn'+s} = \lambda\sum_{k=0}^{n'} p^{n'-k} x_{mk+s} = p\lambda\sum_{k=0}^{n'-1} p^{n'-1-k} x_{mk+s} + \lambda x_{mn'+s} =$$

$$= px_{mn'+s} + \lambda x_{mn'+s} = (\lambda+p)x_{mn'+s}.$$

We conclude therefore that $x_{m(n'+1)+s} = 0$ and then $(I - \lambda R^{(1)}_{m,p})$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

is invertible for every $\lambda \in \mathfrak{p}$.

iii) $\operatorname{Ker} D_{m,p} = \{z=\{z_n\} : z_{n+m} - pz_n = 0,\ p\in\mathfrak{p},\ n,m\in\mathbb{N}\}$ then for

(7) $n = mn'+s$, $z_{mn'+s} = p^{n'} z_s$ $(n'=0,1,\ldots)$, $1\le s\le m$.

So $z = \{z_n\} = (z_1,\ldots,z_n; pz_1,\ldots,pz_m; \ldots; p^{n'}z_1,\ldots,p^{n'}z_m;\ldots)$ $z = z_1(1,0,\ldots,0; p,0,\ldots,0; p,0,\ldots) + \ldots + z_m(0,\ldots,0,1; 0,\ldots,0,p; \ldots)$,

$$z = \sum_{k=1}^{m} z_k \{e_n^{(k)}\}$$ where for $n=mn'+s$, $n'=0,1,\ldots$

$e_{mn'+s} = p^{n'}\delta_{ks}$, δ_{ks} denotes the so-called Kronecker symbol $\delta_{ks} = \begin{cases} 1 & k=s \\ 0 & k\neq s \end{cases}$.

Thus $\dim \operatorname{Ker} D_{m,p} = m$.

iv) We have just proved in (ii) that $R^{(1)}_{m,p})$ is a Volterra operator and

$$(I - \lambda R^{(1)}_{m,p})^{-1} x = y$$ where for $n=mn'+s$ $(n'=1,2,\ldots)$, $1\le s\le m$

$$(8)\qquad y_{mn'+s} = x_{mn'+s} + \lambda \sum_{k=0}^{n'-1} (\lambda+p)^{n'-1-k} x_{mk+s}$$

for $n'=0$, $y_s = x_s$.

In order to obtain the exponential elements we substitute x in (8) by an element of $\operatorname{Ker} D_{m,p}$. Then

$$e_\lambda(z) = (I - \lambda R^{(1)}_{m,p})^{-1}(z) = y$$

for $n=mn'+s$, $n'=1,2,\ldots$ and $1\le s\le m$

$$y_{mn'+s} = z_{mn'+s} + \lambda \sum_{k=0}^{n'-1} (\lambda+p)^{n'-1-k} z_{mk+s} =$$

$$= p^{n'} z_s + \sum_{k=0}^{n'-1} (\lambda+p)^{n'-1-h} p^k z_s ,$$

$y_{mn'+s} = (\lambda+p)^{n'} z_s$ for $n'=0$, $y_s = z_s$.

We recall that if $D\in R(X)$ and $F\in L_0(X)$, then any

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

projection F onto Ker D is an initial operator for D and its corresponding right inverse is given by $R = R_1 - FR_1$ where R_1 is any right inverse of D.

We are interested in those right inverses which are Volterra operators. V(X) will denote the set of all Volterra operators. Suppose that there exists an $R_1 \in \mathfrak{R}_D \cap V(X)$. For every $\lambda \in \mathfrak{p}$ we have

$$(I - \lambda R)x = (I - \lambda R_1 + \lambda FR_1)x = (I - \lambda R_1)x + \lambda FR_1 x.$$

From the fact that $R_1 \in V(X)$ we see that there exist exponential elements $e_\lambda(z) = (I - \lambda R_1)^{-1}(z)$ for every $0 \neq z \in KerD$ and consequently,

$$(I - \lambda R)e_\lambda(z) = (I - \lambda R_1)e_\lambda(z) + \lambda FR_1 e_\lambda(z)$$

but $\lambda R_1 e_\lambda(z) = e_\lambda(z) - z$ and $Fz = z$ for every $z \in KerD$. Then $(I - \lambda R)e_\lambda(z) = Fe_\lambda(z)$ and if $Fe_\lambda(z) \neq 0$ for every $0 \neq z \in KerD$ and every $0 \neq \lambda \in \mathfrak{p}$ then the corresponding right inverse $R = R_1 - FR_1$ is a Volterra operator. In the sequel we shall examine in which conditions we obtain all Volterra operators for $D_{m,p}$. To begin with consider the operator $F_{m,p}$ defined by means of equality

$$(9) \qquad F_{m,p}x = \sum_{k=1}^{m} \left(\sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right) \left\{ e_n^{(k)} \right\}, \quad m,p \text{ are fixed.}$$

Theorem 2: The operator $F_{m,p}$ defined by (9) is an initial operator for $D_{m,p}$ if and only if

$$\sum_{l=0}^{N_k} a_{k,l} p^l = 1 \quad \text{for} \quad 1 \leq k \leq m.$$

Proof: Necessity: Suppose that $F_{m,p}$ is an initial operator for $D_{m,p}$ then $F_{m,p}X = KerD$ and $F_{m,p}^2 = F_{m,p}$, thus

$$F_{m,p}^2 x = F_{m,p}\left(\sum_{k=1}^{m} \left\{ e_n^{(k)} \sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right\} \right) = F_{m,p}y =$$

$$= \sum_{k=1}^{m} \left\{ e_n^{(k)} \sum_{j=0}^{N_k} a_{k,j} x_{mj+k} \right\}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and

$$y_{mj+k} = p^j \sum_{l=0}^{N_k} a_{k,l} x_{ml+k} ,$$

$$F^2_{m,p}x = \sum_{k=1}^{m} \left\{ e_n^{(k)} \sum_{j=0}^{N_k} a_{k,j} p^j \sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right\} =$$

$$= \sum_{k=1}^{m} \left(\sum_{j=0}^{N_k} a_{k,j} p^j \right) \left\{ e_n^{(k)} \sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right\}.$$

Consequently we conclude that $F^2_{m,p}x = F_{m,p}x$ if

$$\sum_{j=0}^{N_k} a_{k,j} p^j = 1, \quad (1 \le k \le m).$$

Sufficiency: Suppose that we have $\sum_{l=0}^{N_k} a_{k,l} p^l = 1$ for $1 \le k \le m$, and let us prove that $F_{m,p}$ is a projection onto $\mathrm{Ker} D_{m,p}$.

Let $x \in X$ be arbitrarily fixed, then by the linearity of $D_{m,p}$ we have

$$D_{m,p} F_{m,p} x = D_{m,p} \left[\sum_{k=1}^{m} \left(\sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right) \left\{ e_n^{(k)} \right\} \right] =$$

$$= \sum_{k=1}^{m} \left(\sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right) D_{m,p} \left\{ e_n^{(k)} \right\}$$

but

$$D_{m,p} \left\{ e_n^{(k)} \right\} = \left\{ y_n^{(k)} \right\}$$

for $n = mn' + s$ $(n' = 0, 1, \ldots)$ $1 \le s \le m$

$$y^{(k)}_{mn'+s} = p^{n'+1} \delta_{ks} - p p^{n'} \delta_{ks} = 0.$$

Consequently $D_{m,p} F_{m,p} = 0$ and $F_{m,p} X \subset \mathrm{Ker} D_{m,p}$.

Now we have to show that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$F_{m,p}^2 = F_{m,p} \quad \text{if} \quad \sum_{l=0}^{N_k} a_{k,l} p^l = 1, \quad 1 \le k \le m.$$

Indeed,

$$F_{m,p}^2 x = F_{m,p} \left(\sum_{k=1}^{m} \left(\sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right) \left\{ e_n^{(k)} \right\} \right) =$$

$$= \sum_{k=1}^{m} \left(\sum_{j=0}^{N_k} a_{k,j} p^j \right) \left\{ e_n^{(k)} \sum_{l=0}^{N_k} a_{k,l} x_{ml+k} \right\},$$

but $\sum_{j=0}^{N_k} a_{k,j} p^j = 1$ for $1 \le k \le m$,

therefore we conclude that $F_{m,p}^2 = F_{m,p}$ and the theorem is proved.

To determine the Volterra right inverses $R_{M,P}$ corresponding to $F_{m,p}$ we apply these operators to exponential elements given by (4):

$$F_{m,p} e_\lambda(z) = \sum_{k=1}^{m} \left(\sum_{l=0}^{N_k} a_{k,l} (\lambda+p)^l z_k \right) \left\{ e_n^{(k)} \right\}.$$

Let us denote by $W_k(\lambda)$ the polynomials in λ for $1 \le k \le m$

$$W_k(\lambda) = \sum_{l=0}^{N_k} a_{k,l} (\lambda+p)^l z_k.$$

Then Volterra operators correspond to the operator $F_{m,p}$ for which $F_{m,p} e_\lambda(z) \neq 0$ for every $0 \neq z \in \mathrm{Ker} D_{m,p}$ namely when at least one coordinate of $F_{m,p} e_\lambda(z)$ is different than zero. Let us write $F_{m,p} e_\lambda(z)$ in the following form

$$F_{m,p} e_\lambda(z) = \sum_{l=0}^{N_1} a_{1,l} (\lambda+p)^l z_1 \{ e_n^{(1)} \} + \sum_{l=0}^{N_2} a_{2,l} (\lambda+p)^l z_2 \{ e_n^{(2)} \} + \ldots$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\ldots+\sum_{l=0}^{N_n} a_{m,l}(\lambda+p)^l z_m\{e_m^{(m)}\} =$$

$$= \Big(\sum_{l=0}^{N_1} a_{1,l}(\lambda+p)^l z_1, \ldots, \sum_{l=0}^{N_m} a_{m,l}(\lambda+p)^l z_m;\ p\sum_{l=0}^{N_1} a_{1,l}(\lambda+p)^l z_1, \ldots$$

$$\ldots, p\sum_{l=0}^{N_m} a_{m,l}(\lambda+p)^l z_m; \ldots \Big).$$

Observe that $\lambda=-p$ can be a root of $W_k(\lambda)$ if $a_{k,0}=0$ for $1\le k\le m$ and then the operators $R_{m,p}$ obtained are not Volterra. So in the sequel we suppose that not all $a_{k,0}$ and not all z_k of $z=\{z_n\}\in \mathrm{Ker}D_{m,p}$ vanish simultaneously.

Theorem 3. Suppose that $X=(s)_{\mathbb{C}}$, $D_{m,p}$, $R^{(1)}_{m,p}$ and $F_{m,p}$ are defined by (1), (2) and (9) respectively.

If $\exists_{1\le i,j\le m}\ i,j$ such that $S_i\cap S_j=\emptyset$, where $S_{i,j}$ denote the set of roots of $W_k(\lambda)$ then $R_{m,p}$ are Volterra operators.

Proof. $W_k(\lambda)$ as a polynomial over $\mathbb{C}$ have respectively N_k+1 roots $(1\le k\le m)$. By the fact that there exist two polynomials with non-common roots there exists at least one coordinate of $F_{m,p}e_\lambda(z)$ different than zero. Hence we conclude that $R_{m,p} = R^{(1)}_{m,p} - F_{m,p}\ R^{(1)}_{m,p}$ are Volterra operators.

Corollary 1. Suppose that $X = (s)_{\mathbb{C}}$ and $D_{m,p}$, $R^{(1)}_{m,p}$, $F_{m,p}$ are defined by (1),(2), and (9) respectively.

If the polynomials $W_k(\lambda)$ are equal for $1\le k\le m$, then $R_{m,p}$ are not Volterra operators.

Theorem 4. Suppose that $X = (s)_{\mathbb{R}}$, $D_{m,p}$, $R^{(1)}_{m,p}$, $F_{m,p}$ are defined by (1),(2),(9) respectively.

If the polynomials $W_k(\lambda)$ are equal for $1\le k\le m$, i.e.

$$W_k(\lambda) = \sum_{l=0}^{N} a_l(\lambda+p)^l$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

and N is odd, then $R_{m,p}$ are not Volterra operators.

Procf. If N is odd then by the continuity of the polynomial $W_k(\lambda)$ and by the theorem of intermediate value there exists λ_0 such that $W_k(\lambda_0) = 0$ and then $F_{m,p}e_{\lambda_0}(z) = 0$. We conclude that $R_{m,p} \notin V(X)$.

Suppose that $X=(S)_{\mathbb{R}}$. If one of the polynomials $W_k(\lambda)$ has no real roots we conclude that $R_{m,p}$ is a Volterra operator. So we shall seek conditions under which $W_1(\lambda) = \sum_{l=0}^{N_1} a_{1,l}(\lambda+p)^l$ has no real roots when N_1 is even and $a_{1,0} \neq 0$. $W_1(\lambda)$ is a polynomial of degree $2q$ and we require that it admits no real roots.

As a polynomial in $\mathbb{C}$, $W_1(\lambda)$ admits $2q$ roots $\alpha_1,\dots,\alpha_q$ and $\bar{\alpha}_1,\dots,\bar{\alpha}_q$ such that $\operatorname{Im}\alpha_j \neq 0$ for $1\le j\le q$. Then

$$W_1(\lambda) = a_{1,N_1}\prod_{j=1}^{q}(\lambda-\sigma_j)(\lambda-\bar{\alpha}_j) = a_{1,N}\prod_{j=1}^{q}(\lambda^2-2\sigma_j\lambda-\tau_j), \tag{10}$$

$\sigma_j = \operatorname{Re}\alpha_j$, $\tau_j = |\alpha_j|^2$, σ_j and τ_j depend on $(a_{1,l})$.

Thus we see that $W_1(\lambda)$ has no real roots iff

$$\Delta_j = \sigma_j^2 - \tau_j < 0 \quad \text{for} \quad 1\le j\le q \tag{11}$$

and we conclude that if $\Delta_j<0$ for $1\le j\le q$ then $R_{m,p}$ are Volterra operators.

So we can announce the following theorem.

Theorem 5. Suppose that $X = (S)_{\mathbb{R}}$, $D_{m,p}$, $R^{(1)}_{m,p}$ and $F_{m,p}$ are defined by (1),(2) and (9) respectively.

If one of the polynomials $W_k(\lambda)$ is of degree N and N is even, i.e. $N=2q$, then this polynomial can be written in the form of (10), and if the conditions (11) are satisfied then $R_{m,p}$ are Volterra operators.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

REFERENCES

[1] **A. Kalfat:** Volterra right inverses for weighted difference operators, Demonstratio Math. 19 (1986), 1089-1093.

[2] **L. Lusternik, V. Sobolev:** Précis d'analyse fonctionnelle. Moscou (1989).

[3] **Nguyen Van Mau:** Characterization of Volterra right inverses (to appear).

[4] **D. Przeworska-Rolewicz:** Algebraic Analysis. PWN - Polish Scientific Publishers and D. Reidel, Warszawa - Dordrecht, 1987.

Added in proof by the Editorial Board:

[5] **D. Przeworska-Rolewicz:** Advantages of one-dimensional kernels, Math. Nachrichten 148 (1990), 143-158.

UNIVERSITE DE TLEMCEN, INSTITUT DES SCIENCES EXACTES, DEPARTMENT DE MATHEMATIQUES, B.P.119, TLEMCEN, ALGERIE

Received June 30, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Adam Grabarski

PROLONGATION OF THE SOLUTION OF A DIFFERENCE SCHEME FOR THE EQUATION $-u''+g(x)u = f(x)$

0. Introduction

In the paper [1] for the differential problem

$$(1)\qquad \begin{cases} -u''+g(x)u = f(x), & x\in(0,1), \\ u(0) = u(1) = 0 \end{cases}$$

a difference scheme has been considered in the form

$$(2)\qquad \begin{cases} -\Delta_i^2 y = f_i - g_i + \frac{h^2}{12}\,\Delta_i^2(f-gy), & i=1,\dots,n-1, \\ y_0 = y_n = 0, \end{cases}$$

where g_i and f_i are the values of the functions g, f at points $x_i=ih$, $h=1/n$, and y_i is the approximation of $u_i=u(x_i)$. The operator Δ_i^2 is defined by

$$\Delta_i^2 y^h = \frac{y_{i-1}-2y_i+y_{i+1}}{h^2}, \qquad y^h = (y_0,\dots,y_n).$$

If $u\in C^6([0,1])$ and $f,g\in C^4([0,1])$, $g(x)\geq 0$, then for h being sufficiently small the scheme (2) has exactly one solution, and

$$(3)\qquad \max_{0\leq i\leq n}|u_i-y_i| \leq c_1\cdot h^4,$$ c_1 - the constant independent of h.

One of the deficiencies of difference schemes is the fact that the approximate solution is obtained only at points of the grid, whereon a particular scheme is determined. Among many others, it makes difficult the comparison of the difference methods to such ones as the Galerkin method or collocation method, wherein an approximate solution is defined on the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

whole interval [0,1].

In the present paper, basing on the solution of the difference scheme (2), we will state a structure of the function determined in the whole interval [0,1], the function wherein the maximum error in the whole interval is of the same order as the error of the differential scheme, i.e. of the order 4. The error estimate within the Sobolev space H^1 will be also stated. Our discussion will be preceded by some general considerations referring to prolongations of solutions for difference schemes and to their errors.

1. General problem

Let U, V and F ($U \subset V \subset F$) be the spaces of the functions being determined within the closed domain Ω, and let V be a normed space. Let Ω_h be a grid on Ω (see e.g. [2] or [3]), whereas U_h and F_h ($U_h \subset F_h$) are to be the linear spaces of the grid functions. For the operators $L : U \longrightarrow F$, $L_h : U_h \longrightarrow F_h$ let us consider the equations

(4) $\quad Lu = f, \quad f \in F,$

(5) $\quad L_h y^h = f^h, \quad f^h \in F_h.$

Assume that equations (4) and (5) have unequivocal solutions in U and U_h respectively. Let X_h be a normed space, and

$$r_h : U \longrightarrow X_h, \quad R_h : F_h \longrightarrow X_h, \quad p_h : X_h \longrightarrow V$$

- the given operators (p_h - operator of prolongation). Let us assume that p_h is a linear operator, and also that

(Z) $\quad \|p_h x^h\|_V \leq K \cdot \|x^h\|_{X_h}$ for any $x^h \in X_h$,

where K is a constant independent of h.

The values $e_h^m(u) = \|r_h u - R_h y^h\|_{X_h}$, $e_h^a(v) = \|v - p_h r_h v\|_V$ and $e_h^p(u) = \|u - p_h R_h y^h\|_V$ are called the method, approximation and prolongation error, respectively. The estimate of the prolongation error can be reduced to the estimate of approximation errors and of method errors. And really

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(6)\quad e_h^p(u) = \|u - p_h R_h y^h\|_V \le \|u - p_h r_h u\| + \|p_h(r_h u - R_h y^h)\| \le$$
$$\le e_h^a(u) + K\|r_h u - R_h y^h\|_{X_h} = e_h^a(u) + K\cdot e_h^m(u).$$

Let us indicate hereby that for maintaining the prolongation error on the same level as the method error, the order of the approximation error is not to be smaller than the order of the method error. Naturally, it imposes some additional requirements as far as the p_h prolongation is concerned.

In the next chapter, an example of prolongation will be stated, being suitable for our task and meeting the (Z) requirement. The examples of other prolongations which meet the (Z) requirement are presented, for instance, in the paper [4].

2. Construction of the operators r_h, R_h and p_h

Let $I=[0,1]$ and $I_h = \{x_i=ih,\ i=0,\dots,n\}$, $h=\frac{1}{n}$. Denote by C_h the space of functions defined on the grid I_h with the norm

$$\|v^h\|_{C_h} = \max_{0\le i\le n} |v_i|,\quad v^h=(v_0,\dots,v_n).$$

Let $F_h=C_h$ and $U_h=\{u^h\in C_h : u_0=u_n=0\}$. Denote by X_h the space of functions $x^h = (v^h,\bar{v}^h) = (v_0,\dots,v_n,\bar{v}_0,\dots,\bar{v}_n)$ with the norm

$$\|x^h\|_{X_h} = \|v^h\|_{C_h} + h\|\bar{v}^h\|_{C_n}.$$

For the function $v\in C^1([0,1])$ we define

$$r_h v = \left(v(x_0),\dots,v(x_n),\ \frac{dv}{dx}(x_0),\dots,\frac{dv}{dx}(x_n)\right),$$

and for the function $v^h = (v_0,\dots,v_n)$ we define

$$R_h v^h = (v_0,\dots,v_n,\delta_0 v^h,\dots,\delta_n v^h),\ \text{respectively,}$$

where $\delta_i v^h$, $i=0,\dots,n$, are certain operators defined on C_h.

Let

$$W(x) = \begin{cases} (x+1)^2(1-2x) & \text{for } x\in[-1,0], \\ (x-1)^2(1+2x) & \text{for } x\in[0,1], \\ 0 & \text{for } x\in R-[-1,1], \end{cases}$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$S(x) = \begin{cases} x(x+1)^2 & \text{for } x\in[-1,0], \\ x(x-1)^2 & \text{for } x\in[0,1], \\ 0 & \text{for } x\in R-[-1,1], \end{cases}$$

and $W_i(x) = W\left(\frac{x-x_i}{h}\right)$, $S_i(x) = hS\left(\frac{x-x_i}{h}\right)$, $i=0,\dots,n$.

For $x^h\in X_h$ we define the operator of prolongation as follows

$$(7)\qquad p_h x^h = \sum_{i=0}^{n} v_i W_i + \sum_{i=0}^{n} \bar{v}_i S_i.$$

From the definition it appears that p_h is linear and

$$(a)\qquad p_h x^h \in \{v\in C^1(I) : v|_{I_i}\in P_3(I_3),\ i=1,\dots,n\},$$

where $I_i=[x_{i-1},x_i]$, while P_3 is the set of polynomials of degree not greater than 3,

$$(b)\qquad (p_h x^h)(x_i) = v_i,\quad i=0,\dots,n,$$

$$(c)\qquad \frac{d}{dx}(p_h x^h)(x_i) = \bar{v}_i,\quad i=0,\dots,n.$$

3. Condition (Z)

Now we will give two examples spaces V, where the condition (Z) is satisfied for the operator p_h defined by (7).

(a) Let $V=C(I)$ with the norm $\|v\|_C = \max_{x\in I}|v(x)|$. Since $W_i(x)\geq 0$ and

$$\sum_{i=0}^{n} W_i(x) \equiv 1 \quad\text{and}\quad \sum_{i=0}^{n} |S_i(x)| \leq \frac{h}{4}$$

then

$$|p_h x^h(x)| \leq \|v^h\|_{C_h} \sum_{i=0}^{n} W_i(x) + \|\bar{v}^h\|_{C_h} \sum_{i=0}^{n} |S_i(x)| \leq$$

$$\leq \|v^h\|_{C_h} + \frac{h}{4}\|\bar{v}^h\|_{C_h} \leq \|x^h\|_{X_h}.$$

Hence

$$(8)\qquad \|p_h x^h\|_C \leq \|x^h\|_{X_h}.$$

(b) Let $V=H^1(I)$ be the Sobolev space with the norm

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$\|v\|_{1,h} = \|v\|_0 + h|v|_1$, where

$$\|v\|_0 = \sqrt{\int_0^1 v^2(x)\,dx}\,, \quad |v|_1 = \sqrt{\int_0^1 \left(\frac{dv(x)}{dx}\right)^2 dx}\,.$$

From the definition of function W_i and S_i we obtain

$$\int_0^1 \left(p_h x^h\right)^2 (x)\,dx = \sum_{i=1}^{n} \sum_{j=\max(0,i-1)}^{\min(n,i+1)} \left[v_i v_j \int_0^1 W_i(x) W_j(x)\,dx + \right.$$

$$\left. + 2 v_i \bar{v}_j \int_0^1 W_i(x) S_j(x)\,dx + \bar{v}_i \bar{v}_j \int_0^1 S_i(x) S_j(x)\,dx \right].$$

Since

$$\int_0^1 W_i(x) W_j(x)\,dx = a_{ij} h,$$

$$\int_0^1 W_i(x) S(x)\,dx = b_{ij} h^2,$$

$$\int_0^1 S_i(x) S_j(x)\,dz = c_{ij} h^3,$$

where a_{ij}, b_{ij} and $c_{i,j}$ are constants independent of h, then using the inequalities $A \cdot B \le \frac{1}{2}(A^2+B^2)$ and $h \sum_{i=0}^{n} 1 \le 2$, we obtain

$$\int_0^1 \left(p_h x^h\right)^2 (x)\,dx \le ah \sum_{i=0}^{n} v_i^2 + bh^3 \sum_{i=0}^{n} \bar{v}_i^2 \le$$

$$\le 2\left(a\|v^h\|_{C_h} + bh^2 \|\bar{v}^h\|_{C_h}^2\right).$$

Hence

$$\int_0^1 \left(p_h x^h\right)^2 (x)\,dx \le K_1^2 \|x^h\|_{X_h}^2,$$

where K_1 is a constant independent of h. Similarly we have

$$\int_0^1 \left(\frac{d}{dx}\, p_h x^h\right)^2 (x)\,dx \le K_2^2 h^{-2} \|x^h\|_{X_h}^2.$$

Finally

(9) $$\|p_h x^h\|_{1,h} \le K \|x^h\|_{X_h},$$

where $K = K_1 + K_2$ is a constant independent of h.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Approximation error

Using the results established in [5] we have

(10) $\|v-p_h r_h v\|_C \le ch^4$ for $v \in H^5(I)$.

Moreover (see e.g. [6]), for $v \in H^4(I)$ we have

$$\|v-p_h r_h v\|_0 \le ch^4, \quad |v-p_h r_h v|_1 \le ch^3.$$

Therefore

(11) $\|v-p_h r_h v\|_{1,h} \le ch^4$ for $v \in H^4(I)$.

5. Method error

We assume that the operators δ_i used in the definition of R_h, satisfy the following conditions

(W) $$\begin{cases} \delta_i v^h - \delta_i w^h = \delta_i (v^h - w^h) \\ |\delta_i v^h| \le c_2 h^{-1} \|v^h\|_{C_h}, \ c_2\text{-independent of } h. \end{cases}$$

Then, for u-being the solution of (1), y^h-being the solution of (2) and

$$u' = \left(\frac{du}{dx}(x_0), \ldots, \frac{du}{dx}(x_n)\right), \quad u^h = \left(u(x_0), \ldots, u(x_n)\right),$$

$$\bar{u} = \left(\delta_0 u^h, \ldots, \delta_n u^h\right), \quad \bar{y}^h = \left(\delta_0 y^h, \ldots, \delta_n y^h\right),$$

we have

$$\|u' - \bar{y}^h\|_{C_h} \le \|u' - \bar{u}\|_{C_h} + \|\bar{u} - \bar{y}^h\|_{C_h} = \|u' - \bar{u}\|_{C_h} +$$

$$+ \max_{0 \le i \le n} |\delta_i (u^h - y^h)| \le \|u' - \bar{u}\|_{C_h} + c_2 h^{-1} \|u^h - y^h\|_{C_h}.$$

Hence the inequality (3) implies that

$$e_h^m(u) = \|r_h u - R_h y^h\|_{X_h} = \|u^h - y^h\|_{C_h} + \|u' - \bar{y}^h\|_{C_h} \le$$

$$\le (1+c_2)\|u^h - y^h\|_{C_h} + h\|u' - \bar{u}\|_{C_h} \le c_1(1+c_2)h^4 + h\|u' - \bar{u}\|_{C_h}.$$

Let

(12) $\delta_0 = \delta_0^{(0)}, \ \delta_i = \delta_i^{(1)}, \ i = 1, \ldots, n-2; \ \delta_{n-1} = \delta_{n-1}^{(2)}; \ \delta_n = \delta_n^{(3)}$

or

(13) $\delta_0 = \delta_0^{(0)}, \ \delta_1 = \delta_1^{(1)}, \delta_i = \delta_i^{(2)}, \ i = 2, \ldots, n-1; \ \delta_n = \delta_n^{(3)},$

where

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\delta_i^{(0)} v^h = \frac{-11v_i+18v_i-9v_{i+2}+2v_{i+3}}{64},$$

$$\delta_i^{(1)} v^h = \frac{-2v_{i-1}-3v_i+6v_{i+1}-v_{i+2}}{64},$$

$$\delta_i^{(2)} v^h = \frac{v_{i-2}-6v_{i-1}+3v_i+2v_{i+1}}{64},$$

$$\delta_i^{(3)} v^h = \frac{-2v_{i-3}+9v_{i-2}-18v_{i-1}+11v_i}{64}.$$

It is easy to show now, that

$$\max_{0\le i\le n} \left| \frac{dv}{dx}(x_i) - \delta_i^{(k)} v^h \right| \le c_3 h^3$$

for $v\in C^4$, and $v^h = (v(x_0),\dots,v(x_n))$. Moreover the conditions (W) are satisfied. Therefore taking δ_i satisfying (12) or (13) we have

(14) $\quad e_h^m(u) \le c\cdot h^4.$

6. Conclusions

We assume that $u\in C^6([0,1])$, $f,g\in C^4([0,1])$ and $g(x)\ge 0$. Let

$$y(x) = \sum_{i=1}^{n-1} y_i W_i(x) + \sum_{i=0}^{n} \bar{y}_i S_i(x),$$

where $\bar{y}_i = \delta_i y^h$, $y^h = (0,y_1,\dots,y_{n-1},0)$ is the solution of (2), δ_i are given by (12) or (13). Then from (6), (8), (10) and (14) yield $\|u-y\|_C \le ch^4$ and from (6), (9), (11) and (14) yield $\|u-y\|_0 + h\left\|\frac{d}{dx}(u-y)\right\|_0 \le ch^4$.

REFERENCES

[1] J. Kołodziejczyk, A. Grabarski: Algorytmy wybranych metod rozwiązywania równania $-u''+g(x)u=f(x)$, Matematyka Stosowana, 32 (1989) 79-96.

[2] С.Г. Михлин, Х.Л. Смолицкий: Приближенные методы решения дифференциальных и интегральных уравнений, Москва, Наука 1965.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

[3] А.А. Самарский, Е.С. Николаев: Методы решения сеточных уравнений, Москва, Наука 1978.

[4] J.K. Kowalski: Application of box splines to the approximation of Sobolew spaces, J. Approx. Theory (to appear).

[5] J. Douglas Jr., T. Dupont: Collocation method for parabolic equations in a single space variable, Lecture Notes Math. 385 (1974).

[6] P. Ciarlet: The finite element method for elliptic problems. North-Holland Publ. Co. Amsterdam 1978.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA
Received July 6, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Marian Majchrowski, Jan Rogulski

AN INVERSE PROBLEM FOR A PARABOLIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

0. Introduction

The parabolic equation $u_t = \alpha\Delta u+F$ governs (at least) two physical problems, namely, evolutions of heat's distribution in a body and diffusion processes. The coefficient α (which can also be a function depending on space and time variables) describes some sort of "resistance" of material in that problems, while F describes an "income" originated from external sources. Several existence and uniqueness theorems for many initial boundary value problems formulated for this equation are well known for years. We refer e.g. to Friedman's book [6] for a background information about this problems and for basic results in this direction.

Inverse problems formulated for this equation are in a sense some kind of problems discussed in Control Theory. For instance, the function F becomes an unknown function (as u itself) and we want to find u and F such that some additional condition for u at a fixed space - point x_0 is satisfied. This sort of problems is discussed in Cannon's papers [2], [3], [4] and the others ([5], [9], [12]).

The crucial role in a representation of a solution u of Fourier problems is played by the function θ_3 (see e.g. [7] and [8]). We refer to [14] for a detailed study of θ_3's properties and for the behaviour of θ_3.

In this paper, we consider an inverse problem for a parabolic system of equations in one-dimensional space. This corresponds (physically) to a situation, where we deal with

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

diffusions of several gases and evolution of their concentrations. We first discuss briefly the existence of solution of the first Fourier problem for systems. Solutions are expressed by means of a matrix function $\mathbb{M}$ which was introduced by Majchrowski (see [10] and [11]) and plays the same role as θ_3 for a single equation. The representation of solution with use of $\mathbb{M}$ allows us to solve an inverse problem by an application of the theory of systems of Volterra integral equations.

1. Formulation of the problem

Let $\mathbb{A}$ be a real $n\times n$ matrix, $\mathbb{H}$ a continuous $n\times n$ matrix-valued function

$$\mathbb{H} : [0,1]\times[0,T]\ni(x,t) \longmapsto \mathbb{H}(x,t)\in\mathbb{R}^{n^2}.$$

and g an $\mathbb{R}^n$ - valued continuous function

$$g : [0,1]\times[0,T]\ni(x,t) \longmapsto g(x,t)\in\mathbb{R}^n.$$

The problem (P) consists in finding of a pair (u,f) of $\mathbb{R}^n$-valued functions

$$u : [0,1]\times[0,T]\ni(x,t) \longmapsto u(x,t)\in\mathbb{R}^n,$$

$$f : [0,T]\ni t \longmapsto f(t)\in\mathbb{R}^n,$$

satisfying the equation

$$(1.1)\qquad \frac{\partial u}{\partial t} = \mathbb{A}\frac{\partial^2 u}{\partial x^2} + \mathbb{H}(x,t)f(x,t) + g(x,t),$$

for $(x,t)\in]0,1[\times]0,T[$, the initial and boundary conditions:

$$(1.2.1)\qquad u(x,0) = \nu(x) \quad \text{for} \quad x\in[0,1]$$

$$(1.2.2)\qquad u(0,t) = \mu_0(t) \quad \text{for} \quad t\in[0,T]$$

$$(1.2.3)\qquad u(1,t) = \mu_1(t) \quad \text{for} \quad t\in[0,T]$$

and an additional condition of the form

$$(1.3)\qquad u(x_0,t) = h(t) \quad \text{for} \quad t\in[0,T],$$

where $x_0\in]0,1[$ is a given point. We require that the known $\mathbb{R}^n$ - valued functions ν, μ_0, μ_1, h in (1.2-3) satisfy the compatibility conditions, i.e.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(1.4) $\nu(0) = \mu_0(0), \quad \nu(1) = \mu_1(0), \quad h(0) = \nu(x_0).$

The list of assumptions for functions H, g, ν, μ_0, μ_1, h and for the matrix A is as follows:

(1.5) $H(0,t) = H(1,t) = 0$ for $t\in[0,T]$,

H has continuous derivatives $\frac{\partial^l H}{\partial x^l}$ for $l=1,2,3,4$ on $]0,1[\times]0,T[$ and

(1.6) $\sup_t \int_0^1 \| \frac{\partial^4 H}{\partial x^4} \| \, dx < \infty,$

(1.7) $\lim_{x\to 0+} \frac{\partial^2 H}{\partial x^2}(x,t) = \lim_{x\to 1-} \frac{\partial^2 H}{\partial x^2}(x,t) = 0,$

(1.8) $H(x_0,t)$ is invertible for every $t\in[0,T]$,

g has continuous derivatives $\frac{\partial g}{\partial x}(x,t)$, $\frac{\partial^2 g}{\partial x^2}(x,t)$ and

(1.9) $\sup_t \int_0^1 \| \frac{\partial^2 g}{\partial x^2}(x,t) \| \, dx < \infty,$

(1,10) $h\in C^1([0,T])$,

(1.11) $\nu, \mu_0, \mu_1 \in C^1([0,T])$,

(1.12) all eigenvalues of A have positive real parts.

2. Parabolic systems of equations

Our aim here is to present some known results concerning parabolic (in Petrovskii's sense, see [10]) systems of equations. In particular, we will sketch briefly a construction of the solution of the first Fourier problem for such system. We refer to [10] for details. We shall use these results further to reduce and simplify the problem (P). Consider now the auxiliary problem (F1).

(2.1) $\frac{\partial v}{\partial t} = A \frac{\partial^2 v}{\partial x^2} + \varphi(x,t)$ for $(x,t)\in]0,1[\times]0,T[$,

(2.2.1) $v(x,0) = \nu(x)$ for $x\in[0,1]$,

(2.2.2) $v(0,t) = f_0(t)$ for $t\in[0,T]$,

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(2.2.3) $v(1,t) = f_1(t)$ for $t\in[0,T]$,

where the known function $\varphi : [0,1]\times[0,T] \longrightarrow \mathbb{R}^n$ is continuous and $\varphi(0,t) = \varphi(1,t) = 0$ for every $t\in[0,T]$. Moreover, assume that φ has continuous derivatives $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial^2\varphi}{\partial x^2}$ on $]0,1[\times]0,T[$ and

$$\sup_t \int_0^1 \| \frac{\partial^2\varphi}{\partial x^2}(x,t) \| \, dx < \infty.$$

Assume the compatibility conditions, i.e., $\nu(0)=f_0(0)$ and $\nu(1)=f_1(0)$. For the sake of simplicity, we assume that ν, f_0 and f_1 are of class C^1. This assumption can be a little weakened, in particular, it is sufficient to assume that ν has all coordinates functions of bounded variation. We are seeking for a continuous function $v : [0,1]\times[0,T] \longrightarrow \mathbb{R}^n$ which has continuous partial derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 v}{\partial x^2}$ on $]0,1[\times]0,T[$.

Let us introduce now a matrix-valued function $\mathbb{M} : \mathbb{R}\times\mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ which plays the same role for the system (2.1) as function θ_3 for a single parabolic equation in Cannon's paper [1]

$$(2.3) \qquad \mathbb{M}(x,t) = I + 2\sum_{k=1}^{\infty} (\cos k\pi x) \exp(-k^2\pi^2 t\mathbb{A}).$$

Theorem 2.4. Under the assumption (1.12) the function $\mathbb{M}$ defined by the formula (2.3) has the following properties:

a) For every $T\ni]0,\infty[$ there exist positive constants $C,a\in\mathbb{R}$ and $m\in\mathbb{N}$ such that

$$\|\exp(-k^2\pi^2 t\mathbb{A})\| \le Ce^{-k^2\pi^2 at} k^{2m} \quad \text{for any } t\in]0,T[\text{ and } k\in\mathbb{N}.$$

b) There exists positive real constant b such that

$$\|\exp(-k^2\pi^2 t\mathbb{A})\| \le b \quad \text{for every } t>0 \text{ and } k\in\mathbb{N}.$$

c) $\mathbb{M} : \mathbb{R}\times\mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ is a C^∞ function and all its partial derivatives can be calculated by term-by-term differentiation of the series (2.3).

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

d) $\lim_{t\to 0^+} \frac{\partial^k \mathbb{M}}{\partial x^k}(x,t) = 0$ for $k=0,1,2,\ldots$, $x\neq 0,\pm 2,\pm 4,\ldots$

Proof. Let $n^2\mathbb{A} = \mathbb{S}+\mathbb{N}$ be a canonical decomposition into a semisimple part $\mathbb{S}$ and a nilpotent part $\mathbb{N}$, where $[\mathbb{S},\mathbb{N}]=0$. Let $\mathbb{S} = \mathbb{B}\mathbb{C}\mathbb{B}^{-1}$ be a canonical Jordan decomposition of a matrix $\mathbb{S}$.

We obtain the standard inequality

$$\|e^{-k^2t\mathbb{S}}\| = \|\mathbb{B}\, e^{-k^2t\mathbb{C}}\, \mathbb{B}^{-1}\| \leq \|\mathbb{B}\|\,\|\mathbb{B}^{-1}\|\, e^{-k^2t\alpha} \tag{2.4.1}$$

where $\alpha = \min\{\mathrm{Re}\,\lambda,\ \lambda \text{ is an eigenvalue of } \mathbb{A}\}$ is a positive constant. From (2.4.1) it follows that for $0\leq t\leq T$

$$\|e^{-k^2t(\mathbb{S}+\mathbb{N})}\| = \left\|e^{-k^2t\mathbb{S}} \sum_{j=0}^{n-1} \frac{(-k^2t)^j}{j!}\mathbb{N}^j\right\| \leq \tag{2.4.2}$$

$$\leq \|\mathbb{B}\|\|\mathbb{B}^{-1}\|\, e^{-k^2t\alpha} \sum_{j=0}^{n-1} \frac{k^{2j}t^j}{j!}\|\mathbb{N}\|^j \leq$$

$$\leq \|\mathbb{B}\|\|\mathbb{B}^{-1}\|\, e^{-k^2t\alpha}\, k^{2(n-1)} \sum_{j=0}^{n-1} \frac{T^j}{j!}\|\mathbb{N}\|^j \quad \text{holds.}$$

Inequality (2.4.2) is equivalent to part a) of Theorem 2.4. To prove the part b), we observe that

$$e^{-k^2t\alpha}\, k^{2j} \leq \frac{(j)^j}{(\alpha t)^j} e^{-j} \quad \text{for all } \alpha,t,j>0 \text{ and } k\in\mathbb{R}.$$

Then from the latter and (2.4.2) we obtain the inequality

$$\|e^{-k^2t(\mathbb{S}+\mathbb{N})}\| \leq \|\mathbb{B}\|\|\mathbb{B}^{-1}\| \sum_{j=0}^{n-1} e^{-k^2t\alpha}\, k^{2j}\, \frac{\|\mathbb{N}\|^j}{j!}\, t^j \leq$$

$$\leq \|\mathbb{B}\|\|\mathbb{B}^{-1}\| \left(\sum_{j=1}^{n-1} \frac{(j\|\mathbb{N}\|)^j}{\alpha^j j!}\, e^{-j+1} \right)$$

equivalent to the part b).

Part c) follows immediately from a) and b). For the proof of d), we refer to [10].■

Let us define now for $(x,t)\in]0,1[\times]0,T[$ the following four functions

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(2.5.1)\qquad v_0(x,t) = -A\int_0^t \frac{\partial M}{\partial x}(x,t-s)f_0(s)\,ds,$$

$$(2.5.2)\qquad v_1(x,t) = A\int_0^t \frac{\partial M}{\partial x}(x-1,t-s)f_1(s)\,ds,$$

$$(2.5.3)\qquad v_2(x,t) = \frac{1}{2}\int_0^1 [M(x-y,t) - M(x+y,t)]\,\nu(y)\,dy,$$

$$(2.5.4)\qquad v_3(x,t) = \frac{1}{2}\int_0^t\int_0^1 [M(x-y,t-s) - M(x+y,t-s)]\varphi(y,s)\,dyds.$$

Theorem 2.6. (cf. [11]). The $\mathbb{R}^n$-valued functions v_0, v_1, v_2, v_3 defined above have unique continuous extensions on $[0,1]\times[0,T]-\{(0,0),(1,0)\}$. These extensions have following values on the boundary:

$$\begin{aligned} &v_0(0,t) = f_0(t), && v_0(1,t) = 0, && v_0(x,0) = 0,\\ &v_1(0,t) = 0, && v_1(1,t) = f_1(t), && v_1(x,0) = 0,\\ &v_2(0,t) = 0, && v_2(1,t) = 0, && v_2(x,0) = \nu(x),\\ &v_3(0,t) = 0, && v_3(1,t) = 0, && v_3(x,0) = 0. \end{aligned}$$

All these functions are of class C^1 on $]0,1[\times]0,T[$ and have continuous second order partial derivatives with respect to x. Moreover v_3 satisfies equation (2.1) and other functions satisfy the homogeneous equation

$$\frac{\partial v}{\partial t} = A\,\frac{\partial^2 v}{\partial x^2}\,.$$

Corollary 2.7. The function $v = v_0+v_1+v_2+v_3$ has a continuous extension on $[0,1]\times[0,T]$ which is a solution of the problem (F1).

3. Existence of a solution of the problem (P)

Let v be a solution of the problem (F1) with $\varphi=0$ and

$$f_j(t) = \mu_j(t) - \int_0^t g(j,s)ds, \quad j=0,1.$$

Theorem 2.6 and Corollary 2.7 guarantee existence of v.

We impose now an additional condition on functions μ_0, μ_1 and g in the following form: we require that there exists

$$\lim_{t\to 0} \frac{\partial v}{\partial t}(x_0,t).$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Now we can substitute

$$u(x,t) = w(x,t) + v(x,t) + x\int_0^t g(1,s)ds + (1-x)\int_0^t g(0,s)\,ds$$

into problem (P). Thus we obtain the following problem (PO) with unknown functions w and f

$$\frac{\partial w}{\partial t} = \mathbb{A}\frac{\partial^2 w}{\partial x^2} + \mathbb{H}f + \gamma \quad \text{for} \quad (x,t)\in]0,1[\times]0,T[, \tag{3.1}$$

$$w(x,0) = 0 \quad \text{for} \quad x\in[0,1], \tag{3.2.1}$$

$$w(0,t) = w(1,t) = 0 \quad \text{for} \quad t\in[0,T], \tag{3.2.2}$$

$$w(x_0,t) = h(t) - v(x_0,t) - x_0\int_0^t g(1,s)ds - (1-x_0)\int_0^t g(0,s)ds \tag{3.3}$$

where

$$\gamma(x,t) = g(x,t) - xg(1,t) - (1-x)g(0,t) \tag{3.4}$$

is a given function vanishing for $x=0$ and $x=1$.

Theorem 3.5. If the functions $\mathbb{H}$, h, g satisfy assumptions (1.5-10) then there exist continuous functions $w : [0,1]\times[0,T] \longrightarrow \mathbb{R}^n$, $f : [0,T] \longrightarrow \mathbb{R}^n$ such that the derivatives $\frac{\partial w}{\partial t}$, $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x^2}$ are defined and continuous on $]0,1[\times]0,T[$ and the pair (w,f) satisfies (3.1-3).

Proof. If a pair (w,f) mentioned above does exist, then, by virtue of Theorem 2.6, we can apply formula (2.5.4) to express w by $\varphi = \mathbb{H}f+\gamma$. Applying condition (3.3), we obtain the following equation for f

$$h(t) - v(x_0,t) - x_0\int_0^t g(1,s)ds - (1-x_0)\int_0^t g(0,s)ds = \tag{3.6}$$

$$= \frac{1}{2}\int_0^t\int_0^1 [\mathbb{M}(x_0-y,t-s) - \mathbb{M}(x_0+y,t-s)][\mathbb{H}(y,s)f(s)+\gamma(y,s)]\,dyds.$$

The above equation is a system of Volterra integral equations of the first kind in the form

$$\int_0^t \mathbb{N}(t,s)f(s)\,ds = \psi(t), \tag{3.7.1}$$

where the matrix-valued function $\mathbb{N}(t,s)$ and the $\mathbb{R}^n$-valued function ψ are defined as follows

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.7.2)\qquad \mathbb{N}(t,s) = \frac{1}{2}\int_0^1 [\mathbb{M}(x_0-y,t-s) - \mathbb{M}(x_0+y,t-s)]\mathbb{H}(y,s)\,dy$$

for $s<t$ and

$$(3.7.3)\qquad \psi(t) = h(t)-v(x_0,t) - x_0\int_0^t g(1,s)\,ds - (1-x_0)\int_0^t g(0,s)\,ds+$$
$$+ \frac{1}{2}\int_0^t\int_0^1 [\mathbb{M}(x_0-y,t-s) - \mathbb{M}(x_0+y,t-s)]\gamma(y,s)\,dyds.$$

By definition (2.3), we have

$$\mathbb{N}(t,s) = 2\int_0^1 \sum_{k=1}^{\infty} \sin k\pi y \sin k\pi x_0\ [\exp(-k^2\pi^2(t-s)\mathbb{A})]\mathbb{H}(y,s)\,dy$$

and we see that for fixed t, s $(t>s)$ the series under above integral is uniformly convergent on $[0,1]$. Hence

$$(3.8.1)\qquad \mathbb{N}(t,s) = \sum_{k=1}^{\infty} \sin k\pi x_0[\exp(-k^2\pi^2(t-s)\mathbb{A})]\ H_k(s),$$

where

$$(3.8.2)\qquad H_k(s) = 2\int_0^t \sin k\pi y\ \mathbb{H}(y,s)\,dy$$

are Fourier matrix coefficients of the expansion of $\mathbb{H}(\cdot,s)$ in sinus series. Integrating (3.8.2) by parts, we can deduce from (1.6) and (1.7) that

$$(3.8.3)\qquad \|H_k(s)\| \le \frac{C}{k^4} \quad \text{for } k\in\mathbb{N},\ C>0.$$

where C is a positive constant. This means that the kernel $\mathbb{N}(x,t)$ has a continuous extension onto the triangle $0\le s\le t\le T$, since the series (3.8.1) is uniformly convergent on it. Moreover, the derivative with respect to the first variable of this kernel, denoted by $\mathbb{N}_t(t,s)$, is also continuous on this closed triangle and

$$\mathbb{N}_t(t,s) = -\pi^2 \sum_{k=1}^{\infty} k^2 \sin k\pi x_0[\exp(-k^2\pi^2(t-s)\mathbb{A})]\ \mathbb{A}\ H_k(s)$$

for $0\le s\le t\le T$.

Now we can differentiate both sides of (3.7.1) and it leads to the following system of Volterra integral equations of the second kind

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\mathbb{N}(t,t)f(t) + \int_0^t \mathbb{N}_t(t,s)f(s)ds = \psi'(t).$$

$\mathbb{N}(t,t)$ is an invertible matrix, since $\mathbb{N}(t,t) = \mathbb{H}(x_0,t)$, by (3.8.1-2) and (1.8) holds. We thus derive the final system of equations for f

$$f(t) + \int_0^t \mathbb{K}(t,s)f(s)ds = \mathbb{H}(x_0,t)^{-1}\psi'(t), \tag{3.9.1}$$

where the kernel

$$\mathbb{K}(t,s) = \mathbb{H}(x_0,t)^{-1}\mathbb{N}_t(t,s) \tag{3.9.2}$$

is continuous on the closed triangle $0 \le s \le t \le T$. Since $\mathbb{H}(x_0,t)^{-1}\psi'(t)$ is continuous, equation (3.9.1) has a unique continuous solution $f : [0,T] \longrightarrow \mathbb{R}^n$. Taking $\varphi = \mathbb{H}f+\gamma$ with this solution f and substituting φ into formula (2.5.4), we obtain the function $w=v_3$ which has together with f all properties listed in Theorem 3.5.■

4. Some remarks on uniqueness

We will discuss here the importance of condition (1.8), i.e., the invertibility of $\mathbb{H}(x_0,t)$. One can attempt to generalize the problem (P) replacing condition (1.3) by a family of conditions

$$u(h_l,t) = h_l(t) \quad \text{for } l=0,1,\dots,k-1$$

where $x_l \in]0,1[$ are distinct and h_l are given $\mathbb{R}^n$-valued functions. In the following example we drop the assumption (1.8).

Example 4.1. Let $k \in N$, $\mathbb{H}(x,t) = (\sin(k+1)\pi x)I$, $\mathbb{A}=I$, $g(x,t) \equiv 0$. $h_0(t) = \dots = h_{k-1}(t) \equiv 0$ and $x_l = \frac{l+1}{k+1}$ for $l=0,\dots,k-1$.

We specify the "generalized" problem (P) as follows

$$\frac{\partial u}{\partial t} = \mathbb{A}\frac{\partial^2 u}{\partial x^2} + \mathbb{H}(x,t)f \quad \text{for } (x,t) \in]0,1[\times]0,T[,$$

$$u(x,0) = 0 \quad \text{for } x \in [0,1],$$

$$u(0,t) = u(1,t) = 0 \quad \text{for } t \in [0,T],$$

$$u(x_l,t) = h_l(t) \quad \text{for } t \in [0,T],\ l=0,1,\dots,k-1$$

with $\mathbb{A}$, $\mathbb{H}$, x_l, h_l described above.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

For every $k \in N$ this problem has infinite dimensional family of smooth solutions. For instance, we can take as such family:

$u(x,t) = \alpha(t)(\sin(k+1)\pi x)a \qquad f(t) = (\pi^2(k+1)^2\alpha(t) + \alpha'(t))a,$

where $a \in R^n$, $\alpha \in C^1([0,T])$ and $\alpha(0)=0$.

Acknowledgement: We are grateful to Professor J. Wolska-Bochenek for her kind interest in this work and encouragement.

REFERENCES

[1] **J.R. Cannon:** Determination of certain parameters in heat conduction problems, J. Math. Anal. Appl. 8 (1964), 188-201.

[2] **J.R. Cannon:** Determination of an unknown heat source from overspecified boundary data, SIAM J. Numer. Anal. 5, (1968), 112-120.

[3] **J.R. Cannon:** A class of inverse problems. Improperly posed problems. San Francisco, 1975.

[4] **J.R. Cannon, R.E. Ewing:** Determination of a source term in a linear parabolic partial differential equation, J. Appl. Math. Phys. 27 (1976), 512-521.

[5] **E.M. Chrzanowski, M. Majchrowski:** An inverse problem with a non-linear boundary condition for a parabolic equation, Demonstratio Math. 10 (1977), 487-495.

[6] **A. Friedman:** Partial differential equations of parabolic type. Prentice-Hall, 1964.

[7] **Ph. Hartman:** the zeroes of the derivatives of the real elliptic θ_3-function, Duke Math. J. 15 (1948), 413-420.

[8] **Ph. Hartman, A. Wintner:** On the solutions of the equation of heat conduction, Amer. J. Math. 72 (1950), 367-395.

[9] **B.F. Jones:** The determination of a coefficient in a parabolic differential equation, J. Math. Mech. 11 (1962), 907-918.

[10] **M. Majchrowski:** On construction of a solution of the

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

parabolic differential equation, Demonstratio Math. 13 (1980), 285-299.

[11] M. Majchrowski: On solutions of Fourier problems for some parabolic systems of partial differential equations, Demonstratio Math. 13 (1980), 675-691.

[12] А.И. Прилепко, В.В. Соловьев: Теоремы разрешимости и метод Ротэ в обратных задачах для уравнения параболического типа, Дифф. Урав. 23 (1987), 1791-1799.

[13] T. Suzuki: Uniqueness and nonuniqueness in an inverse problem for the parabolic equation, J. Diff. Eqs. 47 (1983), 296-316.

[14] A. Wintner: The real elliptic θ_3-function, Duke Math. J. 15 (1948), 407-413.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW, 00-661 WARSZAWA

Received September 12, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Jacek Dąbrowski

ESTIMATION OF DENSITY FUNCTION FOR STATIONARY PROCESSES WITH APPLICATION TO MARKOV PROCESSES

1. Introduction

Let us consider a continuous time parameter strictly stationary, measurable process $\{X_t : t\geq 0\}$ defined on a probability space $(\Omega,\mathcal{F},P)$ that takes values in d-dimensional space R^d. We shall assume that the distribution μ of X_0 has a density function $p(x)$ with respect to Lebesgue measure λ_d in R^d. In this paper we are concerned with kernel type estimation of $p(x)$. The similar problem is considered by Prakasa Rao in [1] Chapter 6 (see also [4]). It is assumed in [1] that $\{X_t : t\geq 0\}$ is a strictly stationary and Markov process satisfying so called $G_2(t,\alpha)$ condition which is somewhat weaker then the Doeblin condition (see Doob [2] p. 221). Under these assumptions and additionally under the assumption that $p(x)$ is a continuous and bounded function, the strong consistency of kernel estimator is proved in [1]. E. Masry and L. Györfi in [5] consider discrete parameter stationary process $\{X_j\}$ which is asymptotically uncorrelated i.e.

$$(1.1)\qquad \sup_{U\in L_2(\mathcal{F}^0_{-\infty}),V\in L_2(\mathcal{F}^\infty_k)} \frac{|\mathrm{cov}\{U,V\}|}{\mathrm{Var}^{1/2}[U]\mathrm{Var}^{1/2}[V]} = \rho(k) \xrightarrow[\text{as } k\to\infty]{} 0.$$

The strong consistency is proved and the rate of convergence of kernel type estimator of $p(x)$ is evaluated in [5] under the following assumption

$$(1.2)\qquad \sum_{k=1}^{\infty} (\log(k))(\log_2(k))\rho^2(k) < \infty.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In this paper we consider a continuous time stationary process which is in a sense also uncorrelated. In fact we assume that the function $r(h,t)$ defined below by (2.7) tends to zero. We do not make any assumption analogous to (1.2) concerning the rate of convergence of $r(h,t)$ to zero. However we assume that this rate is known. Then we can prove strong consistency of kernel type estimator. All results of this paper are valid also in discrete parameter case but we do not formulate them. The proofs are very analogous. As an example we consider a recurrent Markov process with its invariant measure μ having density function $p(x)$. In this case the estimator of $p(x)$ turns out to be also strong consistent.

2. Preliminaries

Let $\{t_n : n=1,2,\dots\}$ be a real sequence such that

(2.1) for every natural n we have $t_{n+1} \geq t_n + 1$ and $t_1 > 0$.

We define the real functions h, h_n, $n=1,2,\dots$ on $[0,\infty)$ by the formulas

$$h(t) = \begin{cases} 1 & \text{if } 0 \leq t < t_1 \\ \frac{1}{n} & \text{if } t_{n-1} \leq t < t_n, \end{cases} \tag{2.2}$$

$$h_n(t) = \begin{cases} h(t) & \text{if } t < t_n, \\ \frac{1}{n+1} & \text{if } t \geq t_n, \ n=2,3,\dots . \end{cases} \tag{2.3}$$

The kernel K is assumed to satisfy

$$\int_{R^d} K(y)\,dy = 1, \tag{2.4}$$

(2.5) K has a bounded support and there exists a constant $L>0$ such that for every $y \in R^d$ we have $0 \leq K(y) \leq L$.

Let $\{X_t : t \geq 0\}$ be a R^d-valued strictly stationary measurable process defined on a probability space $(\Omega, \mathcal{F}, P)$.

The distribution μ of X_t has density $p(x)$ with respect to the Lebesgue measure λ_d on R^d. We consider the kernel estimator of the form

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(2.6)\qquad \hat{p}_t(x) = \left[\int_0^t h(s)ds\right]^{-1} \cdot \int_0^t K\left(\frac{X_s - x}{d\sqrt{h(s)}}\right)ds .$$

From the statistical point of view it is important to determine the sequence $\{t_n: n=1,2,\ldots\}$. One should know when to "switch over" the value of the function $h(t)$. In fact, we will be able to determine the function $h(t)$ if we know the rate of convergence to zero of the following function

$$(2.7)\qquad r(h,t) = E\left\{\left[K\left(\frac{X_0-x}{h}\right) - EK\left(\frac{X_0-x}{h}\right)\right] \times \right.$$

$$\left. \times \left[K\left(\frac{X_t-x}{h}\right) - EK\left(\frac{X_t-x}{h}\right)\right]\right\} \xrightarrow[\text{as } t\to\infty]{} 0.$$

More precisely we make the following assumption

(2.8) Let K be a fixed kernel function satisfying (2.4),(2.5).

For every $1\geq h>0$ we have $\lim_{t\to\infty} r(h,t) = 0$ and the rate of this convergence is known. Since the function $r(h,t)$ depends on the distribution of (X_0,X_t) only, the above condition seems to be weaker then the analogous one given in [5] p. 81 for discrete parameter process.

Let us make the following notations

$$(2.9)\qquad I(t) = K\left(\frac{X_t - x}{d\sqrt{h(t)}}\right), \qquad I_n(t) = K\left(\frac{X_t - x}{d\sqrt{h_n(t)}}\right),$$

$$(2.10)\qquad V(t) = I(t) - EI(t),$$

$$(2.11)\qquad V_n(t) = I_n(t) - EI_n(t),$$

$$(2.12)\qquad Z(t) = \left(\int_0^t h(s)ds\right)^{-1} \cdot \int_0^t V(s)ds,$$

$$(2.13)\qquad Z_n(t) = \left(\int_0^t h_n(s)ds\right) \cdot \int_0^t V_n(s)ds,$$

$$(2.14)\qquad M_p = \left\{ x\in R^d : \lim_{\rho\to 0} \left(\lambda_d\{y:|y-x|\leq\rho\}\right)^{-1} \cdot \right.$$

$$\left. \cdot \int_{\{y:|y-x|\leq\rho\}} |p(y)-p(x)|dy = 0 \right\}.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

3. Main result

Theorem 1. Under the assumptions (2.4), (2.5) and (2.8) for λ_d- almost all $x \in R^d$ (or for all $x \in M_p$) there exists $h(t)$ defined by (2.2) such that we have

$$\lim_{t\to\infty} \hat{p}_t(x) = p(x) \quad \text{a.s.}, \tag{3.1}$$

where the sequence $\{t_n: n=1,2,\ldots\}$ is defined precisely in the proof of the theorem and generally this sequence depends on x.

Let us first prove the following lemma.

Lemma 1. Let the sequence $\{t_n: n=1,2,\ldots\}$ satisfy (2.1). Then under the conditions of Theorem 1 we have

$$\lim_{t\to\infty} E\,\hat{p}_t(x) = p(x) \tag{3.2}$$

for almost all $x \in R^d$ or for all $x \in M_p$.

Proof. We have

$$|E\,\hat{p}_t(x) - p(x)| = \left|\left(\int_0^t h(s)ds\right)^{-1}\left(\int_0^t EI(s)ds - \int_0^t h(s)p(x)ds\right)\right| \le$$

$$\le \left(\int_0^t h(s)ds\right)^{-1}\left(\int_0^t h(s)\,|(h(s))^{-1}E\,I(s) - p(x)|\,ds\right).$$

Since $\int_0^\infty h(s)ds$ in view of (2.1) (2.2), it suffices to show that

$$\lim_{s\to\infty} (h(s))^{-1}\,E\,I(s) - p(x) = 0 \tag{3.3}$$

for almost all $x \in R^d$ (see [1] p. 329).

Using (2.4) and (2.5) one can prove (3.3). The proof is very analogous to the given in [6] (one-dimensional case). The proof of (3.3) in d-dimensional case is given in [7] p. 44. Compare also with [5] p. 83.

Q.E.D.

To prove Theorem 1 we shall need also the following lemma.

Lemma 2. Let $\{Y_0, Y_1, Y_2, \ldots\}$ be a zero-mean strictly

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

stationary, bounded process such that

(3.4) $r(k) = EY_0Y_k \longrightarrow 0$ if $k \longrightarrow \infty$, with $|Y_k| \le L$.

Then for every $\alpha>0$, $\varepsilon>0$ there exists a natural number b such that we have

$$(3.5) \qquad P\left(\sup_{k\ge b} \left| \frac{Y_0+\ldots+Y_{k-1}}{k} \right| > \alpha \right) \le \varepsilon.$$

Remark. The assertion of Lemma 2 follows immediately from the law of large numbers. However we will prove it because of the technique of getting b which plays essential role in a choice of the sequence $\{t_n: n=1,2,\ldots\}$. The number b depends on α, ε and on the rate of convergence of r(k) to zero. This rate of convergence is assumed to be known.

Proof of Lemma 2. Knowing the rate of convergence of r(k) we can choose k_0 such that

$$|r(k)| < \frac{\varepsilon^2\alpha^2}{32} \qquad \text{for } k\ge k_0,$$

and we have

$$(3.6) \qquad \frac{E(Y_0+\ldots+Y_{j-1})^2}{j^2} = \frac{1}{j^2}\sum_{k=0}^{j-1} E\,Y_k^2 +$$

$$+ \frac{2}{j^2}\sum_{k=0}^{j-1}\left(\sum_{i=k+1}^{(k+k_0-1)\wedge(j-1)} EY_kY_i + \sum_{i=(k+k_0)\wedge(j-1)}^{j-1} EY_kY_i \right) \le$$

$$\le \frac{L^2}{j} + \frac{2(k_0-1)L^2}{j} + \frac{\varepsilon^2\alpha^2}{32} < \frac{\varepsilon^2\alpha^2}{16},$$

for $j\ge q = q(k_0,L,\varepsilon,\alpha)$, where $x\wedge y = \min(x,y)$.

Let us define

$$S_k = \frac{Y_{qk}+\ldots+Y_{q(k+1)-1}}{q}, \qquad k=0,1,\ldots,$$

$$A^k(\alpha) = \left\{ \sup_{k\ge1} \frac{S_0+\ldots+S_{k-1}}{k} > \alpha \right\},$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$A_k(\alpha) = \left\{ \inf_{k\geq 1} \frac{S_0+\ldots+S_{k-1}}{k} < \alpha \right\}.$$

The sequence $\{S_k\colon k=0,1,2,\ldots\}$ is also a strictly stationary process and the following inequalities hold

$$(3.7)\qquad \alpha\, P\left(\sup_{k\geq 1} \frac{S_0+\ldots+S_{k-1}}{k} > \alpha \right) \leq \int_{A^k(\alpha)} S_0\, dP,$$

$$(3.8)\qquad \alpha\, P\left(\inf_{k\geq 1} \frac{S_0+\ldots+S_{k-1}}{k} < -\alpha \right) \leq \int_{A_k(-\alpha)} (-S_0)\, dP,$$

(see Doob [2] Chapter 10 § 2).

Let k_1 be a natural number such that $\frac{1}{k_1} < \frac{\alpha}{2\cdot L}$. We define

$$b = q\cdot k_1 .$$

If $k\geq b$, then $k = qk_2+m$, where $k_2\geq k_1$ and $m<q$, and we have

$$(3.9)\qquad P\left(\sup_{k\geq b} \frac{Y_0+\ldots+Y_{k-1}}{k} > \alpha \right) \leq P\left(\sup_{k\geq b} \frac{Y_0+\ldots+Y_{k-1}}{k} > \alpha \right) +$$

$$+ P\left(\inf_{k\geq b} \frac{Y_0+\ldots+Y_{k-1}}{k} < -\alpha \right) \leq$$

$$\leq P\left(\sup_{\substack{k_2\geq k_1 \\ 0\leq m<q}} \frac{Y_0+\ldots+Y_{qk_2-1}}{qk_2} \cdot \frac{qk_2}{qk_2+m} > \frac{\alpha}{2} \right) +$$

$$+ P\left(\sup_{k_2,m} \frac{Y_{qk_2}+\ldots+Y_{qk_2+m}}{qk_2+m} > \frac{\alpha}{2} \right) +$$

$$+ P\left(\inf_{k_2,m} \frac{Y_0+\ldots+Y_{qk_2-1}}{qk_2} \cdot \frac{qk_2}{qk_2+m} < -\frac{\alpha}{2} \right) +$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ P\left(\inf_{k_2,m} \frac{Y_{qk_2}+\ldots+Y_{qk_2+m}}{qk_2+m} < -\frac{\alpha}{2} \right).$$

Since $|Y_k| \le L$ we have

$$(3.10) \qquad \sup_{k_2,m} \left| \frac{Y_{qk_2}+\ldots+Y_{qk_2+m}}{qk_2+m} > \frac{\alpha}{2} \right| \le \frac{q\cdot L}{q\cdot k_2} \le \frac{L}{k_1} < \frac{\alpha}{2}.$$

Using (3.9), (3.10), (3.7), (3.8) and (3.6) we get

$$P\left(\sup_{k\ge b} \left|\frac{Y_0+\ldots+Y_{k-1}}{k}\right| \ge \alpha \right) \le P\left(\sup_{k_2\ge k_1} \frac{S_0+\ldots+S_{k_2-1}}{k_2} > \frac{\alpha}{2} \right) +$$

$$+ P\left(\inf_{k_2\ge k_1} \frac{S_0+\ldots+S_{k_2-1}}{k_2} < -\frac{\alpha}{2} \right) \le \frac{2}{\alpha} \int_{A^k(\frac{\alpha}{2})} S_0 dP + \frac{2}{\alpha} \int_{A^k(-\frac{\alpha}{2})} S_0 dP \le$$

$$\le \frac{4}{\alpha} E|S_0| \le \frac{4}{\alpha} \sqrt{\frac{E(Y_0+\ldots+Y_{q-1})^2}{q^2}} < \frac{4}{\alpha} \sqrt{\frac{\varepsilon^2\alpha^2}{16}} = \varepsilon.$$

This completes the proof of Lemma 2.

Proof of Theorem 1. According to Lemma 1 it suffices to find the sequence $\{t_n: n=1,2,\ldots\}$ such that we have

$$(3.11) \qquad \lim_{t\to\infty} \left(\int_0^t h(s)ds \right)^{-1} \cdot \int_0^t v(s)ds = 0 \qquad \text{a.s.}.$$

For every n we will find t_n such that we have

$$(3.12) \qquad P\left(\sup_{t\ge t_n} |Z(t)| \le \frac{1}{n} \right) \ge 1 - 2^{1-n}.$$

We will define the sequence $\{t_n: n=1,2,\ldots\}$ by induction. Clearly $t_1=1$ will do. We can assume that $t_1,t_2,\ldots,t_{n-1}$ are known and $h(t)$ is defined for $0\le t<t_{n-1}$. It follows from (2.3) that $h_{n-1}(t)$ is defined for all $t\ge 0$.

Let us consider the following process

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.13)\qquad Y_k = \int_{t_{n-1}+k}^{t_{n-1}+k+1} V_{n-1}(s)\,ds, \qquad k=0,1,2,\ldots$$

$\{Y_k: k=0,1,\ldots\}$ is a strictly stationary process with $EY_k=0$. Let us fix $h = \left(\frac{1}{n}\right)^{1/d}$. Then we have

$$EY_0Y_k = \int_{t_{n-1}}^{t_{n-1}+1}\int_{t_{n-1}+k}^{t_{n-1}+k+1} E\,V_{n-1}(u)V_{n-1}(s)\,duds =$$

$$= \int_{t_{n-1}}^{t_{n-1}+1}\int_{t_{n-1}+k}^{t_{n-1}+k+1} r(h,s-u)\,duds \xrightarrow[\text{as } k\to\infty]{} 0,$$

$|Y_k|\le L$ because of (2.5). It follows from the assumption (2.8) that the rate of the above convergence is known. Let us now apply Lemma 2 for the process $\{Y_k: k=0,1,\ldots\}$ and

$$\varepsilon = \frac{2^{-n}}{3}, \quad \alpha = \frac{1}{16n^2}.$$

Then we can find a number b_n such that we have

$$(3.14)\qquad P\left(\sup_{k\ge b_n}\left|\frac{Y_0+\ldots+Y_{k-1}}{k}\right| \le \frac{1}{16n^2}\right) \le \frac{2^{-n}}{3}.$$

Another words we have the following inequality

$$(3.15)\quad P\left(\sup_{k\ge b_n}\left(\int_{t_{n-1}}^{t_{n-1}+k} h_{n-1}(s)\,ds\right)^{-1}\left|\int_{t_{n-1}}^{t_{n-1}+k} V_{n-1}(s)\,ds\right| > \frac{1}{16n}\right) \le \frac{2^{-n}}{3}.$$

Without loss of generality we can assume that $b_n \ge 16Ln^2$. Let $s\ge b_n$ be a real number and $[s]=\text{entier}(s)$. Using (3.15) we have

$$(3.16)\qquad P\left\{\sup_{s\ge b_n}\left(\int_{t_{n-1}}^{t_{n-1}+s} h_{n-1}(u)\,du\right)^{-1}\left|\int_{t_{n-1}}^{t_{n-1}+s} V_{n-1}(u)\,du\right| > \frac{1}{8n}\right\} \le$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$\leq P\left\{ \sup_{s\geq b_n} \left([s]\cdot\frac{1}{n}\right)^{-1} \left| \int_{t_{n-1}}^{t_{n-1}+[s]} V_{n-1}(u)\,du \right| > \frac{1}{16n} \right\} +$$

$$+ P\left\{ \sup_{s\geq b_n} \left(s\cdot\frac{1}{n}\right)^{-1} \left| \int_{t_{n-1}+[s]}^{t_{n-1}+s} V_{n-1}(u)\,du \right| > \frac{1}{16n} \right\} \leq$$

$$\leq \frac{2^{-n}}{3} + P\left\{ \left(b_n\cdot\frac{1}{n}\right)^{-1}\cdot L > \frac{1}{16n} \right\} = \frac{2^{-n}}{3} .$$

Let $u_n>0$ be a real number such that

$$(3.17)\qquad \left(\int_0^{t_{n-1}+u_n} h_{n-1}(s)\,ds \right)^{-1} \left| \int_0^{t_{n-1}} V_{n-1}(s)\,ds \right| \leq$$

$$\leq \left(\int_0^{t_{n-1}} h(s)\,ds + u_n\cdot\frac{1}{n} \right)^{-1} \cdot t_{n-1} L < \frac{1}{8n} .$$

Let us also consider the following process

$$(3.18)\qquad Y'_k = \int_{t_{n-1}+k}^{t_{n-1}+k+1} V'_n(s)\,ds , \quad k=0,1,\ldots ,$$

where

$$V'_n(s) = K\left(\frac{X_s - x}{d\sqrt{1/(n+1)}} \right) - E\,K\left(\frac{X_s - x}{d\sqrt{1/(n+1)}} \right)$$

for all $s\geq 0$.

The process $\{Y'_k : k=0,1,\ldots\}$ is strictly stationary and as in the case of the process $\{Y_k : k=0,1,\ldots\}$ it is easy to see that the rate of convergence $EY'_0Y'_k \longrightarrow 0$ is known. Applying again Lemma 2 for the process $\{Y'_k : k=0,1,\ldots\}$ we can find a natural number b'_n such that

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$(3.19)\quad P\left\{\sup_{k\geq b'_n}\left(\int_{t_{n-1}}^{t_{n-1}+k}[1/(n+1)]ds\right)^{-1}\left|\int_{t_{n-1}}^{t_{n-1}+k}V'_n(s)ds\right|>\frac{1}{16n}\right\}\leq\frac{2^{-n}}{3}.$$

Compare also with (3.14) and (3.15). We can assume that $b'_n\leq 16\cdot L\cdot n(n+1)$. Using (3.19) and the analogous procedure to the one given in (3.16) it is easy to verify that

$$(3.20)\quad P\left\{\sup_{t\geq b'_n}\left(\int_{t_{n-1}}^{t_{n-1}+t}[1/(n+1)]ds\right)^{-1}\left|\int_{t_{n-1}}^{t_{n-1}+t}V'_n(s)ds\right|>\frac{1}{8n}\right\}\leq\frac{2^{-n}}{3}.$$

where t - a real number. We omit the details.

Now we are ready to define the number

$$t_n=\max(t_{n-1}+1,\ t_{n-1}+b_n,\ t_{n-1}+u_n,\ t_{n-1}+b'_n).$$

Since t_n is defined, we can assume that the sequence $\{t_k\}_{k=1}^{\infty}$ and $\{h(s):s\geq 0\}$ are defined. Moreover, using (3.16) and (3.17) we have

$$(3.21)\quad P\left\{\sup_{t\geq t_n}|Z_{n-1}(t)|>\frac{1}{4n}\right\}\leq$$

$$\leq P\left\{\sup_{t\geq t_n}\left(\int_0^t h_{n-1}(s)ds\right)^{-1}\left|\int_0^{t_{n-1}}V_{n-1}(s)ds\right|>\frac{1}{8n}\right\}+$$

$$+P\left\{\sup_{t\geq t_n}\left(\int_{t_{n-1}}^t h_{n-1}(s)ds\right)^{-1}\left|\int_{t_{n-1}}^t V_{n-1}(s)ds\right|>\frac{1}{8n}\right\}\leq\frac{2^{-n}}{3}.$$

Using successively (3.17), (3.16) and (3.20) we obtain the following inequalities

$$P\left\{\sup_{t\geq t_n}|Z(t)|>\frac{1}{n}\right\}\leq P\left\{\sup_{k\geq n}\ \sup_{t_k\leq t<t_{k+1}}|Z(t)|>\frac{1}{n}\right\}\leq$$

$$\leq P\left[\bigcup_{k\geq n}\left\{\sup_{t_k\leq t<t_{k+1}}|Z(t)|>\frac{1}{k}\right\}\right]\leq\sum_{k=n}^{\infty}P\left\{\sup_{t_k\leq t<t_{k+1}}|Z(t)|>\frac{1}{k}\right\}\leq$$

$$\leq\sum_{k=n}^{\infty}\left[P\left\{\left(\int_0^{t_k}h(s)ds\right)^{-1}\left|\int_0^{t_{k-1}}V(s)ds\right|>\frac{1}{2k}\right\}+\right.$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$$+ P\left\{ \sup_{t_k \le t < t_{k+1}} \left(\int_{t_{k-1}}^{t} h(s)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V(s)ds \right| > \frac{1}{2k} \right\} \Big] \le$$

$$\le \sum_{k=n}^{\infty} \Big[0 + P\left\{ \left(\int_{t_{k-1}}^{t_k} h(s)ds \right)^{-1} \left| \int_{t_{k-1}}^{t_k} V(s)ds \right| > \frac{1}{4k} \right\} +$$

$$+ P\left\{ \sup_{t_k \le t < t_{k+1}} \left(\int_{t_{k-1}}^{t} h(s)ds \right)^{-1} \left| \int_{t_k}^{t} V(s)ds \right| > \frac{1}{4k} \right\} \Big] \le$$

$$\le \sum_{k=n}^{\infty} \Big[P\left\{ \sup_{t \ge t_k} \left(\int_{t_{k-1}}^{t} h_{k-1}(s)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V_{k-1}(s)ds \right| > \frac{1}{4k} \right\} +$$

$$+ P\left\{ \sup_{t_k \le t < t_{k+1}} \left(\int_{t_{k-1}}^{t} 1/(k+1)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V'_k(s)ds - \int_{t_{k-1}}^{t_k} V'_k(s)ds \right| > \frac{1}{4k} \right\} \Big] \le$$

$$\le \sum_{k=n}^{\infty} \Big[\frac{2^{-k}}{3} + P\left\{ \sup_{t_k \le t < t_{k+1}} \left(\int_{t_{k-1}}^{t} 1/(k+1)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V'_k(s)ds \right| > \frac{1}{8k} \right\} +$$

$$+ P\left\{ \sup_{t_k \le t < t_{k+1}} \left(\int_{t_{k-1}}^{t} 1/(k+1)ds \right)^{-1} \left| \int_{t_{k-1}}^{t_k} V'_k(s)ds \right| > \frac{1}{8k} \right\} \Big] \le$$

$$\le \sum_{k=n}^{\infty} \Big[\frac{2^{-k}}{3} + P\left\{ \sup_{t \ge t_k} \left(\int_{t_{k-1}}^{t} 1/(k+1)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V'_k(s)ds \right| > \frac{1}{8k} \right\} +$$

$$+ P\left\{ \sup_{t \ge t_k} \left(\int_{t_{k-1}}^{t} 1/(k+1)ds \right)^{-1} \left| \int_{t_{k-1}}^{t} V'_k(s)ds \right| > \frac{1}{8k} \right\} \Big] \le$$

$$\le \sum_{k=n}^{\infty} \left(\frac{2^{-k}}{3} + \frac{2^{-k}}{3} + \frac{2^{-k}}{3} \right) = 2^{-n+1} .$$

Hence we have proved (3.12). Finally, applying Borel-Cantelli Lemma, we obtain (3.11). The proof of the theorem is now complete.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

4. Application to Markov processes

Let us consider an R^d-valued Markov process $\{X_t : t\geq 0, X_0=x_0$ -nonrandom$\}$ with the homogeneous transition probabilities $P(x,t,A)$ and the family of probability measures $\{P_x : x\in R^d\}$ on $(\Omega,\mathcal{F})$ such that

$P(x,t,A) = P_x(X_t\in A)$ (see [3] for precise definition).

We will assume that $\{X_t : t\geq 0\}$ is measurable and recurrent. Let us recall the definition.

Definition ([8] p.234). The Markov process $\{X_t : t\geq 0\}$ is said to be recurrent if for every measurable set $A\subset R^d$ with $\lambda_d(A)>0$ (λ_d -Lebesgue measure in R_d) we have

$$P_x\Big[\int_0^\infty 1_A(X_t)dt = \infty\Big] = 1 \quad \text{for all} \quad x\in R^d.$$

It is well known (see [9]) that in this case the process $\{X_t : t\geq 0\}$ has an invariant measure μ. We will assume that $\mu(R^d)=1$ and μ has almost everywhere positive density function $p(x)$ with respect to λ_d. It is also known (see [9]) that

$$\lim_{t\to\infty} P(x,t,A) = \mu(A).$$

Using the Lebesgue Theorem we obtain

$$(4.1)\qquad \int_{B(x,\rho)} \Big[P\big(z,t,B(x,\rho)\big) - \mu\big(B(x,\rho)\big)\Big]\cdot\mu(dz) \xrightarrow[\text{as } t\to\infty]{} 0\ ,$$

where $B(x,\rho) = \{y : |y-x|\leq\rho\}$.

We consider the kernel function K of the form

$$K(y) = 1_{B(0,\rho_0)}(y),$$

where ρ_0 satisfies $\lambda_d[B(0,\rho_0)]=1$.

We shall prove the following theorem.

Theorem 2. Let $\{X_t : t\geq 0\}$ be a Markov process satisfying the above assumptions. Then there exists the function $h(t)$ defined by (2.2) such that for all $x\in M_p$ and for λ_d- almost all $x_0\in R^d$ we have

$$(4.2)\qquad P_{x_0}\Big\{\lim_{t\to\infty} \hat{p}_t(x) = p(x)\Big\} = 1\ ,$$

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

where $\hat{p}_t(x)$ is defined by (2.6).

If the rate of convergence (4.1) is known for every $0<\rho\le\rho_0$, then we are able to determine the sequence $\{t_n : n=1,2,\dots\}$.

Proof. First let us assume that μ is the initial distribution of the process $\{X_t : t\ge 0\}$. Then $\{X_t : t\ge 0\}$ is a strictly stationary process with respect to the probability measure

$$P_\mu(\cdot) = \int_{R^d} P_x(\cdot)\mu(dx).$$

We will show now that $\lim_{t\to\infty} r(h,t) = 0$, where $r(h,t)$ is defined by (2.7), $0<h\le 1$. We have

$$r(h,t) = E_{P_\mu}\Big\{\Big[\,1_{B(x,\rho_0 h)}(X_0) - E_{P_\mu} 1_{B(x,\rho_0 h)}(X_0)\,\Big] \times$$

$$\times E_{X_0}\Big[\,1_{B(x,\rho_0 h)}(X_t) - E_{P_\mu} 1_{B(x,\rho_0 h)}(X_t)\,\Big]\Big\} =$$

$$= \int_{R^d}\Big\{\Big[\,1_{B(x,\rho_0 h)}(z) - \mu\big(B(x,\rho_0 h)\big)\Big] \times$$

$$\times \Big[P\big(z,t,B(x,\rho_0 h)\big) - \mu\big(B(x,\rho_0 h)\big)\Big]\Big\}\mu(dz) =$$

$$= \int_{B(x,\rho_0 h)} \Big[\,P\big(z,t,B(x,\rho_0 h)\big) - \mu\big(B(x,\rho_0 h)\big)\Big]\mu(dz) \xrightarrow[\text{as } t\to\infty]{} 0\,.$$

Therefore using Theorem 1 for every $x\in M_p$ there exists $h(t)$ defined by (2.2) such that

$$P_\mu\big(\lim_{t\to\infty} \hat{p}_t(x) = p(x)\big) = 1\,.$$

It follows also from Theorem 1 that if the rate of convergence (4.1) is known for $0<\rho\le\rho_0$ then we can determine the sequence $\{t_n : n=1,2,\dots\}$.

Finally we have

$$P_\mu\big(\lim_{t\to\infty} \hat{p}_t(x) = p(x)\big) = \int_{R^d} P_{x_0}\big(\lim_{t\to\infty} \hat{p}_t(x) = p(x)\big)\cdot p(x_0)dx_0\,.$$

Since $p(x_0)>0$, λ_d-almost everywhere, the assertion of Theorem 2 follows immediately.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

In particular, if $\{X_t : t \geq 0\}$ satisfies the Doeblin condition (see [2] p.221), then the rate of convergence (4.1) is exponential.

REFERENCES

[1] B.L.S. Pracasa Rao: Nonparametric functional estimation. Academic Press. New York, London 1983.

[2] J.L. Doob: Stochastic processes. Wiley, New York 1953.

[3] J. Azema, M. Duflo, D. Revuz: Mesure invariante des processus de Markov récurrents, Sem. Cal. Prob. Fac. Sci., Strasbourg III, 1968, Springer Verlag, Lec. Notes in Math, 88, 24-33.

[4] Ishwar V. Basawa, Prakasa Rao: Statistical Inference for Stochastic Processes. London 1980.

[5] E. Masry, L. Györfi: Strong consistency and rates for recursive probability density estimators of stationary processes, J. Mult. Anal. 22 (1987) 79-93.

[6] E. Parzen: On estimation of a probability density function and mode, Ann. Math. Statist. 33 (1962) 1065-1076.

[7] J. Koronacki: Rekurencyjna estymacja gęstości prawdopodobieństwa, Roczniki Polskiego Towarzystwa Matematycznego. Seria III Matematyka Stosowana 15 (1984) 27-90. (in Polish)

[8] M. Duflo, D. Revuz: Propriétés asymptotiques des probabilités de transition des processus de Markov récurrents, Ann. Inst. Poincaré. sect. B. 5(1969) 233-244.

[9] J. Azema, M. Duflo, D. Revuz: Measure invariante sur les classes récurrentes des processus de Markov, Z. Warscheinlichkeitstheorie, 8 (1967) 157-181.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received October 5, 1989.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

DEMONSTRATIO MATHEMATICA
Vol. XXIII No 4 1990

Krzysztof Witczyński

ON PAPPUS'S THEOREM IN THE PROJECTIVE SPACE P_n

The well-known theorem of Pappus in P_2 concerns two triples of distinct points a_{00},a_{01},a_{02} and a_{10},a_{11},a_{12} lying on two distinct straight lines H_0 and H_1, respectively. The theorem says that there exists a straight line P, called Pappus's line, such that the points $Z(a_{00},a_{11}) \cap Z(a_{01},a_{10})$, $Z(a_{01},a_{12}) \cap Z(a_{02},a_{11})$ and $Z(a_{00},a_{12}) \cap Z(a_{02},a_{10})$ belong to P (the symbol $Z(a_k,a_{k+1},\ldots,a_m)$ denotes here the join of the points $a_k,a_{k+1},\ldots,a_m$). This theorem was generalized in [1] as well as in [2].

Notice that in general the point $H_0 \cap H_1$ does not belong to P. However if there exists such a point s that the lines $Z(a_{00},a_{10})$, $Z(a_{01},a_{11})$ and $Z(a_{02},a_{12})$ pass through s, i.e. the triples are in perspective, then $H_0 \cap H_1 \in P$.

The aim of this paper is to generalize the above case of the theorem of Pappus. We shall consider in the paper a projective space P_n over an arbitrary field.

Assume that in P_n we are given hyperplanes $H_0,H_1,\ldots,H_{n-1}$, being in a general position, i.e. $\dim \bigcap_{i=0}^{n-1} H_i = 0$. Let $a_{i0},\ldots,a_{in}$ be such n+1 points beloging to H_i that every n of them are linearly independent, $i = 0,\ldots,n-1$. Let us assume else that there exists such a point s that the points $s, a_{0j},\ldots,a_{n-1j}$ are collinear for all $j \in \{0,1,\ldots,n\}$. In other words, the points a_{ij}, $i \geq 1$, are intersections of the lines $Z(s,a_{0j})$ and the hyperplanes H_i.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

Define

$$S_{ij} = Z(a_{0j},\ldots,a_{kp},\ldots.), \quad j\neq i,$$

where the first index k runs from 0 to n-1, the second one from j to n and next from 0 to j-1, but excluding the value p=i;

and

$$S_i = \bigcap_{\substack{j=0\\ j\neq i}}^{n} S_{ij}.$$

Example. For n=3 we have

$a_{00},a_{01},a_{02},a_{03} \in H_0$; $a_{10},a_{11},a_{12},a_{13} \in H_1$;

$a_{20},a_{21},a_{22},a_{23} \in H_2$;

$S_{01}= Z(a_{01},a_{12},a_{23})$, $S_{02}= Z(a_{02},a_{13},a_{21})$, $S_{03}= Z(a_{03},a_{11},a_{22})$;

$S_{10}= Z(a_{00},a_{12},a_{23})$, $S_{12}= Z(a_{02},a_{13},a_{20})$, $S_{13}= Z(a_{03},a_{10},a_{22})$;

$S_{20}= Z(a_{00},a_{11},a_{23})$, $S_{21}= Z(a_{01},a_{13},a_{20})$, $S_{23}= Z(a_{03},a_{10},a_{21})$;

$S_{30}= Z(a_{00},a_{11},a_{22})$, $S_{31}= Z(a_{01},a_{12},a_{20})$, $S_{32}= Z(a_{02},a_{10},a_{21})$;

$S_0= S_{01} \cap S_{02} \cap S_{03}$; $S_1= S_{10} \cap S_{12} \cap S_{13}$; $S_2= S_{20} \cap S_{21} \cap S_{23}$;

$S_3= S_{30} \cap S_{31} \cap S_{32}$.

Choose in P_n an allowable coordinate system in such a way that the hyperplanes H_i have the following equations

$$H_i : x_i=x_n \quad i=0,\ldots,n-1$$

and, at the same time, the point s has the coordinates $(0,0,\ldots,0,1)$. Let P be the hyperplane represented by the equation $\sum_{i=0}^{n-1} x_i - nx_n = 0$.

Lemma. For every system $a_{01},\ldots,a_{0n-1}$ of linearly independent points in H_0 the respective subspace (point in general) $S_n = \bigcap_{j=0}^{n-1} S_{nj}$ is included in P.

Proof. Let t_{kj} be the k-th coordinate of the point a_{0j},

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

$k,j = 0,\ldots,n-1$. Then the n-th coordinate of this point is t_{0j}. Evidently, the k-th coordinate of the point a_{ij} is equal to t_{kj}, $i,k = 0,\ldots,n-1$, and the n-th coordinate of this point is t_{ij}.

Let us consider the vectors:

$$C_0(t_{00},t_{01},\ldots,t_{0n-1}),\quad C_{n+1}(t_{n-10},t_{01},\ldots,t_{n-3n-2},t_{n-2n-1}),$$
$$C_1(t_{10},t_{11},\ldots,t_{1n-1}),\quad C_{n+2}(t_{n-20},t_{n-11},t_{02},\ldots\ldots,t_{n-3n-1}),$$
$$\vdots\qquad\qquad\vdots\qquad\qquad\vdots$$
$$C_{n-1}(t_{n-10}t_{n-11},\ldots,t_{n-1n-1}),\quad C_{2n-2}(t_{20},t_{31},\ldots,t_{0n-2},t_{1n-1}),$$
$$C_n(t_{00},t_{11},\ldots,t_{n-1n-1}),\quad C_{2n-1}(t_{10},t_{21},\ldots,t_{n-1n-2},t_{0n-1}).$$

It can be immediately seen that the following equality holds:

$$(1)\qquad \sum_{i=0}^{n-1} C_i = \sum_{i=n}^{2n-1} C_i.$$

We shall denote by C_{0m} the $n\times n$ matrix having the vectors $C_1,C_2,\ldots,C_{n-1},C_m$ $m=n,\ldots,2n-1$ as its consecutive columns. By C_{im} will be denoted the matrix obtained from C_{om} by replacing its i-th column by $-C_0$.

One can check easily that the hyperplanes S_{nj} have the following equations:

$$(2)\qquad S_{nj}:\ \sum_{i=0}^{n-1} |C_{in+j}|x_i + |C_{nn}|x_n = 0,\quad j=0,\ldots,n-1.$$

Let now D_{im} denote the matrix obtained from C_{im} by adding $-C_i$ to its i-th column, $i=1,\ldots,n-1$, and let D_{nn} be the matrix obtained from C_{nn} by adding nC_m to its last column, $m=n,\ldots,2n-1$. Finally, let E_{in} be the matrix obtained from D_{in} by replacing its last column by $\sum_{k=n}^{2n-1} C_k$, $i=1,\ldots,n-1$, and let E_{nn} be the matrix D_{nn} with the last column replaced by $n\sum_{k=n}^{2n-1}(C_k-C_0)$.

Notice that, in view of (1), we have

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

(3) $$\det E_{in} = 0 \quad \text{for } i=1,\ldots,n.$$

Let us consider the determinant

$$C = \begin{vmatrix} |C_{0n}| & |C_{1n}| & \ldots & |C_{n-1n}| & |C_{nn}| \\ |C_{0n+1}| & |C_{1n+1}| & \ldots & |C_{n-1n+1}| & |C_{nn}| \\ \vdots & \vdots & & & \vdots \\ |C_{02n-1}| & |C_{12n-1}| & \ldots & |C_{n-12n-1}| & |C_{nn}| \\ 1 & 1 & \ldots & 1 & -n \end{vmatrix}.$$

Obviously, from (2) it follows that the condition $C=0$ is sufficient to the thesis of Lemma be fulfilled. In order to show this equality we add the first column of C multiplied by -1 to the second one. And we repeat this n-1 times with respect to the remaining columns, except the last one. Next we multiply the first column by n and add to the last one. Then we obtain

$$C = (-1)^{n+1} \begin{vmatrix} |D_{1n}| & |D_{2n}| & \ldots & |D_{nn}| \\ |D_{1n+1}| & |D_{2n+1}| & \ldots & |D_{nn+1}| \\ \vdots & \vdots & & \vdots \\ |D_{12n-1}| & |D_{22n-1}| & \ldots & |D_{n2n-1}| \end{vmatrix}.$$

If now we add to the first row the all remaining rows, then we obtain

$$C = (-1)^{n-1} \begin{vmatrix} |E_{1n}| & |E_{2n}| & \ldots & |E_{nn}| \\ |D_{1n+1}| & |D_{2n+1}| & \ldots & |D_{nn+1}| \\ \vdots & \vdots & & \vdots \\ |D_{12n-1}| & |D_{22n-1}| & \ldots & |D_{n2n-1}| \end{vmatrix}.$$

Clearly, with respect to (3), $C=0$. This ends the proof of Lemma.

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

From just proved Lemma we obtain immediately the following generalization of the perspective case of the theorem of Pappus:

Theorem. Let $H_0,\dots,H_{n-1}$ and s,a_{ij} $i=0,\dots,n-1$, $j=0,\dots,n$ be respectively hyperplanes and points satisfying the assumptions formulated in the introduction. Then there exists a hyperplane P such that

$$Z(S_0,S_1,\dots,S_n) \subset P.$$

Remark. Clearly, $\bigcap_{i=0}^{n-1} H_i \in P$.

REFERENCES

[1] K. Witczyński: Generalized Pappus Theorem in the Projective Space P^n, Bull.Acad.Polon.Sci.Ser.Sci.Math.,Astr.Phys. 6 (1979), 705-709.

[2] D. Witczyńska: Pappus' Configuration in the Projective Space P^n, Demonstratio Math. 12(1979), 593-598.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA

Received December 20, 1989.

130274

LIBRARY
Gurukul Kangri Vishwavidyalaya
HARIDWAR

Entered in Data
Signature with Date

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri

CC-0. In Public Domain. Gurukul Kangri Collection, Haridwar

Digitized by Arya Samaj Foundation Chennai and eGangotri